Optimal investment, heterogeneous consumption and the best time for retirement

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OPTIMAL INVESTMENT AND CONSUMPTION
PDE/viscosity solution approach

- Zariphopoulou (1994 SICON): $\pi_t \leq f(X_t)$ and $X_t \geq 0$
- Vila & Zariphopoulou (1997 JET): borrowing constraint
- Oksendal & Sulem (2002 SICON): (fixed and proportional) transaction costs
- Xu & Yi (2016 MCRF): $c_t \leq kX_t + \ell$
Probabilistic/martingale method

- Brennan (1971 JFQA): different borrowing and lending rates
- Bardhan (1994 JEDC): \( c_t \geq \ell \) and \( X_t \geq 0 \)
- Cvitanic & Karatzas (1996 MF): transaction costs
- Elie & Touzi (2008 FS): \( X_t \geq \vartheta \sup_{s \leq t} X_s \)
Monograph

- Pham (2009): Continuous-time stochastic control and optimization with financial applications
- Karatzas & Shreve (2016): Methods of mathematical finance (stochastic modelling and applied probability)
Model formulation
Model features

- Heterogeneous consumptions: basic goods and luxury goods
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- Utility function: two factors, non-concave
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- Mandatory retirement age
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- Mandatory retirement age
- Mixed controls: portfolio, retirement time, consumptions for basic goods and luxury goods
- Complete market setup, also hold for convex constrained trading strategies
Financial assets

- One bond
  \[
  \begin{align*}
  dS_0(t) &= rS_0(t) \, dt, \quad t \geq 0, \\
  S_0(0) &= s_0 > 0.
  \end{align*}
  \]

- \(n\) stocks
  \[
  \begin{align*}
  dS_i(t) &= S_i(t) \left( b_i \, dt + \sum_{j=1}^{n} \sigma_{ij} \, dB_j(t) \right), \quad t \geq 0, \\
  S_i(0) &= s_i > 0.
  \end{align*}
  \]

The parameters \(r, \mu\) and \(\sigma\) are all constant and \(\sigma\) is nonsingular.
Wealth process and controls

\[
\begin{aligned}
dX(t) &= (rX(t) + \pi(t) \cdot \mu + I(t) \mathbb{1}_{\{t \leq \tau\}} - c(t) - g(t)) \, dt \\
&\quad + \pi(t) \cdot \sigma \, dB(t), \\
X_0 &= x_0.
\end{aligned}
\]

- \(I(\cdot)\): the income process (given)
- \(\pi(\cdot)\): the investment strategy
- \(\tau\): the retirement time, no later than the mandatory retirement age \(T\)
- \(c(\cdot)\): the consumption rate on the basic goods
- \(g(\cdot)\): the consumption rate on the luxury goods
Target

Find a feasible strategy \((\pi, c, g, \tau)\) to maximize

\[
\mathbb{E}\left[ \int_{0}^{+\infty} e^{-\rho t} u(c(t), g(t)) \, dt - \int_{0}^{\tau} e^{-\rho t} L(t) \, dt \right]
\]

- \(u(\cdot, \cdot)\): the heterogeneous utility function, non-concave for luxury goods
- \(\rho\): the discount factor
- \(L(\cdot)\): the labor cost process (given, deterministic)
Non-concave utility maximisation

- Carpenter (2002 JF): Does option compensation increase managerial risk appetite?
- Guan, Li, Xu and & Yi (2017 MCRF): A stochastic control problem and related free boundaries in finance
- Bian, Chen & Xu (2019 SIFIN): Utility maximization under trading constraints with discontinuous utility
Multi-consumption goods

- Breeden (1979 JFE): An intertemporal asset pricing model with stochastic consumption and investment opportunities
- Madan (1988 JET): Risk measurement in semimartingale models with multiple consumption goods
- Lakner (1989 PhD thesis): Consumption/investment and equilibrium in the presence of several commodities
- Ait-Sahalia, Parker and Yogo (2004 JF): Luxury goods and the equity premium
Multi-consumption goods (cont’d)

