Maximum principle for one kind of discrete-time stochastic optimal control problem and its applications

Zhen WU

School of Mathematics, Shandong University

Joint work with Dr. Feng ZHANG (SDUFE)
Outline

1. Introduction
2. Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle
3. Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game
4. Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem
5. Conclusions and more extensions
Outline

1 Introduction

2 Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle

3 Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game

4 Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem

5 Conclusions and more extensions
The maximum principle (MP) is a milestone of optimal control theory.

Discrete-time stochastic optimal control problems

- **Discrete-time** optimal control problems are as important as **continuous-time** ones.
  - **Difference equations** are in their own right important mathematical models.
  - In many situations, it is sufficient or natural to describe a system by a **discrete-time model**.
    - **Signal values** are sometimes only available at certain times.
    - **Discretization** of the dynamics of continuous-time problems.
    - **Computer manipulation**.

- **Difficulties in studying discrete-time stochastic optimal control problems:**
  - **Differentiation and integration operations** no longer work;
  - well-known tools of **Itô's Calculus** do not necessarily hold (**Itô's formula**, **BDG inequality**, etc.).
To turn the original problem into a deterministic one and then use a matrix version of MP:
A.Beghi and D.D’ Alessandro (J. Optim. Theory Appl., 1998),
M.Ait Rami, X.Chen and X.Y.Zhou (J. Global Optim., 2002).

To use the completion of squares for LQ problems:
J.B.Moore, X.Y.Zhou and A.E.B.Lim (Syst. Control Lett., 1999),

Oswaldo L.V.Costa, D.Li, J.F.Zhang, H.S.Zhang, W.H.Zhang, ...

Works for discrete-time stochastic optimal controls
Comments on one literature


- **Main contributions:**
  - necessary as well as sufficient conditions are established for one kind of discrete-time stochastic optimal control problem;
  - one kind of new backward stochastic difference equation is introduced to be the adjoint equation.

- **Some deficiencies:**
  - the assumptions are strict, excluding the quadratic cost functional;
  - the convex perturbation is not appropriately defined;
  - the necessary condition does not necessarily hold for general convex control domains;
  - the square-integrability of the solution to the adjoint equation is not guaranteed.
Results and contributions of this talk

This talk considers one kind of discrete-time stochastic optimal control problem with convex control domains.

- **Main results:**
  - necessary MP
  - sufficient MP (verification theorem)
  - extensions and applications

- **Main contributions:**
  - discrete-time stochastic MP in a rigorous version
  - proof in a clear and concise way
  - road for further related topics
Outline

1. Introduction

2. Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle

3. Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game

4. Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem

5. Conclusions and more extensions
Notations

- \((\Omega, \mathcal{F}, \mathbb{P})\): a probability space
- For fixed \(N \in \mathbb{N}\), 
  \(\mathcal{T} := \{0, 1, \cdots, N - 1\}\), 
  \(\mathcal{T} := \{1, 2, \cdots, N\}\).
- Noises: \(W_k = (W_k^1, \cdots, W_k^d) \top \in \mathbb{R}^d\), \(k \in \mathcal{T}\).
- Filtration \(\{\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_N\} \subset \mathcal{F}\):

  \[\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_k = \sigma\{W_0, W_1, \cdots, W_{k-1}\}, \quad k \in \mathcal{T}.\]
Noises and admissible controls

- Assume that for some $\alpha > 2$,

$$\mathbb{E} \left[ |W_k|^\alpha |\mathcal{F}_k| \right] = C(\alpha) < \infty, \quad \forall k \in \mathcal{T}. \quad (1)$$

