Optimal global regularity for elliptic equations which is degenerate or singular on the boundary

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The Original Problem

\[(MA)\]

$$\det D^2 u = F(x, u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial \Omega$$

where \(\Omega\) is a bounded convex domain in \(\mathbb{R}^n\), and \(F\) satisfies the assumption \((H)\):

\[F(x, t) \in C(\Omega \times (-\infty, 0)) \text{ is non-decreasing in } t \text{ for any } x \in \Omega\]

and

\[0 < F(x, t) \leq A d_x^{\beta-n-1} |t|^{-\alpha}, \forall (x, t) \in \Omega \times (-\infty, 0)\]

for some constants \(A > 0, \beta \geq n + 1 \text{ and } \alpha \geq 0\), where \(d_x = \text{dist}(x, \partial\Omega)\).
Motivations (1)

When $F(x, t) = |t|^{-(n-2)}$ and $u$ is a solution to problem $(MA)$, then

- $(-u)^{-1} u x_i x_j dx_i dx_j$ is a Hilbert metric in $\Omega$
  — [Loewner-Nirenberg: 1974]
- The Legendre transform
  \[ y = Du(x), \quad u^*(y) = x \cdot y - u(x). \]

The graph of $u^*$ defines an Affine hyperbolic spheres
- Affine hyperbolic sphere is a well-known important model in Affine geometry as well as a fundamental model in Affine Sphere Relativity
  — [Minguzzi: CMP, 2017]
Motivations (2)

• When $F = f(x)|t|^{-p}$, Problem $(MA)$ is the Projection of the equation 

$$
det D^2u = f(x)u^{-p} \text{ in } S^n \subset R^{n+1}$$

on the plane $\{x_{n+1} = -1\}$ from the unit sphere $S^n$, which is $L_p$-Minkowski problem in the affine geometry


• Also, for general $F$, Problem $(MA)$ may be obtained from the constructing non-homogeneous complete Einstein-Kähler metrics on a tubular domain

— [Cheng-Yau: CPAM, 1986].
Cheng-Yau’s Results

Cheng and Yau in [Cheng-Yau: CPAM, 1977] proved that if $\Omega$ is a strictly convex $C^2$-domain and $F \in C^k_k$ ($k \geq 3$) satisfies (H), then the problem $(MA)$ admits an unique convex solution $u \in C^{k+1,\varepsilon}(\Omega) \cap C(\bar{\Omega})$ for any $\varepsilon \in (0, 1)$.
Questions

• **Q 1**: For the Affine Hyperbolic Sphere (AHS) \( \det D^2 u = |u|^{-n-2} \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \), what is the (optimal) boundary regularity of the Affine Hyperbolic Sphere? —-(mentioned a few times by S. T. Yau)

• **Q 2**: What is the existence, uniqueness and the optimal boundary regularity for the problem (MA) even if \( F \) is not in \( C^3 \), or \( \partial \Omega \) is not in \( C^2 \), or \( \Omega \) is not strictly convex?
Answer to Q1 (Yau’s Question)

Theorem 1 [J-Wang: JDG, 2013] Suppose $\Omega$ is a bounded, uniformly convex domain in $\mathbb{R}^n$ with $C^{k,\alpha}$ boundary, where $3 \leq k \leq n + 2$ and $\alpha \in (0, 1)$. Then the graph $M_v$ is $C^{k,\alpha}$ up to its boundary.

Theorem 2 [J-Wang-Zhao: JDE, 2017] If $n$ is even, then the graph $M_v$ is $C^\infty$ up to its boundary if $\partial\Omega \in C^\infty$. But if $n$ is odd, the result in Theorem 1 is optimal.
Regularity results for uniformly elliptic case

\[ \det D^2 u = f(x) \text{ in } \Omega, \quad u = \phi \text{ on } \partial \Omega, \]

where \( \Omega \) is bounded and strictly convex.