• Wachter & Yogo (2010 RFS): Why do household portfolio shares rise in wealth?
• Koo, Roh & Shin (2017 JIA): An optimal consumption, gift, investment, and voluntary retirement choice problem with quadratic and HARA utility
• Campanale (2018 B.E.JM): Luxury consumption, precautionary savings and wealth inequality
Data from Ait-Sahalia, Parker and Yogo (2004 JF)

**Figure:** The growth rate for PCE nondurables and services and sales of luxury retailers against excess stock returns.
Data from Ait-Sahalia, Parker and Yogo (2004 JF) (Cont’d)

**Figure**: The growth rate for PCE nondurables and services, the growth rate for sales of luxury retailers, and excess stock returns.
Mixed control with PDE

- Choi & Shim (2006 MF): Disutility, optimal retirement, and portfolio selection
- Lim & Shin (2008 QF): Optimal investment, consumption and retirement decision with disutility and borrowing constraints
- Guan, Li, Xu & Yi (2017 MCRF): A stochastic control problem and related free boundaries in finance
Mixed control with RBSDE

- Buckdahn & Li (2011 AMAS): Stochastic differential games with reflection and related obstacle problems for Isaacs equations
- Hamadene & Lepeltier (2000 SPTA): Reflected BSDEs and mixed game problem
Methods

• Combine heterogeneous consumptions to a single total consumption
• Turn the non-concave utility into a concave utility
• Post-retirement problem: stationary life-time problem, explicit solution
• Pre-retirement problem: a nonlinear variational inequality
• Dual method: turn nonlinear variational inequalities into linear ones
Dual approach in probability

- Bismut (1973 JMAA): Conjugate convex functions in optimal stochastic control
- Shreve & Xu (1992 AAP): A duality method for optimal consumption and investment under short-selling prohibition. I. general market coefficients; and II. constant market coefficients
- Cvitanic & Karatzas (1992 AAP): Convex duality in constrained portfolio optimization
Dual approach in probability/PDE

- Hugonnier & Kramkov (2004 AAP): Optimal investment with random endowments in incomplete markets
- Hugonnier, Kramkov & Schachermayer (2005 MF): On utility-based pricing of contingent claims in incomplete markets
- Xu & Yi (2016 MCRF): An optimal consumption-investment model with constraint on consumption
- Guan, Li, Xu and & Yi (2017 MCRF): A stochastic control problem and related free boundaries in finance
Optimal stopping

- Shiryaev (1978): Optimal stopping rules
- Barndorff-Nielsen & Shiryaev (2010): Change of time and change of measure
- Dai & Xu (2011 MF): Optimal redeeming strategy of stock loans with finite maturity
- Xu & Zhou (2013 AAP): Optimal stopping under probability distortion
- Xu & Yi (2019 MOR): Optimal redeeming strategy of stock loans under drift uncertainty
Model reformulation
Definition

The overall utility is

$$\bar{u}(k) = \sup_{c, \ g \geq 0, \ c + g = k} u(c, g).$$

Assume it satisfies the Inada conditions with power growth rate

- \( \lim_{k \to +\infty} \bar{u}(k) = +\infty \)
- \( \lim_{k \to 0^+} \bar{u}'(k) = +\infty \)
- \( \bar{u}(k) \ll k^p \) with \( 0 < p < 1 \)
Example 1: \( u(c, g) = u_1(c) + u_2(g) \)

- If \( u(c, g) = u_1(c) + u_2(g) \) is increasing and strictly concave in both \( c \) and \( g \), and

\[
\lim_{c \to 0^+} u_1'(c) = \lim_{g \to 0^+} u_2'(g) = +\infty.
\]

Then \( \bar{u}(\cdot) \) is globally concave.
Example 2: \( u(c, g) = \sqrt{c} + \sqrt{(g - a)^+} \)