- Let $U_0, \cdots, U_{N-1}$ be nonempty convex subsets of $\mathbb{R}^m$. $u = \{u_0, \cdots, u_{N-1}\}$ is called an admissible control, if
  - $u_k$ is an $U_k$-valued, $\mathcal{F}_k$-measurable r.v. for each $k \in \mathcal{T}$;
  - for some $\beta > 2$,

$$\mathbb{E} \sum_{k=0}^{N-1} |u_k|^\beta < \infty. \quad (2)$$

Denote by $\mathcal{U}$ the set of admissible controls.
Problem formulation

The discrete-time stochastic optimal control problem, Problem (OCP), is to find \( u^* = \{u_0^*, \cdots, u_{N-1}^*\} \in \mathcal{U} \) such that

\[
J(u^*) = \inf_{u \in \mathcal{U}} J(u),
\]

where the state \( X \) is defined by

\[
\begin{align*}
X_{k+1} &= b(k, X_k, u_k) + \sigma(k, X_k, u_k)W_k, \quad k \in \mathcal{T}, \\
X_0 &= x_0 \in \mathbb{R}^n,
\end{align*}
\]

and the cost \( J \) is given by

\[
J(u) = \mathbb{E} \left\{ \sum_{k=0}^{N-1} l(k, X_k, u_k) + h(X_N) \right\}.
\]

Such \( u^* \) is called an optimal control; the corresponding system state is denoted by \( X^* = \{X_k^*, k \in \mathcal{T}\} \).
Assumptions

The coefficients are defined by

\[ b(k, \cdot, \cdot, \cdot) : \mathbb{R}^n \times U_k \times \Omega \to \mathbb{R}^n, \quad \sigma(k, \cdot, \cdot, \cdot) : \mathbb{R}^n \times U_k \times \Omega \to \mathbb{R}^{n \times d}, \]
\[ l(k, \cdot, \cdot, \cdot) : \mathbb{R}^n \times U_k \times \Omega \to \mathbb{R} \quad \text{and} \quad h(\cdot, \cdot) : \mathbb{R}^n \times \Omega \to \mathbb{R} \quad \text{for} \ k \in \mathcal{T}. \]

\textbf{(H1)} \ b(k, x, v) \text{ and } \sigma(k, x, v) \text{ are } \mathcal{F}_k\text{-measurable for any } (k, x, v), \text{ and continuously differentiable in } (x, v) \text{ with bounded derivatives. In addition, } |b(k, 0, 0)| + |\sigma(k, 0, 0)| \leq C \text{ for some } C > 0.

\textbf{(H2)} \ l(k, x, v) \text{ is } \mathcal{F}_k\text{-measurable and } h(x) \text{ is } \mathcal{F}_N\text{-measurable for any } (k, x, v); \ (l, h) \text{ are continuously differentiable in } (x, v), \text{ and the derivatives are bounded by } C(1 + |x| + |v|). \text{ Besides,}
\[ \mathbb{E} \left[ \sum_{k=0}^{N-1} |l(k, 0, 0)| + |h(0)| \right] < \infty. \]
Solvability of the state equation

Set $\gamma = \min\{\alpha, \beta\}(>2)$.

**Lemma 1.** Under (H1), Eq. (3) admits a unique solution $X = \{X_k, k \in T\}$ such that $X_k \in L^\gamma(\Omega, \mathcal{F}_k, \mathbb{P}; \mathbb{R}^n)$ for each $k$. Moreover, there exists $C = C(\alpha, \gamma, N) > 0$ satisfying

$$
\mathbb{E} \sum_{k=1}^{N} |X_k|^\gamma \leq C\mathbb{E} \left\{ 1 + |x_0|^\gamma + \sum_{k=0}^{N-1} |u_k|^\gamma \right\}. 
$$

\begin{equation}
\tag{4}
\end{equation}

**Proof.** The existence of $X$ is obvious. We only need to check (4). In fact, by (H1),

$$
\mathbb{E}|X_{k+1}|^\gamma \leq C(\gamma)\mathbb{E}\left\{ [1 + |X_k|^\gamma + |u_k|^\gamma] 
+ [1 + |X_k|^\gamma + |u_k|^\gamma]|W_k|^\gamma \right\}, \quad k \in T.
$$
Solvability of the state equation

