Multidimensional solitons in optics, ultracold gases, and beyond: Predictions, creation, and expectations

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(1) Introduction
Formation of localized structures in the form of solitons (solitary waves) in many physical settings is accounted for by the interplay of nonlinear self-attraction (alias self-focusing) of physical fields (e.g., electromagnetic waves in photonics, or macroscopic wave functions, alias matter waves, in Bose-Einstein condensates, BECs) and basic linear effects, such as dispersion or diffraction (in BEC, the dispersion corresponds to the quantum-mechanical kinetic energy).

The concept of solitons was introduced in 1965, in the context of the Korteweg – de Vries (KdV) equation:

The **KdV** equation was derived as a basic model for the propagation of *gravity waves* on the surface of a shallow layer of inviscid water, and also as a commonly adopted model for the propagation of *ion-acoustic waves* in plasmas. The standard form of the **KdV** equation:

\[
\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.
\]

A family of exact soliton solutions of this equation:

\[
u = -\frac{2\alpha^2}{\cosh^2\left(\alpha(x - 4\alpha^2 t)\right)},
\]

where \(\alpha\) is an arbitrary positive parameter. The entire soliton family is *stable*. 
Arguably, the most important model which gives rise to solitons is the *nonlinear Schrödinger (NLS) equation*:

\[ i \frac{\partial \Psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial x^2} - |\Psi|^2 \Psi. \]

It generates a family of bright-soliton solutions with two free parameters - amplitude \( \eta \) and velocity \( c \):

\[ \Psi(x, t) = e^{icx + i(\eta^2 - c^2)t/2} \frac{\eta}{\cosh(\eta(x - ct))}. \]

With the increase of the input’s peak power, self-trapping of an input pulse into a **fundamental** or **higher-order** soliton (**breather**) was observed:
Standard telecommunications fibers can carry *soliton streams*, which may be used to transmit data in fiber-optical telecom networks. The bit-rate of up to **100 GB/s per channel** can be easily achieved, using currently available technologies. The *single* so far built soliton-based *commercial* telecom link, about **3,000** km long, was installed in Australia (between Adelaide and Perth) in **2003**. Later, interests of telecom developers have switched to other (non-soliton) technologies.
Another famous realization of solitons was demonstrated in BEC loaded in nearly one-dimensional (“cigar-shaped”) trapping potentials (in this context, the NLS equation is usually called the Gross-Pitaevskii (GP) equation). Bright matter-wave solitons were first created in the condensate of $^7$Li atoms:


Then, bright solitons in a less anisotropic trap were created in $^{85}$Rb:

The famous experimental picture of the atomic density distribution in a chain of 7 matter-wave solitons with unequal amplitudes in $^7$Li (from the work of R. Hulet et al.):
The objective of the tutorial is to present an overview of fundamental models which give rise to self-trapped modes in the form of stable two-and three-dimensional (2D and 3D) solitons (including ones with intrinsic vorticity), and physical realizations of the models. The realizations of multidimensional solitons are chiefly provided, like in the 1D case, by nonlinear optics and matter waves in BEC.
An old (published in 2005) but still relevant review of the topic of multidimensional solitons (with a recent invited update, published in 2016):

REVIEW ARTICLE

Spatiotemporal optical solitons

Boris A Malomed¹, Dumitru Mihalache², Frank Wise³ and Lluis Torner⁴

A more recent short review of the topic:

**Multidimensional solitons: Well-established results and novel findings**

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A new review, published in *Nature Reviews Physics* 1, 185-197 (2019), is focused on novel theoretical and experimental findings:

**New Frontiers in Multidimensional Self-Trapping of Nonlinear Fields and Matter**

Yaroslav V. Kartashov, Gregory E. Astrakharchik, Boris A. Malomed, and Lluis Torner
A still more recent review article, specifically focused on 2D and 3D solitons with embedded vorticity (which is one of the central topics of the present tutorial):

Invited Review Article

(INVITED) Vortex solitons: Old results and new perspectives

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As the stability is the main issue for 2D and 3D settings (unlike 1D models, where solitons are generically stable), the rest of the talk is structured according to basic mechanisms which provide for stabilization of the multidimensional solitons.

The subsequent presentation is organized as follows:

(2) A brief review of solitons (fundamental and vortical ones) created by the cubic nonlinearity in the 2D space, which are unstable.

(3) Relatively old material: Stable 2D and 3D vortex solitons in models with the cubic-quintic (CQ) nonlinearity.