- Assume \( 0 < c_1 \leq f(x) \leq c_2 \) and \( f \in C^k \ (k \geq 2) \), and \( \partial \Omega, \phi \) are sufficiently smooth. The \( C^{k+1} \)-regularity for the solution was obtained by
  - Clabi (1958)
  - Caffarelli-Nirenberg-Spruck (1983)
  - Tian (1983), Trudinger-Urbas (1983), \cdots, etc
Assume $0 < c_1 \leq f(x) \leq c_2$ and $f \in C^\alpha$

- The **interior** $C^{2,\alpha}$-regularity for the solution was obtained by
  — [Caffarelli: Ann Math, 1990] for $\alpha \in (0, 1)$, which was re-proved by [J-Wang: Siam J Math Anal, 2007] for $\alpha \in [0, 1]$

- The **boundary** $C^{2,\alpha}$-regularity was obtained by
  — [Trudinger-Wang: Ann Math, 2008] when and $\partial \Omega, \phi \in C^3$
  — [Savin: JAMS, 2012] when $\partial \Omega, \phi \in C^{2,\alpha}$.
Regularity results for degenerate elliptic case

• If $f \geq 0$ and $f^{1/(n-1)} \in C^{1,1}$, then $u \in C^{1,1}$.

• If $f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the regularity of the solution and its asymptotic expansion near the origin was studied by
  — [Rios C., Sawyer E.T., Wheeden R.L: Adv Math, 2006]
  — [Savin: CPAM, 2010]

• Assume $0 < c_1 \leq f(x) \leq c_2$ and $f \in C^\alpha$ ($\alpha \in (0, 1]$). The global $C^{2,\alpha}$-regularity for the problem
  \[ \det D^2u = f(x)(dx)\alpha \] in $\Omega$, $u = \phi$ on $\partial\Omega$,
  was obtained by
  — [Savin: Invent Math, 2017].
• The $C^\infty(\bar{\Omega})$ solution to the Eigenvalue Problem

\[ \det D^2u = (-\lambda u)^n \] in $\Omega$, $u = 0$ on $\partial\Omega$,

was obtained by
— [Hong-Huang-Wang: CPDE, 2011] when $n = 2$

• Suppose that $\mu(u) > 0$ is nondecreasing in $u$, $p > n + 1$, $\alpha \in [0, 2(p - n - 1))$, and $\partial\Omega \in C^{1,1}$. The $C^2(\Omega) \cap C^\delta(\bar{\Omega})$ (for some $\delta \in (0, 1)$) solution for the problem

\[ \det D^2u = \mu(u)(d_x)^\alpha(1 + |Du|^2)^p \] in $\Omega$, $u = \phi$ on $\partial\Omega$

was obtained by
— [Chen: Lecture Notes Math, No 1306, 1986]
— [Urbas: Invent Math, 1986]
Q2: How are about the existence, uniqueness and the optimal boundary regularity of the solution to the

\[(MA) \quad \det D^2u = F(x, u) \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial\Omega\]

when $F$ is not in $C^3$, or $\partial\Omega$ is not in $C^2$, or $\Omega$ is not strictly convex? Here, $F$ satisfies the assumption $(H)$:

$F(x, t) \in C(\Omega \times (-\infty, 0))$ is non-decreasing in $t$ for any $x \in \Omega$

and

\[0 < F(x, t) \leq Ad_x^{\beta-n-1}|t|^{-\alpha}, \; \forall (x, t) \in \Omega \times (-\infty, 0)\]

for some constants $A > 0$, $\beta \geq n + 1$ and $\alpha \geq 0$, where $d_x = dist(x, \partial\Omega)$. 
Answer to Question 2

Theorem 3 [J-Li-Tu: Preprint, 2018]  
Supposed that $\Omega$ is a bounded convex domain in $\mathbb{R}^n$ and $F(x,t)$ satisfies $(H)$. Let

$$
\gamma_1 := \begin{cases} 
\frac{\beta-n+1}{n+\alpha}, & \text{if } \beta < \alpha + 2n - 1, \\
\text{any number in } (0,1), & \text{if } \beta \geq \alpha + 2n - 1.
\end{cases}
$$

Then problem $(MA)$ admits an unique convex (Alexandrov) generalized solution $u \in C^{\gamma_1}(\overline{\Omega})$. Furthermore, $u \in C^{2,\gamma_1}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty,0))$. 
Improved the Regularity for \((a, \eta)\) type domain

Denote

\[ x = (x_1, x_2, \ldots, x_n) = (x', x_n), \quad x' = (x_1, \ldots, x_{n-1}) \]