By (1), Hölder’s inequality yields \( \mathbb{E} [|W_k|^{\gamma} | \mathcal{F}_k] \leq C(\alpha, \gamma) =: C_1 \). Then

\[
\mathbb{E} [|X_k|^{\gamma} | W_k|^{\gamma}] = \mathbb{E} \{ |X_k|^{\gamma} \mathbb{E} [|W_k|^{\gamma} | \mathcal{F}_k] \} \leq C_1 \mathbb{E} |X_k|^{\gamma}.
\]

Similarly, \( \mathbb{E} [|u_k|^{\gamma} | W_k|^{\gamma}] \leq C_1 \mathbb{E} |u_k|^{\gamma} \). So, there exists \( C_2 := C_2(\alpha, \gamma) \) s.t.

\[
\mathbb{E} |X_{k+1}|^{\gamma} \leq C_2 \mathbb{E} \{ 1 + |X_k|^{\gamma} + |u_k|^{\gamma} \}, \quad k \in T.
\]

Then, by induction we get for \( k \in T \),

\[
\mathbb{E} |X_k|^{\gamma} \leq \sum_{j=1}^{k} (C_2)^j + (C_2)^k |x_0|^{\gamma} + \sum_{j=0}^{k-1} (C_2)^{k-j} \mathbb{E} |u_j|^{\gamma}.
\]

Thus we can draw the conclusion.

Remark. The conditions (1) and (2) are not sufficient to get the \( L^r \)-estimate of \( X_k \) \( (k \geq 2) \) for \( r > \gamma \). In fact, if \( \xi \in L^p \) and \( \eta \in L^q \), then generally we can only get \( \xi \eta \in L^r \) for \( r \leq pq/(p+q)(< \min\{p, q\}) \).
Convex perturbation

Let us define the convex perturbation.

For any \( \mu = \{\mu_0, \cdots, \mu_{N-1}\} \in U \), define

\[
\begin{align*}
\nu_k &= \mu_k - u_k^*, \\
u &= \{\nu_0, \cdots, \nu_{N-1}\}, \\
\nu^\varepsilon &= \nu_k - u_k^*, \\
u^\varepsilon &= \{\nu^\varepsilon_0, \cdots, \nu^\varepsilon_{N-1}\}.
\end{align*}
\]

Then \( \nu^\varepsilon \in U \).

Similar to (4), it’s easy to check that

\[
\mathbb{E} \sum_{k=0}^{N} |X^\varepsilon_k - X_k^*|^2 \leq C\varepsilon^2 \mathbb{E} \sum_{k=0}^{N-1} |\nu_k|^2,
\]

where \( C \) is a positive constant independent of \( \varepsilon \).
The variational equation

Let us introduce (the variational equation)

\[
Z_{k+1} = \left[ b_x^T(k)Z_k + b_v^T(k)\nu_k \right] \\
+ \sum_{i=1}^{d} \left[ (\sigma_x^i(k))^T Z_k + (\sigma_v^i(k))^T \nu_k \right] W_k^i, \quad k \in \mathcal{T},
\]

(5)

where \( \sigma = (\sigma^1, \cdots, \sigma^d) \).

Eq. (5) admits a unique solution \( Z = \{Z_k, k \in \mathcal{T}\} \) such that

\[ Z_k \in L^2(\Omega, \mathcal{F}_k; \mathbb{P}; \mathbb{R}^n) \text{ for each } k. \]

Moreover, there exists \( C > 0 \) such that

\[ \mathbb{E} \sum_{k=1}^{N} |Z_k|^2 \leq C \mathbb{E} \sum_{k=0}^{N-1} |\nu_k|^2. \]
The variational equation

As a linear stochastic difference equation, the variational equation (5) could be explicitly solved. In fact, (5) is equivalent to

\[ Z_{k+1} = M_k Z_k + R_k, \quad k \in T; \quad Z_0 = 0, \]

where

\[ M_k = b_x^\top(k) + \sum_{i=1}^{d} (\sigma^i_x(k))^\top W^i_k, \quad R_k = b_v^\top(k) \nu_k + \sum_{i=1}^{d} (\sigma^i_v(k))^\top W^i_k \nu_k. \]