(4) A new model and novel results: Stable 2D and 3D composite solitons in two-component spin-orbit (SO)-coupled BEC.

(5) Newest theoretical and experimental results: the prediction and actual creation of 3D and 2D matter-wave solitons (“quantum droplets”) stabilized by quantum fluctuations (represented by the so-called Lee-Huang-Yang correction to the GP equations).

(6) Conclusions.
(2) 2D solitons produced by the NLS equation with the cubic nonlinearity,
\[ iu_t + \frac{1}{2}(u_{xx} + u_{yy}) + |u|^2u = 0. \]

2D soliton solutions with integer vorticity $S$ and chemical potential $\mu$ are looked for, in the polar coordinates $(r, \theta)$, as
\[ u = \exp(-i\mu t + iS\theta) \ U(r), \]
where function $U(r)$ is a solution of a radial ordinary differential equation,
\[ \mu U + \frac{1}{2} [U'' + \frac{1}{r}U' - (S^2/r^2)U] + U^3 = 0. \]

**Boundary conditions** (b.c.): soliton solutions must decay as $\exp(-(-2 \mu^{1/2})r)$ at $r \to \infty$, and as $r^S$ at $r \to 0$.

In the case of $S = 0$, the b.c. is $U'(r = 0) = 0$. 
A typical radial profile of the soliton solution with $S = 0$ (fundamental soliton):
A radial profile of the vortex soliton with $S = 1$: 

$U^2$ vs. $r$
Fundamental (zero-vorticity, $S = 0$) solitons [alias *Townes’ solitons*, R.Y. Chiao, E. Garmire & C. H. Townes, Phys. Rev. Lett. 13, 479 (1964)] are *unstable* against the *collapse* (i.e., catastrophic self-compression of the wave function, which leads to formation of a singularity after a finite evolution time).
The numerically simulated \textit{collapse process} (the evolution of the radial cross section of the collapsing \textit{Townes’ soliton} is displayed here):
The generalization in the form of *Townes’ solitons* with embedded vorticity, alias the *topological charge*, \( S \geq 1 \), was introduced later:


**The theory of spiral laser beams in nonlinear media**

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All vortex solitons are strongly unstable against spontaneous splitting. Examples of the instability for $S = 1$ and $S = 2$: 

![Diagram showing examples of the instability for $S = 1$ and $S = 2$.]
The families of the Townes’ solitons and their vortex counterparts are **degenerate**: due to the specific **conformal symmetry** of the 2D NLS equation with the cubic nonlinearity, the norm of each family (with $S = 0,1,2$, etc.) takes a **single value**, which **does not** depend on the soliton’s **chemical potential**, $\mu$:

$$N_S = 2\pi \int_0^\infty U^2 (r; \mu) r dr.$$  

For $S = 0$ (the **fundamental Townes' solitons**),

$$N_0 \approx 5.85$$

[an analytical **variational approximation**,

M. Desaix, D. Anderson, and M. Lisak,

This degenerate value of the norm of the Townes’ soliton separates **collapsing** \((N > N_0)\) and **decaying** \((N < N_0)\) localized solutions of the 2D NLS equation. As any **separatrix** solution, the Townes soliton is **unstable** against small perturbations. Thus, the **critical collapse** sets in above a final **threshold value** of the norm, \(N_{\text{thr}} \equiv N_0\). Any **stabilizing mechanism**, added to the simplest 2D NLS equation, acts by letting the norm of 2D solitons take values **below** the **threshold value**, hence they **cannot** undergo the **collapse**. However, in the 3D case, the collapse is **supercritical**, with zero **threshold**.
(3) 2D and 3D Systems with the cubic-quintic nonlinearity

The stabilization of 2D and 3D fundamental and vortical solitons can be provided by a combination of competing self-focusing cubic and self-defocusing quintic nonlinear terms.
In optics, the 3D NLS equation can be written as an equation governing the spatiotemporal evolution of the electromagnetic field:

\[ i \frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial \tau^2} + |u|^2 u - |u|^4 u = 0. \]

Stationary solutions with integer vorticity, \( S \geq 0 \), and propagation constant (wavenumber) \( k \), are looked for in the cylindrical coordinates as

\[ u(x, y, \tau, z) = U(r, \tau) \exp(ikz + iS\theta), \quad r \equiv \sqrt{x^2 + y^2}, \]

with \( U(r, \tau) \) satisfying the radial equation:

\[
\left( \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} - \frac{S^2}{r^2} U + \frac{\partial^2 U}{\partial \tau^2} \right) + U^3 - U^5 = kR
\]
The stability of fundamental solitons (S = 0) in the framework of this equation is obvious. A nontrivial problem is the stability of vortex solitons against splitting by azimuthal perturbations. For 2D vortex solitons, this possibility was first reported in