**Definition.** *Supposed that \(\Omega\) is a bounded convex domain in \(\mathbb{R}^n\), and \(x_0 \in \partial \Omega\). We say \(x_0\) is \((a, \eta)\) type if there are numbers \(a \in [1, +\infty)\) and \(\eta > 0\), after translation and rotation transforms, we have

\[ x_0 = 0 \quad \text{and} \quad \Omega \subseteq \{x \in \mathbb{R}^n | x_n \geq \eta |x'|^a\}. \]

*\(\Omega\) is called \((a, \eta)\) type domain if every point of \(\partial \Omega\) is \((a, \eta)\) type.*
Remarks on \((a, \eta)\) type domain

**Remark 1.** The convexity requires that the number \(a\) should be no less than 1. The less is \(a\), the more convex is the domain. There is no \((a, \eta)\) type domain for \(a \in [1, 2)\), although part of \(\partial \Omega\) may be \((a, \eta)\) type for \(a \in [1, 2)\).

**Remark 2.** \((2, \eta)\) type domain is equivalent to the domain satisfies exterior sphere condition.
Hölder exponent can be described by the convexity for $\Omega$.

**Theorem 4 [J-Li-Tu: Preprint, 2018]** Supposed that $\Omega$ is $(a, \eta)$ type domain in $\mathbb{R}^n$ with $a \in [2, +\infty)$, and $F$ satisfies (H). Let

$$
\gamma_2 := \begin{cases} 
\frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)}, & \text{if } \beta < \alpha + 2n - 1 - \frac{2n-2}{a}, \\
\text{any number in } (0, 1), & \text{if } \beta \geq \alpha + 2n - 1 - \frac{2n-2}{a}.
\end{cases}
$$

Then the convex generalized solution obtained in Theorem 3 satisfies

$$
u \in C^{\gamma_2}(\overline{\Omega}).$$

Furthermore $u \in C^{2, \gamma_2}(\Omega)$ if $F(x, t) \in C^{0, 1}(\Omega \times (-\infty, 0))$.

**Remark** This result was obtained by J-Li in JDE, 2018 for $F \equiv t^{-n-2}$.
The boundary regularity of Theorems 3 and 4 is optimal

Consider the equation for affine hyperbolic sphere

$$\det D^2 u = \frac{1}{|u|^{n+2}} \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega$$

The $\gamma_1 = \frac{1}{n+1}$ for general convex domain, and $\gamma_2 = \frac{1}{2}$ for $(2, \eta)$ type domain, any of which can not be improved. In fact,

(1) If $\Omega = B_1^{n-1}(0) \times R^+$, then the solution is

$$u(x) = -\frac{(n + 1)^{\frac{1}{2}}}{n} \frac{1}{n^{\frac{n+1}{2}}} [1 - (x_1 + \cdots + x_{n-1})^2]^{\frac{n}{2(n+1)}};$$

(2) If $\Omega = B_1(0)$, then $u(x) = -\sqrt{1 - |x|^2}$. 
The sketch of the proof of Theorem 3-(1)

- **Lemma:** Let $\Omega$ be a bounded convex domain and $u$ be a convex function with $u|_{\partial\Omega} = 0$. If there is a $\gamma \in (0, 1]$ and a $M > 0$ such that

$$|u(y)| \leq M d_y \gamma, \quad \forall y \in \Omega$$

where $d_y = \text{dist}(y, \partial\Omega)$, then

$$|u|_{C^\gamma(\overline{\Omega})} \leq M [1 + \left(\frac{\text{diam}(\Omega)}{2}\right)^\gamma].$$

- For any point $y \in \Omega$, letting $z \in \partial\Omega$ be the nearest boundary point to $y$. Since problem (MA) is invariant under translation and rotation transforms, we assume $z = 0$, $\Omega \subseteq R^n_+$ and the line $yz$ is the $x_n$ axis.
The sketch of the proof of Theorem 3-(2)

• Let

\[ W(x) = -M x_n^\gamma \cdot \sqrt{N^2 l^2 - r^2} \]

where \( l = \text{diam}(\Omega) \) and \( r = \sqrt{x_1^2 + \ldots + x_{n-1}^2} \). Choosing positive constants \( \gamma, M, N \) and after tedious calculation we find that \( W \) is an sub-solution to problem (MA).