Then using induction yields

\[ Z_k = \sum_{i=0}^{k-1} M_i^{k-1} R_i, \quad (6) \]

where \( M^j_i = M_j M_{j-1} \cdots M_{i+1} \) when \( i < j \), and \( M^j_j = I_n \) when \( i = j \).
The variational inequality

**Lemma 2.** Under (H1) and (H2), it holds that

\[
\mathcal{L} J(u^*) := \mathbb{E} \left\{ \sum_{k=0}^{N-1} \left[ \langle l_x(k), Z_k \rangle + \langle l_v(k), \nu_k \rangle \right] + \langle h_x(X^*_N), Z_N \rangle \right\} \geq 0.
\]

The proof is similar to that of the *continuous-time* counterpart.

Define the Hamiltonian \( H \) by

\[
H(k, x, v, p, q) = \langle b(k, x, v), p \rangle + \sum_{i=1}^{d} \langle \sigma^i(k, x, v), q^i \rangle + l(k, x, v)
\]

with \( q = (q^1, \cdots, q^d) \). Note that \( \sum_{i=1}^{d} \langle \sigma^i, q^i \rangle = tr \left[ \sigma^\top q \right] \).
The adjoint equation

Introduce the following adjoint equation:

\[

definition start
\begin{align*}
P_k &= \mathbb{E}\left\{ H_x(k + 1, X_{k+1}^*, u_{k+1}^*, P_{k+1}, Q_{k+1}) \bigg| \mathcal{F}_k \right\}, \\
Q_k &= \mathbb{E}\left\{ H_x(k + 1, X_{k+1}^*, u_{k+1}^*, P_{k+1}, Q_{k+1}) W_k^\top \bigg| \mathcal{F}_k \right\}, \\
k &= 0, 1, \ldots, N - 2, \\
P_{N-1} &= \mathbb{E}[h_x(X_N^*) \big| \mathcal{F}_{N-1}], \\
Q_{N-1} &= \mathbb{E}[h_x(X_N^*) W_{N-1}^\top \big| \mathcal{F}_{N-1}].
\end{align*}
\]

(7)
The adjoint equation

This adjoint equation, introduced first by Lin and Zhang (IEEE TAC, 2015), is a kind of backward stochastic difference equation, which is quite different from the continuous-time counterpart.

Note that Cohen and Elliott (SPA, 2010; SICON, 2011) also studied one kind of backward stochastic difference equation and obtained the existence and uniqueness result. However, our adjoint equation is different from the equations studied in the above two references. On the one hand, they have different forms. On the other hand, our adjoint equation is introduced naturally as the dual equation of the variational equation, while in our opinion, the equations studied in the above two references are not appropriate to serve as the adjoint equations of Problem (OCP).
The main concern of the adjoint equation (7) is the integrability of its solution \( \{(P_k, Q_k), k \in T\} \).

For later use we need these integrability conditions:

\[
\begin{aligned}
\mathbb{E}[|P_k| + |Q_k|] < \infty, & \quad k \in T, \\
\mathbb{E}[(|P_{k+1}| + |Q_{k+1}|)|W_k|] < \infty, & \quad k = 0, \ldots, N - 2, \\
\mathbb{E}[(|P_k| + |Q_k|)|X_k| + (|P_k| + |Q_k|)|u_k|] < \infty, & \quad k \in T.
\end{aligned}
\] (8)

Let us point out that the first two integrability conditions in (8) comes from the adjoint equation itself, while the third condition is needed in the prove of necessary as well as sufficient MP.
The classical square-integrability conditions of the noises and admissible controls are insufficient for our purpose.