- If \( u(c, g) = \sqrt{c} + \sqrt{(g - a)^+} \) for some \( a > 0 \). Then

\[
\bar{u}(k) = \sup_{c, g \geq 0, \ c + g = k} u(c, g) = \begin{cases} 
\sqrt{k}, & 0 \leq k \leq 2a; \\
\sqrt{2k - 2a}, & k > 2a.
\end{cases}
\]
Example 2: \( u(c, g) = \sqrt{c} + \sqrt{(g - a)^+} \)

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\[
\bar{u}(k) = \sup_{c, g \geq 0, \quad c + g = k} u(c, g) = \begin{cases} 
\sqrt{k}, & 0 \leq k \leq 2a; \\
\sqrt{2k - 2a}, & k > 2a.
\end{cases}
\]

- The concave envelope of \( \bar{u}(\cdot) \) is given by

\[
\tilde{u}(k) = \begin{cases} 
\sqrt{k}, & 0 \leq k < a; \\
\frac{1}{2\sqrt{a}}(k + a), & a \leq k \leq 3a; \\
\sqrt{2k - 2a}, & k > 3a.
\end{cases}
\]
Example 2: \( u(c, g) = \sqrt{c} + \sqrt{(g - a)^+} \) (cont’d)

**Figure:** Non-concave \( \bar{u}(\cdot) \) and its concave envelope \( \tilde{u}(\cdot) \).
New formulation

• The new wealth process follows

\[
\begin{align*}
\frac{dX(t)}{dt} &= (rX(t) + \pi(t) \cdot \mu + I(t) \mathbb{1}_{\{t \leq \tau\}} - k(t)) \, dt \\
&\quad + \pi(t) \cdot \sigma \, dB(t),
\end{align*}
\]

\[X_0 = x_0,\]

• \(k(\cdot) = c(\cdot) + g(\cdot)\): the total consumption process

• The new target is

\[
\sup_{\tau, k, \pi} \mathbb{E} \left[ \int_0^{+\infty} e^{-\rho t} \bar{u}(k(t)) \, dt - \int_0^\tau e^{-\rho t} L(t) \, dt \right]
\]

• \(\bar{u}\): the overall utility, non-concave in general
POST-RETIREMENT PROBLEM
The wealth process after retirement $t > \tau$ follows

$$dX(t) = (rX(t) + \pi(t) \cdot \mu - k(t)) \, dt + \pi(t) \cdot \sigma \, dB(t).$$

Define the value function for the post-retirement problem

$$V(x) = \sup_{k, \pi} E \left[ \int_{\tau}^{+\infty} e^{-\rho(t-\tau)} \tilde{u}(k(t)) \, dt \mid X(\tau) = x \right]. \quad (1)$$

- Difficulty: non-concave utility
- Approach: dual method
Dual utility

- Define

\[ h(y) = \sup_{k \geq 0} (\tilde{u}(k) - ky), \quad y > 0. \]

- Then

\[ h(y) = \sup_{k \geq 0} (\tilde{u}(k) - ky), \]
\[ \tilde{u}(k) = \inf_{y > 0} (h(y) + ky). \]
Example 2: \( u(c, g) = \sqrt{c} + \sqrt{(g - a)^+} \) (cont’d)

- \( h(y) = \frac{1}{4y} + (\frac{1}{4y} - ay)^+ \).
- The supreme is attained at
  \[
  (c^*(y), g^*(y)) = \begin{cases} 
  \left( \frac{1}{4y^2}, a + \frac{1}{4y^2} \right), & 0 < y < \frac{1}{2\sqrt{a}}; \\
  \left( \frac{1}{4y^2}, 0 \right), & y \geq \frac{1}{2\sqrt{a}}.
  \end{cases}
  \]
- Either \( c^*(y) + g^*(y) \leq a \) or \( \geq 3a \).
- Never optimal to consume \( a < c^*(y) + g^*(y) < 3a \).
- Either \( g^*(y) = 0 \) or \( g^*(y) \geq 2a \).
- Never optimal to consume \( 0 < g^*(y) < 2a \) for luxury goods.
Figure: Consumption of basic and luxury goods under nonhomothetic utility
**Dual value function**