M. Quiroga-Teixeiro and H. Michinel

Stable azimuthal stationary state in quintic nonlinear optical media

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Accurate results for the same 2D problem have been reported in the following paper:

DOI: 10.1007/s00332-002-0475-3

Spectrally Stable Encapsulated Vortices for Nonlinear Schrödinger Equations

R. L. Pego\textsuperscript{1} and H. A. Warchall\textsuperscript{2,3}
Experimentally, the stability of (2+1)D fundamental ($S = 0$) solitons in an optical cubic-quintic medium (actually, it is a fluid) was demonstrated relatively recently (stable vortex solitons supported by the cubic-quintic nonlinearity have not yet been created in the experiment):

Robust Two-Dimensional Spatial Solitons in Liquid Carbon Disulfide

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Georges Boudebs, Hervé Leblond, and Vladimir Skarka
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(Received 29 March 2012; published 2 January 2013)

The excitation of near-infrared (2 + 1)D solitons in liquid carbon disulfide is demonstrated due to the simultaneous contribution of the third- and fifth-order susceptibilities. Solitons propagating free from diffraction for more than 10 Rayleigh lengths although damped, were observed to support the proposed soliton behavior. Numerical calculations using a nonlinear Schrödinger-type equation were also performed.
The 3D setting with the cubic-quintic nonlinearity
A challenging problem is to construct 3D vortex solitons in the cubic-quintic medium, and analyze their stability. Theoretically, this was done long ago in:

Stable Spinning Optical Solitons in Three Dimensions

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(Received 2 July 2001; published 4 February 2002)
The basic results for the stability are summarized by plots which show wavenumber $-\mu$ of the 3D solitons vs. their total norm (energy, in terms of optics). Note that the Vakhitov-Kolokolov \textit{(necessary) stability criterion} holds on top branches, $d(\text{wave number})/d(\text{energy}) > 0$

$$E = \iiint U^2(x, y, t) dx dy dt$$
The original (classical) paper by N.G. Vakhitov and A.A. Kolokolov (1973):

STATIONARY SOLUTIONS OF THE WAVE EQUATION
IN A MEDIUM WITH NONLINEARITY SATURATION

N. G. Vakhitov and A. A. Kolokolov

In the linear (in the perturbation) approximation the authors find the stability criterion for the principal mode of the nonlinear wave equation. The structure and stability of natural modes having cylindrical and spherical symmetry is investigated in the case of a medium with nonlinearity saturation.

1. Stationary Solutions Having Cylindrical Symmetry

In the parabolic approximation the envelope of the electric field of the light beam can be described by the equation [6]

\[ i \frac{\partial E}{\partial z} + \Delta_\perp E + f(|E|^2) E = 0, \tag{1} \]

where \( \Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \); \( f(|E|^2) \) describes the nonlinear part of the permittivity.

An example of simulated *recovery* of a strongly perturbed *stable* soliton with intrinsic vorticity $S = 1$ (*doughnut*):
An example of the simulated instability of the 3D doughnut soliton with vorticity $S = 2$: splitting in three fragments:
(4) Novel results: Stable two- and three-dimensional composite solitons in spin-orbit (SO)-coupled self-attractive BEC
Basic results are presented here for 2D solitons as per the following papers:

**Creation of two-dimensional composite solitons in spin-orbit-coupled self-attractive Bose-Einstein condensates in free space**

Hidetsugu Sakaguchi and Ben Li  
Department of Applied Science for Electronics and Materials, Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Kasuga, Fukuoka 816-8580, Japan  
(Received 12 December 2013; published 26 March 2014)

**Vortex solitons in two-dimensional spin-orbit coupled Bose-Einstein condensates: Effects of the Rashba-Dresselhaus coupling and Zeeman splitting**

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(Received 18 February 2016; published 2 September 2016)
The concept of *emulation* (alias *simulation*) of complex physical effects, known in condensed-matter physics, by much simpler settings available in **BEC** (*matter waves*) and **photonics** (*optical waves*), has drawn a great deal of interest:

Lately, a new topic has emerged in the framework of this approach: the emulation of spin-orbit (SO) interactions in semiconductors, such as those accounted for by the *Rashba* and *Dresselhaus* Hamiltonians, by *mapping* the spinor wave function of electrons into the pseudo-spinor (two-component) wave function of a binary BEC gas:


Here, our objective is to construct **self-trapped** (localized) **stable 2D vortical modes** in the SO-coupled BEC with **attractive** nonlinearities. At the first glance, this objective seems **absolutely impossible**, as one may expect that such solitons are destabilized by the **critical collapse**, similarly to the above-mentioned **Townes’ solitons** of the NLS equation with the **self-attractive** cubic term.
(4b) The model

The system of GP equations for the two-component wave function \((\Phi_+, \Phi_-)\) of the binary BEC coupled by the SO terms of the \textbf{Rashba type} with strength \(\lambda\) (it may be scaled to 1), coefficient of the SPM self-attraction \(\equiv 1\), coefficient of the XPM inter-component attraction \(\gamma \geq 0\) (the trapping potential \(\sim \Omega^2\) will be dropped, the aim being to construct \textbf{stable solitons} in \textbf{free space}): 

\[
\begin{align*}
    i \frac{\partial \Phi_+}{\partial t} &= -\frac{1}{2} \nabla^2 \Phi_+ - (|\Phi_+|^2 + \gamma |\Phi_-|^2) \Phi_+ \\
    &\quad + \lambda \left( \frac{\partial \Phi_-}{\partial x} - i \frac{\partial \Phi_-}{\partial y} \right) + \frac{1}{2} \Omega^2 (x^2 + y^2) \Phi_+, \\
    i \frac{\partial \Phi_-}{\partial t} &= -\frac{1}{2} \nabla^2 \Phi_- - (|\Phi_-|^2 + \gamma |\Phi_+|^2) \Phi_- \\
    &\quad - \lambda \left( \frac{\partial \Phi_+}{\partial x} + i \frac{\partial \Phi_+}{\partial y} \right) + \frac{1}{2} \Omega^2 (x^2 + y^2) \Phi_-,
\end{align*}
\]
(4c) Semi-vortex states

The coupled GP equations admit a family of solutions for semi-vortices, with vorticities \( m_+ = 0 \) in one component, and \( m_- = 1 \) in the other. The exact ansatz for these solutions, compatible with the underlying equations (\( \mu < 0 \) is the chemical potential):

\[
\phi_+ (x, y, t) = \exp(-i \mu t) f_+ (r),
\]
\[
\phi_- (x, y, t) = \exp(-i \mu t + i \theta) r f_+ (r),
\]

with \( f_\pm (r) \) taking finite values \( f_\pm (0) \) at \( r = 0 \), and decaying \( \sim \exp(-\sqrt{-2 \mu} r) \) at \( r \to \infty \).
A numerically found cross-section (along $y = 0$) of the two components, $|\varphi_{\pm}|$, for a **stable semi-vortex**, which was obtained, by means of the **imaginary-time integration**, as a stationary soliton in the **free space**:
The numerically found dependence between the total norm of the semi-vortices and their chemical potential demonstrates that (1) the norm of the semi-vortex indeed falls below the threshold value necessary for the onset of the collapse: \( N(\mu) < N_{\text{thr}} \equiv N(\mu \to -\infty) \approx 5.85 \); (2) there is no finite minimum value of the norm necessary for the existence of the semi-vortex; (3) the dependence satisfies the Vakhitov-Kolokolov criterion, \( d\mu/dN < 0 \), which is a necessary condition for the stability:
In the limit of \( N \to N_{\text{thr}} \approx 5.85 \), the semi-vortex degenerates into the usual (unstable) Townes’ soliton in the first component, with the chemical potential \( \mu \to -\infty \), leaving the second (vortical) component empty.

Direct simulations demonstrate that the entire family of the semi-vortices is completely stable.

In this system, the SO-coupling terms break the conformal invariance of the 2D NLS equations with the cubic self-attraction. This leads to lifting the degeneracy of the norm, pushing the norm of the semi-vortices to \( N < N_{\text{thr}} \), thus securing their stability against the onset of the collapse.