- If \( \{W_k\} \) are only square-integrable \( (\alpha = 2) \), then \( X_k, 1 \leq k \leq N \), are at most square-integrable even when \( \{u_k\} \) are bounded. In this case, \( (P_k, Q_k), 1 \leq k \leq N - 2 \), are not square-integrable. Thus \( \mathbb{E}[(|P_{k+1}| + |Q_{k+1}|)|W_k|] \) and \( \mathbb{E}[(|P_k| + |Q_k|)|X_k|] \) are not well defined. So, a stronger integrability condition on \( \{W_k\} \) is needed.

- If \( \{u_k\} \) are only square-integrable \( (\beta = 2) \) and \( \{W_k\} \) are not bounded, then \( X_k, 1 \leq k \leq N \), are at most square-integrable, and \( (P_k, Q_k), 1 \leq k \leq N - 2 \), are still not square-integrable. Thus \( \mathbb{E}[(|P_k| + |Q_k|)|X_k|] \) and \( \mathbb{E}(|P_k| + |Q_k|)|u_k| \) are not well defined. So, a stronger integrability condition on \( \{u_k\} \) is needed.
Based on the previous discussions, let us assume moreover

(H3) \( \alpha \geq N + 1 \) and \( \beta = 2\alpha(\alpha - N + 1)^{-1} \).

**Lemma 3.** Under (H1)-(H3), the integrability conditions in (8) hold true.

**Proof.** As a corollary of Hölder’s inequality or Young’s inequality, the following assertion holds true: if \( \xi \in L^a \) and \( \eta \in L^b \) with \( a, b > 0 \), then \( \xi \eta \in L^{\frac{ab}{a+b}} \), and \( \xi \eta \in L^1 \) if \( \frac{1}{a} + \frac{1}{b} \leq 1 \) or \( \frac{ab}{a+b} \geq 1 \).

Denote \( \theta_k = \frac{\alpha\gamma}{\alpha+\gamma(N-1-k)} \) for \( k \in \mathcal{T} \). Note that \( \theta_k \) is increasing in \( k \) and \( \theta_k \leq \gamma \).

Let us use induction to show that

\[
|P_k| \in L^{\theta_k}, \quad |Q_k| \in L^{\theta_{k-1}}, \quad k = 1, \ldots, N - 1. \tag{9}
\]
The adjoint equation

Since $|h_x(X_*^N)| \leq C(1 + |X_*^N|)$, $|X_*^N| \in L^\gamma$ and $|W_{N-1}| \in L^\alpha$, we have $|P_{N-1}| \in L^\gamma$ and $|Q_{N-1}| \in L^{\frac{\alpha \gamma}{\alpha + \gamma}}$. Thus (9) holds for $k = N - 1$.

Let us use induction to proceed. Assume

$$|P_{j+1}| \in L^{\theta_{j+1}}, \quad |Q_{j+1}| \in L^{\theta_j} \quad (10)$$

for some $j = 1, \ldots, N - 2$. Firstly, note that

$$|P_j| \leq C \mathbb{E}[\left(1 + |X_{j+1}^*| + |u_{j+1}^*| + |P_{j+1}| + |Q_{j+1}|\right) |F_j|].$$

Since $|X_{j+1}^*| \in L^\gamma$, $|u_{j+1}^*| \in L^\beta$, $|P_{j+1}| \in L^{\theta_{j+1}}$, $|Q_{j+1}| \in L^{\theta_j}$, and $\theta_j < \theta_{j+1} \leq \gamma \leq \beta$, it follows that $P_j \in L^{\theta_j}$. Secondly, we have

$$|Q_j| \leq C \mathbb{E}[\left(1 + |X_{j+1}^*| + |u_{j+1}^*| + |P_{j+1}| + |Q_{j+1}|\right) |W_j||F_j|].$$

Note that $|W_j| \in L^\alpha$, $|X_{j+1}^*||W_j| \in L^{\frac{\alpha \gamma}{\alpha + \gamma}}$, $|u_{j+1}^*||W_j| \in L^{\frac{\alpha \beta}{\alpha + \beta}}$, $|P_{j+1}^*||W_j| \in L^{\theta_{j+1}}$, $|Q_{j+1}^*||W_j| \in L^{\theta_j-1}$. Since $\theta_{j-1} < \theta_j \leq \frac{\alpha \gamma}{\alpha + \gamma} \leq \frac{\alpha \beta}{\alpha + \beta} < \alpha$, we have $|Q_j| \in L^{\theta_j-1}$. 
The adjoint equation