• By comparison principle for generalized solutions, we have

\[ |u(y)| \leq |W(y)| \leq M N l y_n^{\beta-n+1} = M N l d y^{\beta-n+1} \]

which, together with the Lemma, implies the following
The sketch of the proof of Theorem 3-(3)

- **A Priori Estimate:** Under the assumptions of Theorem 3, if \( u \in C(\overline{\Omega}) \) is a convex generalized solution to problem \((MA)\), then \( u \in C^{\gamma_1}(\overline{\Omega}) \) and

\[
|u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha, \beta, A, diam(\Omega), n).
\]

- The above method can be used to prove Theorem 4, but constructing the sub-solution to problem \((MA)\) is much more complicated. Its form is

\[
W(x_1, \ldots, x_n) = -\left[ \left( \frac{x_n}{\varepsilon} \right)^{\frac{2}{a}} - x_1^2 - \ldots - x_{n-1}^2 \right]^{\frac{1}{b}}
\]

where \( b \) and \( \varepsilon \) need to be chosen according three cases

\[ a = 2; \quad a \geq \frac{2\alpha+2}{\beta-n+1}; \quad 2 < a < \frac{2\alpha+2}{\beta-n+1} \quad \text{if} \quad \frac{2\alpha+2}{\beta-n+1} > 2. \]
The sketch of the proof of Theorem 3-(4)

• Suppose that \( \Omega \) is bounded convex but \( F(x, t) \in C^k(\Omega \times (-\infty, 0)) \) \((k \geq 3)\) satisfies (H).

Choose a sequence of bounded and strictly convex domains \( \{\Omega_i\} \) such that

\[
\Omega_i \in C^2 \quad \text{and} \quad \Omega_i \subseteq \Omega_{i+1}, \quad i = 1, 2, \ldots, \quad \bigcup_{i=1}^{\infty} \Omega_i = \Omega.
\]

Then by Cheng-Yau’s result, there exists a convex generalized solution \( u_i \) to problem (MA) in the domain \( \Omega_i \) for each \( i \). Set \( u_i \equiv 0 \) in \( \mathbb{R}^n \setminus \Omega_i \) and extend \( u \) in \( \mathbb{R}^n \). By the a priori estimate, we have the uniform estimations

\[
|u_i|_{C^{\gamma_1}(\Omega)} = |u_i|_{C^{\gamma_1}(\Omega_i)} \leq C(\alpha, \beta, A, \text{diam}(\Omega), n),
\]
The sketch of the proof of Theorem 3-(5)

which implies that there is a subsequence, still denoted by itself, convergent to a $u$ in the space $C(\overline{\Omega})$ and

$\textbf{Hölder estimate} \quad |u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha, \beta, A, \text{diam}(\Omega), n)$.

By the well-known convergence result for convex generalized solutions, we see that $u$ is a convex generalized solution to problem (MA).

• Drop the restriction on the smoothness for $F$.

Suppose $F_j \in C^k(\Omega \times (-\infty, 0))$ ($k \geq 3$) satisfy satisfies (H) as above, and $F_j$ locally uniform convergence to $F$ in as $j \to \infty$. Then by the above result, for each $j$, there exists a convex generalized solution $u_j \in C^{\gamma_1}(\overline{\Omega})$ to problem (MA) with $F$ replaced by $F_j$. 
The sketch of the proof of Theorem 3-(6)

Moreover, by the Hölder estimate we have

\[ |u_j|_{C^{\gamma_1}(\Omega)} \leq C(\alpha, \beta, A, \text{diam}(\Omega), n) \]

for all \( j \). Using this estimate, we obtain a generalized solution \( u \) to problem \((MA)\), which is the limit of a subsequence of \( u_j \) in the space space \( C(\Omega) \). Furthermore, the solution \( u \) still satisfies the Hölder estimate. The uniqueness for \((MA)\) is directly from the comparison principle.