Actually, the semi-vortex solitons realize the system’s ground state at \( N < N_{\text{thr}} \), which does not exist in the absence of the SO coupling. The collapse still occurs at \( N > N_{\text{thr}} \).
(4d) The stabilization of 3D solitons by the SO coupling.
The 3D model with the SO coupling (in particular, of the \textit{Weyl type}) is based on the following system of GP equations for the spinor (two-component) \textit{wave function}, with the vector of Pauli matrices, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$, and $g \equiv +1$, $\eta > 0$ (the \textit{self-attractive} nonlinearity, which gives rise to the \textit{supercritical collapse} in 3D):

$$\left[ i \frac{\partial}{\partial t} + \frac{1}{2} \nabla^2 + i \lambda \nabla \cdot \sigma 
+ g \begin{pmatrix} |\psi_+|^2 + \eta |\psi_-|^2 & 0 \\ 0 & |\psi_-|^2 + \eta |\psi_+|^2 \end{pmatrix} \right] \Psi = 0.$$
These \textbf{GP} equations can be derived from the corresponding \textbf{Hamiltonian}:

\begin{align*}
E_{\text{tot}} &= E_{\text{kin}} + E_{\text{soc}} + E_{\text{int}}, \\
E_{\text{kin}} &= \frac{1}{2} \int d^3r \, \Psi^\dagger p^2 \Psi, \quad E_{\text{soc}} = \lambda \int d^3r \, \Psi^\dagger (p \cdot \sigma) \Psi, \\
E_{\text{int}} &= -\frac{g}{2} \int d^3r \left( |\psi_+|^4 + |\psi_-|^4 + 2\eta |\psi_+\psi_-|^2 \right),
\end{align*} 

(1)
A possibility of the existence of metastable 3D solitons can be predicted by means of a simple qualitative consideration.

Dimensional analysis — If $L$ is a characteristic size of the self-trapped condensate, an estimate for amplitudes of the wave functions with norm $N = \int d^3r (|\psi_+|^2 + |\psi_-|^2)$ is $(|\psi_\pm|)_{\text{max}} \sim \sqrt{NL^{-3/2}}$. Therefore, the three terms in Eq. (1) scale with $L$ as

$$E_{\text{tot}}/N \sim c_{\text{kin}}L^{-2} - c_{\text{soc}}\lambda L^{-1} - \left(c_{\text{int}}^{(\text{self})} + c_{\text{int}}^{(\text{cross})} \eta \right) NL^{-3},$$

(2)

with positive coefficients $c_{\text{kin}}$, $c_{\text{soc}}$, and $c_{\text{int}}^{(\text{self/cross})}$. Evidently, Eq. (2) gives rise to a local minimum of $E_{\text{tot}}(L)$ at finite $L$, provided that

$$0 < \lambda N < c_{\text{kin}}^2/ \left[ 3 \left( c_{\text{int}}^{(\text{self})} + c_{\text{int}}^{(\text{cross})} \eta \right) c_{\text{soc}} \right],$$

(3)
An illustration of this: for fixed $N$, a metastable soliton may exist as a local minimum of the total energy (the blue line), but not as a ground state (red: no SO coupling; green: the SO coupling is too strong):
Examples of (meta) **stable** 3D solitons, with $N = 8$ [(a), for $\eta = 0.3$ – a semi-vortex](#);(b), for $\eta = 1.5$ – a *mixed mode*]:
(5) The newest addition: stabilization of 3D and 2D “superfluid droplets” by quantum fluctuations.

Corrections to the mean-field theory for BEC with self-repulsion, generated by quantum fluctuations around the mean-field state, were derived in 1957 by Lee, Huang, and Yang (LHY):

PHYSICAL REVIEW VOLUME 106, NUMBER 6 JUNE 15, 1957

Eigenvalues and Eigenfunctions of a Bose System of Hard Spheres and Its Low-Temperature Properties

T. D. Lee, Columbia University, New York, New York

AND

Kerson Huang and C. N. Yang, Institute for Advanced Study, Princeton, New Jersey

(Received March 19, 1957)

It is shown that the pseudopotential method can be used for an explicit calculation of the first few terms in an expansion in power of \((\rho a^3)\) of the eigenvalues and the corresponding eigenfunctions of a system of Bose particles with hard-sphere interaction. The low-temperature properties of the system are discussed.
Recently, a model was elaborated for a two-component BEC with the usual mean-field repulsive self-interaction in each component, and stronger attraction between the components. The collapse driven by the dominant cross-attraction is arrested by the LHY corrections, which are effectively represented by self-repulsive quartic terms, added to the GP equation with the usual (mean-field) cubic ones.
The theoretical elaboration of the model, in both 3D and 2D settings:

Quantum Mechanical Stabilization of a Collapsing Bose-Bose Mixture

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(Received 28 June 2015; published 7 October 2015)

Ultradilute Low-Dimensional Liquids

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(Received 24 May 2016; revised manuscript received 28 July 2016; published 1 September 2016)
Assuming a symmetric configuration, with *equal wave functions* of both components, $\Psi_1 = \Psi_2 \equiv \Psi$, the system of coupled GP equations may be reduced to the *single* one:

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2} \nabla^2 \Psi - |\Psi|^2 \Psi + \gamma |\Psi|^3 \Psi,$$

in 3D, where $\gamma > 0$ is an effective strength of the *quartic* term which represents the LHY correction.