With these preparations, the integrability conditions in (8) hold if
\[ |Q_1||W_0|, |Q_1||X_1|, |Q_2||X_2| \in L^1. \]

It’s easy to check that the above holds true if
\[ \gamma(\alpha - N + 1) \geq 2\alpha. \] (11)

Since \( \gamma = \min\{\alpha, \beta\} \), (11) is equivalent to
\[ \left\{ \begin{array}{l}
\alpha > N - 1, \\
\alpha(\alpha - N + 1) \geq 2\alpha, \\
\beta(\alpha - N + 1) \geq 2\alpha. 
\end{array} \right. \]

Solving it yields
\[ \alpha \geq N + 1, \quad \beta \geq 2\alpha(\alpha - N + 1)^{-1}. \]

Thus, it’s proper and sufficient to choose \( \alpha \geq N + 1 \) and
\( \beta = 2\alpha(\alpha - N + 1)^{-1} \), which is just (H3).
The adjoint equation

Remark. (i). The proof of Lemma 3 shows the reason that we need and how we worked out the assumption (H3). Under the preconditions (H1)-(H2), (H3) is almost the minimum condition that ensures (8).

(ii). Lemma 3 guarantees that the adjoint equation is well defined with a unique solution \( \{(P_k, Q_k), k \in T\} \), and all the expectations involving \( \{(P_k, Q_k), k \in T\} \) in this talk are well defined.

(iii). It is expected that both \( \alpha \) and \( \beta \) could be small; however, it’s not the case. In fact, (H3) implies that \( 2 < \beta \leq \alpha \); in addition, the smaller \( \beta \) is, the larger \( \alpha \) is.
The adjoint equation

**Lemma 4.** Under (H1)-(H3), the variational equation (5) and the adjoint equation (7) admits the following duality:

$$
\mathbb{E} \sum_{k=0}^{N-1} \left< b_v(k) P_k + \sum_{j=1}^{d} \sigma^j_v(k) Q^j_k, \nu_k \right> = \mathbb{E} \sum_{k=1}^{N} \left< l_x(k), Z_k \right>.
$$

(12)

In fact, the adjoint equation (7) can be solved explicitly to get

$$
\begin{align*}
P_k &= \sum_{i=k+1}^{N} \mathbb{E} \left[ (M_k^{i-1})^\top l_x(i) \big| \mathcal{F}_k \right], \\
Q_k &= \sum_{i=k+1}^{N} \mathbb{E} \left[ (M_k^{i-1})^\top l_x(i) W_k^\top \big| \mathcal{F}_k \right].
\end{align*}
$$

(13)

Then Lemma 4 could be proved by interchanging the order of summations and making use of the properties of conditional expectations. Lemma 4 shows the main role played by \((P, Q)\). (13) is the origin of the adjoint equation (7).
**Theorem 1.** Assume (H1)-(H3). If \( u^* = \{ u_k^*, k \in \mathcal{T} \} \) is an optimal control of Problem (DTC), then it satisfies

\[
\langle H_v(k, X_k^*, u_k^*, P_k, Q_k), v - u_k^* \rangle \geq 0, \quad \forall v \in U_k, k \in \mathcal{T},
\] (14)

**Proof.** By (12), the variational inequality becomes

\[
\mathbb{E} \sum_{i=0}^{N-1} \langle H_v(i, X_i^*, u_i^*, P_i, Q_i), \mu_i - u_i^* \rangle \geq 0, \quad \forall \mu = \{ \mu_i, i \in \mathcal{T} \} \in \mathcal{U}.
\]