• It remains to prove \( u \in C^{2, \gamma_1}(\Omega) \) if \( F(x, t) \in C^{0,1}(\Omega \times (-\infty, 0)) \). The Hölder estimate implies \( F(x, u(x)) \in C^{\gamma_1}(\Omega) \). Hence we can use the Caffarelli’s local \( C^{2,\alpha} \) regularity to obtain \( u \in C^{2, \gamma_1}(\Omega) \). [Caffarelli: Ann Math, 1990; Jian-Wang: Siam J Math Anna, 2007]
Finding proper affine hyperspheres with mean curvature $H$ which is asymptotic to a cone in $\mathbb{R}^{n+k+1}$ is reduced to solve

\begin{equation}
\det D^2 u = \frac{[x \cdot \nabla u(x) - u(x)]^{-k}}{[Hu(x)]^{n+k+2}} \quad \text{in } \Omega \subset \mathbb{R}^n,
\end{equation}

\begin{equation}
u = 0 \quad \text{on } \partial \Omega
\end{equation}

where $\Omega$ is a bounded convex domain containing the origin, $H < 0$ and $k \geq 0$ are constants.

- Haodi Chen and Genggeng Huang in [JDE: 267 (2019)] proved that $(PAS)$ admits a unique convex solution $u \in C(\bar{\Omega}) \cap C^\infty(\Omega)$. 
Application to proper affine hyperspheres-2

Since \( \Omega \) contains the origin, \( u \) is convex and \( u = 0 \) on \( \partial \Omega \), then

\[
x \cdot \nabla u(x) - u(x) \geq -u(0) > 0.
\]

Therefore \( f(x, u) := \frac{[x \cdot \nabla u(x) - u(x)]^{-k}}{[Hu(x)]^{n+k+2}} \) satisfies

\[
0 < f(x, u) \leq \frac{(-u(0))^{-k}}{(-H)^{n+k+2}} |u(x)|^{-n-k-2}.
\]

By Theorem 4, we have

**Theorem 5** \( \) Supposed that \( \Omega \) is \((a, \eta)\) type domain in \( \mathbb{R}^n \) with \( a \in [2, +\infty) \). Let \( \gamma_3 = \frac{a+n-1}{a(n+1+k/2)} \). Then the convex solution to \((AFS)\) satisfies \( u \in C^{\gamma_3}(\overline{\Omega}) \).
Our method can be applied to Chaplygin gas and minimal graph

\[ \Delta u - \frac{u_i u_j}{1 + |\nabla u|^2} u_{ij} = -\frac{n}{u} \text{ in } \Omega \]

\[ u = 0 \text{ on } \partial \Omega \]

\[ u > 0 \text{ in } \Omega, \]

Where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \).

- \( n = 2 \): two dimensional Riemann problem with four-shock data (the vortex and the saddle) for Chaplygin gas

- \( n \geq 2 \): the graph of \( u \) defines a minimal graph in hyperbolic space
The Existence and Uniqueness

• There is a unique solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ if $\Omega \in C^2$ and the mean curvature $H|_{\partial\Omega} \geq 0$.
  —[Lin, Invent Math 1989]

• There is a unique solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ if $n = 2$ and $\Omega$ is piecewise $C^2$-convex domain and the curvature $K|_{\partial\Omega} > 0$.
Questions

• **Q 3**: Is the solution \( u \) Hölder continuous up to the boundary?

• Answered by [Lin: Invent Math 1989] if \( \Omega \in C^2 \) and curvature \( H|_{\partial \Omega} > 0 \);

• Answered by [Han-Shen and Yue Wang: Car Var PDE, 2016] if \( \Omega \) is piecewise \( C^2 \) and curvature \( H|_{\partial \Omega} > 0 \);
Concave Solutions

Assume that $\Omega$ is a bounded convex domain. Then the problem admits a unique solution $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$, and $u$ is concave. Moreover,

$$u \in C^{\frac{1}{n+1}}(\overline{\Omega})$$

This result was proved by Qing Han, Weiming Shen and Yue Wang in Car Var PDE, 2016
The regularity depends on the convexity.

Applying our method and constructing the super-solution in the form

\[ W(x_1, ..., x_n) = \left( \left( \frac{x_n}{\varepsilon} \right)^{\frac{2}{a}} - x_1^2 - ... - x_{n-1}^2 \right)^{\frac{1}{b}} \]

where \( b \geq 2 \) and \( \varepsilon > 0 \) are to be determined, we obtain

**Theorem 6 [J-You Li: Preprint, 2018]** Let \( \Omega \) be \((a, \eta)\) type domain with \( a \in [2, +\infty) \). Then

\[ u \in C^{\frac{1}{\bar{a}}}(\overline{\Omega}) \]

where \( \bar{a} = \max\{\frac{1}{a}, \frac{1}{n+1}\} \).
Thank You!