Analysis on steady subsonic solutions with both fixed and free boundaries

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Motivation

- The well-posedness theory for unsteady compressible Euler equations is widely open
- An important problem in the transonic flows
Three Dimensional Euler System and Divergent Nozzles

The three-dimensional steady full Euler system reads as

\[
\begin{aligned}
\text{div } (\rho \mathbf{u}) &= 0, \\
\text{div } (\rho \mathbf{u} \otimes \mathbf{u} + P \mathbf{l}_n) &= 0, \\
\text{div } (\rho (\frac{1}{2} |\mathbf{u}|^2 + e) \mathbf{u} + P \mathbf{u}) &= 0,
\end{aligned}
\]

where \( \mathbf{u} = (u_1, u_2, u_3) \), \( \rho \), \( P \), \( e \) and \( S \) stand for the velocity, density, pressure, internal energy and specific entropy, respectively. The equation of state, the internal energy \( e \), and the sound speed are given by

\[
P = A \rho^\gamma e^{\frac{S}{cv}}, \quad e = \frac{P}{(\gamma - 1) \rho}, \quad c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}.
\]

The nozzle wall \( \Gamma^2 \) can be represented by

\[
\sqrt{x_2^2 + x_3^2} = x_1 \tan(\theta_0 + \epsilon f(r)), \quad x_1 > 0, \quad r_1 < r < r_2
\]

and \( \theta_0 \in (0, \frac{\pi}{2}) \) and \( f \) is a smooth \( C^2, \alpha \) function defined on \([r_1, r_2]\).
Given $U_b^-(r_1) > c(\rho_b(r_1), S_b^-) > 0$ and $P_b(r_1), S_b^-$,

$$(u^-, P_b^-, S_b^-)(x) = (U_b^-(r_1)e_r, P_b^-(r_1), S_b^-)$$

at $r = r_1$,

there exists two positive constants $P_1$ and $P_2$ such that if the pressure $P_e \in (P_1, P_2)$ is posed at the exit $r = r_2$, there exists a unique spherical symmetric transonic shock solution

$$(u_b^\pm, P_b^\pm, S_b^\pm)(x) = (U_b^\pm(r)e_r, P_b^\pm(r), S_b^\pm),$$

(3)

to (1) defined in

$$\Omega^-_{un} = \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 \leq x_1^2 \tan^2 \theta_0, r \in (r_1, r_b)\}$$

and

$$\Omega^+_{un} = \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 \geq x_1^2 \tan^2 \theta_0, r \in (r_b, r_2)\},$$

where $r = r_b \in (r_1, r_2)$ is a shock wave, and

$$[\rho U_b] = 0, \quad [\rho_b U_b^2 + P_b] = 0, \quad S_b^+ > S_b^-,$$

where $[f]$ denotes the jump of $f$ at $r = r_b$. 
Introduce the spherical coordinates

\[ x_1 = r \cos \theta, \quad x_2 = r \sin \theta \cos \varphi, \quad x_3 = r \sin \theta \sin \varphi. \quad (4) \]

and decompose the velocity \( \mathbf{u} = U_1 \mathbf{e}_r + U_2 \mathbf{e}_\theta + U_3 \mathbf{e}_\varphi \). The axisymmetric solutions do not depend on \( \varphi \) so that the Euler system reads

\[
\begin{aligned}
\partial_r (r^2 \rho U_1 \sin \theta) + \partial_\theta (r \rho U_2 \sin \theta) &= 0, \\
\rho U_1 \partial_r U_1 + \frac{1}{r} \rho U_2 \partial_\theta U_1 + \partial_r P - \frac{\rho (U_2^2 + U_3^2)}{r} &= 0, \\
\rho U_1 \partial_r U_2 + \frac{1}{r} \rho U_2 \partial_\theta U_2 + \frac{1}{r} \partial_\theta P + \frac{\rho U_1 U_2}{r} - \frac{\rho U_3^2}{r} \cot \theta &= 0, \\
\rho U_1 \partial_r (r U_3 \sin \theta) + \frac{1}{r} \rho U_2 \partial_\theta (r U_3 \sin \theta) &= 0, \\
\rho U_1 \partial_r S + \frac{1}{r} \rho U_2 \partial_\theta S &= 0.
\end{aligned}
\quad (5)
\]
The perturbed nozzle is
\[ \Omega = \{(r, \theta, \varphi) : r_1 < r < r_2, 0 \leq \theta \leq \theta_0 + \epsilon f(r), \varphi \in [0, 2\pi]\}, \]
where \( f \in C^{2,\alpha}([r_1, r_2]) \) satisfying
\[ f(r_1) = f'(r_1) = 0. \] (6)