- Define

\[
\hat{V}(y) = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} h(Y(t)) \, dt \left| Y(0) = y \right. \right],
\]

where

\[
dY(t) = Y(t)((\rho - r) \, dt + \vartheta \cdot dB(t)).
\]

- Define the concave conjugate function of \(\hat{V}\) by

\[
\mathcal{V}(x) = \inf_{y > 0} \left( \hat{V}(y) + xy \right), \quad x > 0.
\]
Verification theorem for the post-retirement problem

Theorem 1

The function $\mathcal{V}$ is the same as the value function $V$ of the post-retirement problem.
Example 3:  \( u(c, g) = \frac{(c-a)^{1-\phi}}{1-\phi} + \frac{(g+b)^{1-\psi}}{1-\psi} \) (Ait-Sahalia et al.)

- The utility function is

\[
u(c, g) = \frac{(c-a)^{1-\phi}}{1-\phi} + \frac{(g+b)^{1-\psi}}{1-\psi}.
\]

- Then

\[
h(y) = \frac{\phi}{1-\phi} y^{1-\frac{1}{\phi}} - ay + \frac{1}{1-\psi} b^{1-\psi} + \left( \frac{\psi}{1-\psi} y^{1-\frac{1}{\psi}} + by - \frac{1}{1-\psi} b^{1-\psi} \right) \mathbb{1}_{\{y < b-\psi\}}.
\]

- The dual value function is

\[
\hat{V}(y) = C_1y^{1-\frac{1}{\phi}} + C_2y + C_3 + (C_4y^{1-\frac{1}{\psi}} + C_5y + C_6) \mathbb{1}_{\{y < b-\psi\}}.
\]
Pre-retirement problem
Investment, consumption and best retirement time

Pre-retirement problem

Problem formulation

Value function

- The wealth process before retirement $t \leq \tau$ follows

$$dX(t) = (rX(t) + \pi(t) \cdot \mu + I(t) - k(t)) \, dt + \pi(t) \cdot \sigma \, dB(t).$$

- The pre-retirement problem is

$$\sup_{k, \pi, \tau} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} \left( \bar{u}(k(t)) - L(t) \right) \, dt + e^{-\rho \tau} V(X(\tau)) \right].$$

- Difficulty: non-concave utility, mixed controls
- Approach: dual method

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Verification theorem for the pre-retirement problem

**Theorem 2**

If \( w \) is a classical solution of the variational inequality (VI)

\[
\begin{align*}
\min \left\{ - \sup_{k, \pi} \left\{ (\partial_t + \mathcal{L})W - \rho W + \bar{u}(k) - L(t) \right\}, W - V \right\} &= 0, \\
W(T, x) &= V(x), \quad (t, x) \in S := [0, T) \times (0, \infty);
\end{align*}
\]

where

\[
\mathcal{L} = \frac{1}{2} \| \pi \cdot \sigma \|^2 \partial_{xx} + (rx + \pi \cdot \mu + I(t) - k) \partial_x.
\]

Then \( w \) is the value function of the pre-retirement problem.
Consider the following dual variational inequality
\[
\begin{cases}
\min \left\{ -\partial_t \hat{W} - \frac{1}{2} \| \vartheta \|_2^2 y^2 \partial_{yy} \hat{W} - (\rho - r) y \partial_y \hat{W} + \rho \hat{W} \ight. \\
\left. - y l(t) - h(y) + L(t), \hat{W} - \hat{V} \right\} = 0,
\end{cases}
\]
\[
\hat{W}(T, y) = \hat{V}(y).
\]
\[(t, y) \in S; \]  