The dimension reduction $3D \rightarrow 2D$ (under the action of strong confinement in the transverse direction) gives rise to the following 2D *quasi-GP* equation:

$$\frac{\partial \Psi}{\partial t} = -\frac{1}{2} \nabla^2 \Psi + \ln(|\Psi|^2) |\Psi|^2 \Psi.$$
Recently, experimental creation of quasi-2D (oblate) “droplets”, with aspect ratio $\sim 10:1$, zero vorticity, and $\sim 10,000 \ ^{39}\text{K}$ atoms per droplet, was reported in the following papers: Science 359, 301 (2018),
Bright Soliton to Quantum Droplet Transition in a Mixture of Bose-Einstein Condensates

P. Cheiney, C. R. Cabrera, J. Sanz, B. Naylor, L. Tanzi, and L. Tarruell*
ICFO—Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain
Another experimental result: the creation of nearly isotropic 3D quantum droplets (with negligible confinement in any direction) in the same atomic species, $^{39}\text{K}$:
Getting back to the theory: vortex-droplet states, with embedded vorticity $S$, were very recently constructed in the framework of the 2D reduced model, $\Psi = \exp(-i\mu t + iS\theta)U(r)$.

Y. Li, Z. Chen, Z. Luo, C. Huang, H. Tan, W. Pang, and B. A. Malomed,


The vortex droplets have their stability regions up to $S = 5$ (an example of the evolution (splitting) of an unstable vortex mode with $S = 1$ is shown too):
Also found were (in a small parameter regions) stable modes with *hidden vorticity*, $\Psi_{1,2} = \exp(-i\mu t \pm i\theta)U(r)$, in the system of coupled **GP** equations which *do not coalesce* into a single equation:
A further ramification of the topic: prediction of stable semi-discrete fundamental and vortical solitons in an array of tunnel-coupled quasi-1D traps filled by the binary BEC, under the action of the LHY terms:

PHYSICAL REVIEW LETTERS 123, 133901 (2019)

Semidiscrete Quantum Droplets and Vortices

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Another recent theoretical result: prediction of stability regions (by means of a systematic numerical analysis, augmented by analytical approximations) for fully 3D droplets, with embedded vorticity $S = 1$ and $2$:
Examples of splitting of an unstable vortical droplet, with $\mu = -0.04$ (the top row), and of the evolution of a stable one (the bottom row), with $\mu = -0.16$. In both cases, $S = 1$. 
Vortical droplets with $S = 2$ may be stable too, but only at very large values of the norm. Examples of the splitting of an unstable droplet ($\mu = -0.04$) and evolution of a stable one ($\mu = -0.183$) with $S = 2$: 

![Diagram showing the evolution of droplets with $S = 2$.]
(6) Conclusions

Recent theoretical and experimental studies have led to the prediction and, in some cases, experimental creation of stable self-trapped modes in the form of fundamental and vortex solitons in 2D and 3D geometries. Especially interesting are the predicted possibilities for the creation of (meta)stable semi-vortices by means of the SO (spin-orbit) coupling for the binary BEC with cubic attractive interactions (something which was previously considered absolutely impossible), as well as the theoretically predicted and experimentally realized creation of 3D superfluid droplets, stabilized by quantum fluctuations.
Generally, the theoretical studies of multidimensional solitons have advanced much farther than the experimental work. Creation of stable 2D and 3D solitons in real experiments remains a challenging objective.

As concerns further development of the theory, an intriguing possibility is the elaboration of complex topological self-trapped states in 3D, such as hopfions, i.e, vortex tori with an intrinsic twist of the toroidal core. Relatively simple 3D models make it possible to produce hopfions, both unstable and stable ones:
Twisted Toroidal Vortex Solitons in Inhomogeneous Media with Repulsive Nonlinearity

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Another species of “exotic” self-trapped topologically organized states, which may be stable: hybrids built of mutually displaced vortices and antivortices (+S and -S), with broken axial symmetry (New J. Phys. 16, 063035 (2014)).