Suppose the supersonic incoming flow at the inlet \( r = r_1 \) is given by
\[ \Phi_{en} = (U_1^-, U_2^-, U_3^-, P^-, S^-) = \Phi_b^- + \epsilon \Psi(\theta), \] (7)
where \( \Phi_b^- = (U_b^-(r), 0, 0, P_b^-(r), S_b^-) \) and \( \Psi(\theta) \in (C^{2,\alpha}([0, \theta_0]))^5 \).

At the exit of the nozzle, the end pressure is prescribed by
\[ P^+(x) = P_e + \epsilon P_0(\theta) \text{ on } r = r_2, \] (8)
here \( \epsilon > 0 \) is sufficiently small, and \( P_0 \in C^{1,\alpha}([0, 2\theta_0]) \).
Denote the transonic shock surface by $S$ and the upstream and downstream flows by $x_1 = \eta(x_2, x_3)$ and $(u^\pm, P^\pm, S^\pm)(x)$, respectively. Then the Rankine-Hugoniot conditions on $S$ become

\[
\begin{align*}
\left[(1, -\nabla_{x'}\eta(x')) \cdot \rho u\right] &= 0, \\
\left[((1, -\nabla_{x'}\eta(x')) \cdot \rho u)u\right] + (1, -\nabla_{x'}\eta(x'))^t[P] &= 0, \\
\left[(1, -\nabla_{x'}\eta(x')) \cdot (\rho(e + \frac{1}{2}|u|^2) + P)u\right] &= 0,
\end{align*}
\]

where $\nabla_{x'} = (\partial_{x_2}, \partial_{x_3})$. Moreover, the physical entropy condition is also satisfied

\[
S^+(x) > S^-(x), \quad \text{on} \quad x_1 = \eta(x_2, x_3).
\]
**Theorem 1** (Weng, Xie, Xin) Given the supersonic incoming flow \( \Phi_{en} \) satisfying the certain compatibility conditions, the transonic shock problem has a unique solution

\[
\Phi^+ = (U_1^+, U_2^+, U_3^+, P^+, S^+)(r, \theta) \quad \text{and} \quad \xi(\theta)
\]

(i) \( \xi(\theta) \in C_{3,\alpha; (0, \theta_*)}^{(-1-\alpha; \{\theta_*\})} \) and

\[
\| \xi(\theta) - r_b \|_{3,\alpha; (0, \theta_*)}^{(-1-\alpha; \{\theta_*\})} \leq C_0 \epsilon, \tag{11}
\]

where \( (r_*, \theta_*) \) stands for the intersection circle of the shock surface with the nozzle wall and \( C_0 \) is a positive constant depending only on the supersonic incoming flow.

(ii) \( \Phi^+(r, \theta) \in C_{2,\alpha; R_+}^{(-\alpha; \Gamma_{w,s})} \), and

\[
\| \Phi^+ - \Phi_b^+ \|_{2,\alpha; R_+}^{(-\alpha; \Gamma_{w,s})} \leq C_0 \epsilon, \tag{12}
\]

where

\[
\Gamma_{w,s} = \{(r, \theta) : \xi(\theta) \leq r \leq r_2, \theta = \theta_0 + \epsilon f(r)\}.
\]
Know Results and Remarks

**Known Results**

- Potential flows: G.-Q. Chen and Feldman (Dirichlet condition for velocity potential at the exit), Xin and Yin (the problem is in general ill-posedness given the exit pressure), Bae and Feldman (Non-isentropic potential flows)

- flat nozzle for the Euler system: G. Q. Chen et al for velocity boundary conditions at the exit, S. X. Chen etc for the particular pressure at the exit

- Divergent nozzle for the Euler system: Li-Xin-Yin for 2D and 3D axisymmetric without swirl, S. X. Chen for 2D case

**Remark**

- The nozzle wall $\Gamma^2$ can depend on both $r$ and $\theta$.