(5)
Dual value function

Related optimal stopping problem

\[ \hat{W}(t, y) = \sup_{t \leq \tau \leq T} \mathbb{E}\left\{ \int_{t}^{\tau} e^{-\rho(s-t)} \left( I(s) Y(s) + h(Y(s)) - L(s) \right) ds \right. \]
\[ \left. \quad + e^{-\rho(\tau-t)} \hat{V}(Y(\tau)) \middle| Y(t) = y \right\}, \]

where the underlying process \( Y(\cdot) \) follows a GBM

\[ dY(t) = Y(t)((\rho - r) dt + \vartheta \cdot dB(t)). \]
Dual value function: Existence

Theorem 3 (Existence)

The problem (5) has a solution \( \hat{\mathcal{W}} \), which is convex and decreasing in \( y \). Moreover, \( \hat{\mathcal{W}} \), \( \partial_y \hat{\mathcal{W}} \) are continuous in \( S \), \( \partial_t \hat{\mathcal{W}} \), \( \partial_{yy} \hat{\mathcal{W}} \) are bounded in any bounded subdomain of \( S \); the free boundary, defined by the boundary of \( \{ \hat{\mathcal{W}} = \hat{\mathcal{V}} \} \), is Lipschitz in both time and space variable.

Idea to prove: The existence of the solution can be proved by standard penalty method. For the proof of the Lipschitz continuity of the free boundary, we refer to Nystrom (2007).
Theorem 4 (Comparison principle)

Let \( u_i(t, y) \), \( i = 1, 2 \), be the solutions of the following VIs

\[
\begin{align*}
\min \left\{ - (\partial_t + \mathcal{M}) u_i - f_i(t, y), \ u_i - g_i(t, y) \right\} &= 0, \\
u_i(T, y) &= h_i(y), \quad (t, y) \in S,
\end{align*}
\]

where \( \mathcal{M} \) is a linear elliptic operator on \( y \). If \( f_1 \geq f_2, \ g_1 \geq g_2, \ h_1 \geq h_2, \) and \( |u_1(t, y)| + |u_2(t, y)| \leq Ce^{Cy^2} \) in \( S \), for some \( C > 0 \), then

\[
u_1(t, y) \geq u_2(t, y), \quad (t, y) \in S.
\]
Value function of the pre-retirement problem

As a consequence, we have

**Corollary 5**

*The dual variational inequality (5) has a unique solution \( \hat{W} \).*
As a consequence, we have

**Corollary 5**

The dual variational inequality (5) has a unique solution \( \hat{\mathcal{W}} \).

**Theorem 6**

Let

\[
\mathcal{W}(t, x) = \inf_{y > 0} \left( \hat{\mathcal{W}}(t, y) + xy \right), \quad (t, x) \in S.
\]

Then \( \mathcal{W} \) is the value function of the pre-retirement problem.
OPTIMAL RETIREMENT REGION
Variational inequality

- Define
  \[ \mathbb{W}(t, y) := e^{-\rho t} (\hat{W}(t, y) - \hat{V}(y)), \quad (t, y) \in S. \]

- Then
  \[
  \begin{aligned}
  &\min \left\{ -(\partial_t + \mathbb{L})\mathbb{W} - e^{-\rho t} (yI(t) - L(t)), \quad \mathbb{W} \right\} = 0, \\
  &\mathbb{W}(T, y) = 0, \\
  &\mathbb{W}(T, y) = 0, \quad (t, y) \in S;
  \end{aligned}
  \]

  where
  \[ \mathbb{L} := \frac{1}{2} \| \vartheta \|^2 y^2 \partial_{yy} + (\rho - r)y \partial_y. \]
• Define the retirement region

\[ \mathcal{R} = \{(t, y) \in S \mid \mathbb{W}(t, y) = 0\}, \]

and the working region

\[ \mathcal{C} = \{(t, y) \in S \mid \mathbb{W}(t, y) > 0\}. \]

• Then

\[ \mathcal{R} = \{(t, y) \in S \mid y \leq b(t)\}, \]

\[ \mathcal{C} = \{(t, y) \in S \mid y > b(t)\}, \]

where the free boundary \( b(t) = \inf\{y > 0 \mid \mathbb{W}(t, y) > 0\} \).
Properties of the free boundary

• We have \( b(t) \leq \frac{L(t)}{I(t)} \) for all \( t \in [0, T] \).
Properties of the free boundary

- We have \( b(t) \leq \frac{L(t)}{l(t)} \) for all \( t \in [0, T] \).