- There is another result on the stability of transonic shock for 3D axisymmetric case with swirl via a different approach by Park after we uploaded the paper
There is a singular factor \( \sin \theta \) in the density equation of (5), the standard Lagragian coordinate used by Li-Xin-Yin is not invertible near the axis \( \theta = 0 \).

Observation: \( \sin \theta \) is of order \( O(\theta) \) near \( \theta = 0 \). Define \((\tilde{y}_1, \tilde{y}_2) = (r, \tilde{y}_2(r, \theta))\) such that

\[
\begin{align*}
\frac{\partial \tilde{y}_2}{\partial r} &= -r \rho^{\pm} U_2^{\pm} \sin \theta, \\
\frac{\partial \tilde{y}_2}{\partial \theta} &= r^2 \rho^{\pm} U_1^{\pm} \sin \theta, \\
\text{if } (r, \theta) \in \overline{R}_\pm, \\
\tilde{y}_2(r_1, 0) &= 0, \\
\tilde{y}_2(r_2, 0) &= 0.
\end{align*}
\]

It is clear that \( \tilde{y}_2 \geq 0 \) in \( \overline{R_-} \cup \overline{R_+} \). Setting

\[
\begin{align*}
y_1 &= \tilde{y}_1 = r, \\
y_2 &= \tilde{y}_2^{\frac{1}{2}}(r, \theta).
\end{align*}
\]

The transformation \( \mathcal{L} : (r, \theta) \in \bar{R} \mapsto (y_1, y_2) \in \bar{D} \) satisfies

\[
\det \begin{pmatrix}
\frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta} \\
\frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta}
\end{pmatrix} = \frac{r^2 \rho U_1 \sin \theta}{2y_2} \geq C_3 > 0. \tag{13}
\]
Reformulated System and the Domain

The reformulated system can be written as

\[
\begin{align*}
\partial_{y_1}\left(\frac{2y_2}{y_1^2\rho U_1 \sin \theta}\right) - \partial_{y_2}\left(\frac{U_2}{y_1 U_1}\right) &= 0, \\
\partial_{y_1}\left(U_1 + \frac{P}{\rho U_1}\right) - \frac{y_1 \sin \theta}{2y_2} \partial_{y_2}\left(\frac{P U_2}{U_1}\right) - \frac{2P}{y_1 \rho U_1} - \frac{P U_2 \cos \theta}{y_1 \rho U_1^2 \sin \theta} - \frac{(U_2^2 + U_3^2)}{y_1 U_1} &= 0, \\
\partial_{y_1}(y_1 U_2) + \frac{y_1^2 \sin \theta}{2y_2} \partial_{y_2} P - \frac{U_3^2}{U_1} \cot \theta &= 0, \\
\partial_{y_1}(y_1 U_3 \sin \theta) &= 0, \\
\partial_{y_1}B &= 0.
\end{align*}
\]

The nozzle wall \( \Gamma_{w,s} \) is straighten to be \( \Gamma_{w,y} = (\phi(M), r_2) \times \{M\} \).
Put \( \varpi = \frac{U_2}{U_1} \), then one has

\[
\begin{aligned}
\partial y_1 \varpi - \frac{y_1 \rho U_1 \varpi \sin \theta}{2y_2} \partial y_2 \varpi - \frac{\varpi}{y_1} \varpi^2 \cot \theta + \frac{y_1 \sin \theta}{2y_2 U_1} \partial y_2 P \\
- \frac{\varpi}{\rho c^2(\rho, S)} \partial y_1 P - \frac{U_2^2}{y_1 U_1^2} \cot \theta = 0, \\
\partial y_1 P - \frac{\rho c^2(\rho, S) U_1}{y_1(c^2(\rho, S) - U_1^2)} \left( \frac{y_1^2 \rho U_1 \sin \theta}{2y_2} \right) \partial y_2 \varpi + \varpi \cot \theta \\
- \frac{y_1 \rho c^2(\rho, S) U_1 \varpi \sin \theta}{2y_2(c^2(\rho, S) - U_1^2)} \partial y_2 P - \frac{\rho c^2(\rho, S) U_1^2}{y_1(c^2(\rho, S) - U_1^2)} (\varpi^2 + 2) \\
- \frac{\rho c^2(\rho, S) U_3^2}{y_1(c^2(\rho, S) - U_1^2)} = 0.
\end{aligned}
\]

The corresponding boundary conditions become

\[
\begin{aligned}
\varpi(y_1, 0) = 0, \quad \forall y_1 \in [r_1, r_2], \\
\varpi(y_1, M) = \epsilon y_1 f'(y_1), \quad \forall y_1 \in [r_1, r_2], \\
P(r_2, y_2) = P_e + \epsilon P_0(\theta(r_2, y_2)), \quad \forall y_2 \in [0, M].
\end{aligned}
\]
Fix the Domain

Introduce the coordinate transformation

\[ z_1 = \frac{y_1 - \psi(y_2)}{r_2 - \psi(y_2)} N, \quad z_2 = y_2, \quad N = r_2 - r_b \]

so that the free boundary becomes a fixed boundary. Setting

\[
\begin{align*}
W_1(z) &= \tilde{U}_1(z) - \tilde{U}_0^+(z_1), \quad W_2(z) = \tilde{\omega}(z), \\
W_3(z) &= \tilde{U}_3(z), \quad W_4(z) = \tilde{P}(z) - \tilde{P}_b^+(z_1), \\
W_5(z) &= \tilde{S}(z) - S_b^+, \quad W_6(z_2) = \psi(z_2) - r_b.
\end{align*}
\]  

(14)

After this coordinate transformation, the equation for the shock becomes

\[
\psi'(z_2) = \frac{2z_2}{\sin \theta} \left( \frac{(\tilde{U}_b^+(0) + W_1) W_2 - U_2^-(r_b + W_6(z_2), z_2)}{(r_b + W_6(z_2))((\tilde{P}_b^+(0) + W_4) - P^-(r_b + W_6(z_2), z_2))} \right),
\]

where the functions are evaluated at \((0, z_2)\).
Define the solution class

\[ \Xi_\delta = \left\{ \mathbf{W} : \| \mathbf{W} \|_{\Xi_\delta} = \sum_{i=1}^{5} \| W_i \|_{2,\alpha;E+}^{(-\alpha;\Gamma_{W,z})} + \| W_6 \|_{3,\alpha;(0,M)}^{(-1-\alpha;\{M\})} \leq \delta; \right\} \]

\[ \partial_{z_2} W_j(z_1, 0) = 0, \; j = 1, 3, 4, 5; \; W'(0) = W_6^{(3)}(0) = 0; \]

\[ W_2(z_1, 0) = \partial_{z_2}^2 W_2(z_1, 0) = W_5(z_1, 0) = 0 \right\}. \]

Given any \( \hat{\mathbf{W}} \in \Xi_\delta \), we will develop an iteration to produce a new \( \mathbf{W} \in \Xi_\delta \) so we get a mapping \( T \) from \( \Xi_\delta \) to itself by choosing suitable small \( \delta \). To design a good iteration, we first need to find the explicit form of the leading linear order term, and all the \( \mathbf{W} \) in the remaining nonlinear error terms will be replaced by \( \hat{\mathbf{W}} \) and finally the error terms should be bounded by \( C(\| \hat{\mathbf{W}} \|_{\Xi_\delta}^2 + \epsilon) \).
It is easy to derive that
\[
\partial_{z_1} W_5 = 0, \quad \partial_{z_1} \tilde{B} = 0, \quad \forall z \in [0, N] \times [0, M). \tag{16}
\]
Furthermore, one has
\[
\begin{cases}
\partial_{z_1} \left[ (r_b + z_1 + \frac{N-z_1}{N} W_6(z_2)) W_3 \sin \theta(z_1, z_2) \right] = 0, \\
W_3(0, z_2) = U_3^-(r_0 + W_6(z_2), z_2).
\end{cases} \tag{17}
\]
The equation for the shock can be written as
\[
W_6'(z_2) = \frac{2z_2}{\sin \theta (r_b + W_6(z_2))} \left( \tilde{U}_b(0) + W_1 \right) W_2 - U_2^-(r_b + W_6(z_2), z_2)
\]
\[
\left( \tilde{P}_b^+(0) - P_b^-(r_b) + W_4 - (P^- - P_b^-(r_b)) \right),
\]
where \( W_i \) are evaluated at \((0, z_2)\) and \( P^- \) is evaluated at the corresponding point on the shock.
The elliptic modes can be governed by a problem for second order equation