- Because \( W \) is independent of \( u(\cdot) \), the free boundary \( b(\cdot) \) is \textit{irrelevant} to the individual’s utility function! It is universal.
Hypothesis on growth condition

Hypothesis 1

We have \( \frac{L'(t)}{L(t)} \geq \rho \geq \frac{l''(t)}{l(t)} \) for \( t \in [T - \ell, T] \) with \( \ell \) a positive constant \( \leq T \).

- For a young person, his marginal labor cost is decreasing as he gets more skilled.
Hypothesis on growth condition

Hypothesis 1

We have $\frac{L'(t)}{L(t)} \geq \rho \geq \frac{l''(t)}{l(t)}$ for $t \in [T - \ell, T]$ with $\ell$ a positive constant $\leq T$.

- For a young person, his marginal labor cost is decreasing as he gets more skilled.
- For an older one, his marginal labor cost is increasing as he becomes ageing with less energy and more burdens such as illness, family issue, child care.
Hypothesis: income process, labor cost process
Theorem 7

Assume Hypothesis 1 holds. Then \( b(t) \) is increasing for \( t \in [T - \ell, T] \) with the terminal value

\[
b(T-) := \lim_{t \to T} b(t) = \frac{L(T)}{l(T)}.
\]
Monotonicity of the free boundary

Figure: The two regions $\mathcal{R}$ and $\mathcal{C}$ under Hypothesis 1
A numerical example

- Define

\[ L(t) = \begin{cases} 
   a_0 + a_1 t + \frac{1}{2} a_2 t^2, & \text{if } t \leq T - \ell; \\
   e^{Kt}, & \text{if } t > T - \ell,
\end{cases} \]

where

\[ a_0 = e^{K(T - \ell)} \left( 1 - K(T - \ell) + \frac{1}{2} K^2 (T - \ell)^2 \right), \]
\[ a_1 = Ke^{K(T - \ell)} (1 - K(T - \ell)), \]
\[ a_2 = K^2 e^{K(T - \ell)}. \]
A numerical example (cont’d)

• Choose $1/K < T - \ell$ so that $L(\cdot)$ is first decreasing and then increasing.
• Set $I(t) = Ce^{K't}$.
• Choose the following parameters

\[ K = 2, \quad K' = 0.4, \quad C = 8, \quad \ell = 0.7, \quad T = 2, \quad \rho = 0.5. \]

They satisfy all the requirements and Hypothesis 1.
A numerical example (cont’d)

Figure: The functions $L(\cdot)$ and $I(\cdot)$. 
A numerical example (cont’d)

Figure: The non-monotone free boundary $b(\cdot)$. 
Optimal consumption

- Consume only basic goods when the wealth is small
Optimal consumption

- Consume only basic goods when the wealth is small
- Consume basic goods and make savings when the wealth is intermediate
Optimal consumption

- Consume only basic goods when the wealth is small
- Consume basic goods and make savings when the wealth is intermediate
- Consume small portion in basic goods and large portion in luxury goods when the wealth is large
Optimal retirement time

- Prefer to work for young people
Optimal retirement time

- Prefer to work for young people
- Prefer to retire near mandatory retirement age
Optimal retirement time

- Prefer to work for young people
- Prefer to retire near mandatory retirement age
- Not universal: different wealth levels for individuals with different preferences
Optimal retirement time

- Prefer to work for young people
- Prefer to retire near mandatory retirement age
- Not universal: different wealth levels for individuals with different preferences
- Universal: same level marginal consumption utilities for different individuals, determined only by market parameters and income process and labor cost process
Thank you for your attention!