\[ \begin{align*}
& \partial_{z_1} \left( \frac{\lambda_4(z_1)}{\lambda_2(z_1)} \partial_{z_1} \phi \right) - \left\{ a \lambda_6(z_1) + \frac{d}{dz_1} \left( \frac{\lambda_4(z_1)\lambda_3(z_1)}{\lambda_2(z_1)} \right) \right\} \left( \phi(0, z_2) - \frac{W_6(M)}{a} \right) \\
& \quad + \frac{\lambda_5(z_1)}{\lambda_1(z_1)} \left( \frac{\sin \theta_b(z_2)}{2z_2} \partial_{z_2} \left( \frac{\sin \theta_b(z_2)}{2z_2} \partial_{z_2} \phi \right) + \frac{\kappa_b \cos \theta_b(z_2)}{2z_2} \partial_{z_2} \phi \right) = F, \\
& \partial_{z_1} \phi(0, z_2) + \beta \left( \phi(0, z_2) - \frac{W_6(M)}{a} \right) = G, \\
& \partial_{z_1} \phi(N, z_2) = \epsilon \lambda_2(N) P_0(\hat{\theta}(N, z_2)) - \int_{z_2}^{M} G_1(N, s) ds, \\
& \partial_{z_2} \phi(z_1, 0) = 0, \\
& \partial_{z_2} \phi(z_1, M) = -\frac{2M}{\sin \theta_b(M)} \lambda_1(z_1) \epsilon (r_0 + z_1 + \frac{N - z_1}{N} \hat{W}_6(M)) f'.
\end{align*} \]

The solvability condition for this problem determines the location of the shock.
Jet Problems

General Jet Problems for Two Dimensional Flows

A Simpler Case for Two Dimensional Flows
Steady Euler System

2D steady Euler System:

\[
\begin{cases}
\text{div}(\rho u) = 0, \\
\text{div}(\rho u \otimes u) + \nabla p = 0,
\end{cases}
\]  

(19)

where \( p = p(\rho) \). If we denote \( p'(\rho) = c^2(\rho) \), and

\[
A = \begin{pmatrix}
\frac{uc^2(\rho)}{\rho} & c^2(\rho) & 0 \\
\rho & c^2(\rho) & 0 \\
0 & \rho u & 0
\end{pmatrix}, \\
B = \begin{pmatrix}
\frac{vc^2(\rho)}{\rho} & 0 & c^2(\rho) \\
0 & \rho v & 0 \\
c^2(\rho) & 0 & \rho v
\end{pmatrix}, \\
U = \begin{pmatrix}
\rho \\
u \\
v
\end{pmatrix}
\]

then, 2-D system can be written as

\[AU_{x_1} + BU_{x_2} = 0.\]

\[
det(\lambda A - B) = 0 \implies \lambda_1 = \frac{v}{u}, \quad \lambda_{\pm} = \frac{uv \pm c(\rho)\sqrt{u^2 + v^2 - c^2(\rho)}}{u^2 - c^2}.
\]
Boundary Conditions

- The nozzle walls are assumed to be impermeable

\[ (u, v) \cdot \vec{n} = 0, \quad \text{on } \partial \Omega, \]

where \( \vec{n} \) is the unit outer normal of the nozzle walls.

- The mass flux crossing any section transversal to the \( x_1 \)-axis remains a positive constant \( m_0 \),

\[ \int_S (\rho u, \rho v) \cdot \vec{l} \, dS = m, \]

where \( \vec{l} \) is the unit normal of \( S \) in the positive \( x_1 \)-direction.

- Prescribe horizontal velocity of the flow in the upstream,

\[ u(x_1, x_2) \rightarrow u_0(x_2) \quad \text{as } x_1 \rightarrow -\infty. \]

Remark: One can also prescribe the Bernoulli function in the upstream.
Jet Problem

**Problem** Given the incoming horizontal velocity $u_0$ and the total flux $m$, find $(\rho, u, v)$, the free boundary $\Gamma$, and the outer pressure $p_e$ such that $\Gamma$ connects with $S_1$, $(\rho, u_1, u_2)$ satisfies the Euler system (19) in $\Omega$, and

$$p(\rho) = p_e \quad \text{and} \quad (u_1, u_2) \cdot n = 0 \quad \text{on} \quad \Gamma,$$

where $\Omega$ is the region bounded by $S_0$, $S_1$, and $\Gamma$.

**Major Progress:**

- Early works: Gilbarg, Serrin, ...
- Alt, Caffarelli, Friedman (JDE, 1985): Existence of an irrotational solution via variational formulation (some recent reformulation by Lili Du, etc);
- Wang and Xin: Existence of a subsonic and sonic jet for potential flows via hodograph transformation
Main Results on Subsonic Flows with Jet

**Theorem 2** (Shi, Tang, Xie) Suppose that

\[ S_1 = \{(x_1, x_2) | x_1 = \xi(x_2), x_2 \in [1/2, 1]\} \] and

\[ S_0 = \{(x_1, 0) : x_1 \in \mathbb{R}\}. \]

Without loss of generality, we assume

\[ \lim_{x_2 \to 1} \xi(x_2) = -\infty. \]

There exists an \( \epsilon_0 > 0 \) such that

\[ u_0'(1) = 0, \quad |u_0'| + |u_0''| \leq \epsilon_0. \] (23)

There exists an \( m_{cr} \) such that as long as \( m > m_{cr} \), the jet problem has a unique solution. Furthermore, at far field, the free boundary has a representation \( x_2 = k(x_1) \) satisfying

\[ \lim_{x_1 \to \infty} k(x_1) = \bar{a} \]

where \( \bar{a} \) is unique determined by \( m \) and \( \bar{u}_1 \).

**Remarks:**

- Jets and cavities for 2D full Euler and 3D axisymmetric Euler system

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Subsonic Flows with Physical Boundaries
Proposition 1

\[ AU_{x_1} + BU_{x_2} = 0 \iff \begin{cases} 
(\rho u)_{x_1} + (\rho v)_{x_2} = 0, \\
(u, v) \cdot \nabla \left( \frac{u^2 + v^2}{2} + h(\rho) \right) = 0, \\
(u, v) \cdot \nabla \left( \frac{\omega}{\rho} \right) = 0,
\end{cases} \tag{24} \]

where \( \omega = v_{x_1} - u_{x_2} \), if the given flows satisfy

\[ u > 0 \text{ in } \Omega, \tag{25} \]

and the following asymptotic behavior

\[ u, \rho \text{ and } v_{x_2} \text{ are bounded, while } v, v_{x_1} \text{ and } \rho_{x_2} \to 0, \text{ as } x_1 \to -\infty. \]
Stream Function

Stream function $\psi$:

$$
\psi_{x_1} = -\rho v, \quad \psi_{x_2} = \rho u. \quad \Rightarrow \quad \nabla \perp \psi \cdot \nabla \left( \frac{u^2 + v^2}{2} + h(\rho) \right) = 0,
$$

where $\nabla \perp = (-\partial_{x_2}, \partial_{x_1})$.

$$
h(\rho) + \frac{|\nabla \psi|^2}{2\rho^2} = h(\rho) + \frac{1}{2}(u^2 + v^2) = B(\psi). \quad (26)
$$

In the upstream,

$$
\psi = \int_0^{X_2} \rho_0 u_0(s)ds \quad \Rightarrow \quad X_2 = \kappa(\psi). \quad (27)
$$

Set

$$
f(\psi) = u'_0(\kappa(\psi)), \quad \text{and} \quad F(\psi) = u_0(\kappa(\psi)). \quad (28)
$$

Then $f$ and $F$ are well-defined on $[0, m]$. Furthermore,

$$
f(\psi) = \rho_0 F(\psi)F'(\psi). \quad (29)$$
### Representation of Density and Vorticity

\[
(h(\rho) + \frac{|\nabla \psi|^2}{2\rho^2})(x_1, x_2) = (h(\rho) + \frac{u^2 + v^2}{2})(-\infty, \kappa(\psi))
\]

\[
= h(\rho_0) + \frac{F^2(\psi(x_1, x_2))}{2}.
\]

\[
\rho = H(|\nabla \psi|^2, \psi) = J \left(|\nabla \psi|^2, h(\rho_0) + \frac{F^2(\psi)}{2}\right), \quad (30)
\]

\[
\nabla \psi \cdot \nabla \left(\frac{\omega}{\rho}\right) = 0 \Rightarrow \frac{\omega}{\rho}(x_1, x_2) = -\frac{f(\psi(x_1, x_2))}{\rho_0} = -F(\psi)F'(\psi).
\]

\[
(31)
\]

The density \( \rho \) can be represented by

\[
\rho = H(|\nabla \psi|^2, \psi).
\]

One has the following boundary conditions

\[
\psi = 0 \text{ on } S_1, \text{ and } \psi = m \text{ on } S_2. \quad (32)
\]
Stream Function Formulation for the Jet Problem

Using the stream function formulation, the jet problem can be formulated into the following boundary value problem

\[
\nabla \cdot \left( g(\|\nabla \psi\|^2, \psi) \nabla \psi \right) - \frac{F(\psi)F'(\psi)}{g(\|\nabla \psi\|^2, \psi)} = 0 \text{ in } \{\psi < m\},
\]

\[
\psi = 0 \text{ on } \mathbb{R} \times \{0\},
\]

\[
\psi = m \text{ on } S_1 \cup \partial \{\psi < m\},
\]

\[
\|\nabla \psi\| = \Lambda \text{ on } \partial \{\psi < m\}
\]

and we also ask \(\psi\) satisfies

\[
\|\nabla \psi\|^2 < \Sigma^2(\psi) \text{ on } \{\psi < m\},
\]

where \(g = 1/H\) and \(p(H(\Lambda^2, m)) = p_e\).
Lemma 1 Let \( \psi \) be a minimizer of the problem

\[
\min_{\psi \in K_{\mu,R}} J_{\mu,R}^\epsilon(\psi),
\]

with

\[
K_{\mu,R} := \{ \psi \in H^1(\Omega_{\mu,R}) : \psi = \phi_{\mu,R} \text{ on } \partial \Omega_{\mu,R} \}.
\]

\[
J_{\mu,R}^\epsilon(\psi) := \int_{\Omega_{\mu,R}} G_\epsilon(|\nabla \psi|^2, \psi) + \lambda_\epsilon^2 \chi_{\{\psi < m\}} \, dx,
\]

where

\[
G_\epsilon(t, z) := \frac{1}{2} \int_0^t g_\epsilon(\tau, z) d\tau + \frac{1}{\gamma} \left( g_\epsilon(0, z)^{-\gamma} - g_\epsilon(0, m)^{-\gamma} \right)
\]

and

\[
\lambda_\epsilon^2 := 2\partial_t G_\epsilon(\Lambda^2, m)\Lambda^2 - G_\epsilon(\Lambda^2, m).
\]

Then \( \psi \) is a weak solution to the equation in (33) and satisfies the boundary conditions in (33) in the weak sense.
Let $\psi$ be a minimizer for (34).

- $\psi$ is a supersolution, i.e.

$$
\int_\Omega \partial_p G(\nabla \psi, \psi) \cdot \nabla \zeta + \partial_z G(\nabla \psi, \psi) \zeta \geq 0, \text{ for all } \zeta \geq 0, \zeta \in C_0^\infty(\Omega).
$$

- If $0 \leq \psi_0 \leq m$ on $\partial \Omega$, then

$$
0 \leq \psi \leq m.
$$

- $\psi \in C_{loc}^{0,\alpha}(\Omega)$ for any $\alpha \in (0, 1)$. Moreover,

$$
\|\psi\|_{C^{0,\alpha}(K)} \leq C(m, K, \epsilon_0, \lambda, \alpha, n) \text{ for any } K \subseteq \Omega.
$$
Let $\psi$ be a supersolution in the sense of (27). Let $\phi$ be a solution
\[ \int_{\Omega} \partial_p G(\nabla \phi, \phi) \cdot \nabla \zeta + \partial_z G(\nabla \phi, \phi) \zeta = 0, \text{ for all } \zeta \in C_0^\infty(\Omega), \]
and $\phi \leq \psi$ on $\partial \Omega$. Then if $\epsilon_0$ is sufficiently small, we have $\phi \leq \psi$ in $\Omega$.

Let $x_0 \in \{ \psi < m \}$ such that $\text{dist}(x_0, \Gamma_\psi) \leq \min\{1, \frac{1}{4} \text{dist}(x_0, \partial \Omega)\}$. Then if $\epsilon_0$ is sufficiently small, there exists $C > 0$ such that
\[ \psi(x_0) \geq m - C\lambda \text{dist}(x_0, \Gamma_\psi). \]
Let $\psi$ be a minimizer for (34). Then

- $\psi \in C_{loc}^{0,1}(\Omega)$.
- For any $p > 1$ and any $0 < r < 1$, there exists a constant $c_r > 0$ such that for any $B_R \subset \Omega$ with $R \leq 1$, if

  \[
  \frac{1}{R} \left( \frac{1}{|B_R|} \int_{B_R} |m - \psi|^p \right)^{1/p} \leq c_r \lambda,
  \]

  then $\psi = m$ in $B_{rR}$.
- Assume that $u_0$ satisfies (23). Then

  \[
  \psi_0(-\mu, x_2) < \psi(x_1, x_2) < \psi_0(R, x_2), \text{ for all } (x_1, x_2) \in \Omega_{\mu, R}.
  \]

- $\psi$ is the unique minimizer and furthermore, $\partial_{x_1} \psi \geq 0$. 
Inspired by the unique continuation results by Koch and Tataru, we have the following proposition.

**Proposition 2** Let $\psi, \psi_0 \in W_{loc}^{1, 2}(\mathbb{R} \times [0, \bar{\xi}]), \bar{\xi} > 0$, be two solutions to the Cauchy problem

$$\nabla \cdot \partial_p G(\nabla \psi, \psi) + H(\nabla \psi, \psi) = 0 \text{ in } \mathbb{R} \times (0, \bar{\xi}),$$

$$\psi = m, \quad \partial_{x_2} \psi = \Lambda \text{ on } \mathbb{R} \times \{\bar{\xi}\},$$

where $m, \Lambda$ are constants. Assume that $\mathbb{R}^2 \times \mathbb{R} \ni (p, z) \mapsto G(p, z)$ are $C^2$ and $(p, z) \mapsto H(p, z)$ are $C^1$. Then $\psi_0 = \psi$. 
We combine the comparison principle and unique continuation type results.

- If $\Lambda_n \to \Lambda$, then $\psi_{\Lambda_n} \to \psi_{\Lambda}$ uniformly in $\Omega_{\mu,R}$ and $k_{\Lambda_n}(x_2) \to k_{\Lambda}(x_2)$ for each $\tilde{a} < x_2 \leq 1/2$.

- If $\Lambda > 0$ is large, then the free boundary $\Gamma_{\mu,R,\Lambda}$ is nonempty and it satisfies $k_{\Lambda}(1/2) < 0$; if $\Lambda$ is small, then $k_{\Lambda}(1/2) > 0$.

- $N \cup \Gamma$ is $C^1$ in a $\{\psi < m\}$-neighborhood of $A$ (the connecting point).
Summary

▶ Stability of transonic shocks for 3D axisymmetric solutions
▶ Subsonic flow with jet

Ongoing Projects

▶ Stability of transonic shocks under 3D perturbations for the exit pressure
▶ Well-posedness for 3D jet for potential flows
▶ 2D problem with both transonic shock and jet
▶ ...

Chunjing Xie  Subsonic Flows with Physical Boundaries
Thanks!