For any \( k \in \mathcal{T}, v \in U_k \) and \( A \in \mathcal{F}_k \), define \( \mu_k = v \chi_A + u_k^* \chi_{\bar{A}} \), and \( \mu_i = u_i^* \) if \( i \neq k \). Then \( \mu \in \mathcal{U} \). Applying to the last inequality gives

\[
\mathbb{E} [\langle H_v(k, X_k^*, u_k^*, P_k, Q_k), v - u_k^* \rangle \chi_A] \geq 0.
\]

Since \( \langle H_v(k, X_k^*, u_k^*, P_k, Q_k), v - u_k^* \rangle \) is \( \mathcal{F}_k \)-measurable and \( A \in \mathcal{F}_k \) is arbitrarily chosen, the result follows directly.
Theorem 2. Assume (H1)-(H3). Let \((u^*, X^*)\) be an admissible pair and \((P, Q)\) the solution of the adjoint equation (7). Assume that \(h(\cdot)\) and \(H(k, \cdot, \cdot, P_k, Q_k)\) are convex functions for each \(k \in \mathcal{T}\). Then \(u^*\) is an optimal control of Problem (DTC) if it satisfies (14).

Proof. For admissible pair \((u, X)\), set \(\hat{b}(k) = b(k, X_k, u_k) - b(k, X_k^*, u_k^*)\) and \(\hat{\sigma}(k) = \sigma(k, X_k, u_k) - \sigma(k, X_k^*, u_k^*)\). Then

\[
J(u) - J(u^*) = \mathbb{E} \sum_{k=0}^{N-1} [H(k, X_k, u_k, P_k, Q_k) - H(k, X_k^*, u_k^*, P_k, Q_k)]
\]

\[
- \mathbb{E} \sum_{k=0}^{N-1} \left[ \langle \hat{b}(k), P_k \rangle + \sum_{j=1}^{d} \langle \hat{\sigma}^j(k), Q^j_k \rangle \right] + \mathbb{E} [h(X_N) - h(X_N^*)].
\]

Note that for the first two expectations to be well defined, the third integrability condition in (8) is needed.
Sufficient MP

If \( h \) and \( H \) are convex in \((x, v)\) and (14) holds, then

\[
J(u) - J(u^*) \geq \mathbb{E} \sum_{k=1}^{N} \langle l_x(k), \hat{X}_k \rangle - \mathbb{E} \sum_{k=0}^{N-1} \left[ \langle \alpha_k, P_k \rangle + \sum_{j=1}^{d} \langle \beta^j_k, Q^j_k \rangle \right],
\]

where \( \hat{X} = X - X^* \), \( \alpha_k = \hat{b}(k) - b_x^\top(k) \hat{X}_k \) and \( \beta^j_k = \hat{\sigma}^j(k) - (\sigma^j_x(k))^\top \hat{X}_k \). Similar to (6), we have \( \hat{X}_k = \sum_{i=0}^{k-1} M_i^{k-1} \delta_i \)

for \( k \in \mathbb{T} \), where \( \delta_k = \alpha_k + \sum_{j=1}^{d} \beta^j_k W^j_k, k \in \mathbb{T} \).

Finally, in view of (13), by interchanging the order of summations and using the properties of conditional expectations we can show that the right-hand side of the last inequality is equal to zero.
Outline

1. Introduction

2. Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle

3. Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game

4. Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem

5. Conclusions and more extensions
Let us consider a system described by

\[
\begin{cases}
X_{k+1} = b(k, X_k, u_{1,k}, u_{2,k}) + \sigma(k, X_k, u_{1,k}, u_{2,k})W_k, \\
X_0 = x_0 \in \mathbb{R}^n,
\end{cases}
\]  

and introduce two cost functionals

\[
J_i(u_1, u_2) = \mathbb{E} \left\{ \sum_{k=0}^{N-1} l_i(k, X_k, u_{1,k}, u_{2,k}) + h_i(X_N) \right\}, \quad i = 1, 2.
\]

Player \(i\) hopes to minimize the cost \(J_i\) by selecting an appropriate control \(u_i^*\), \(i = 1, 2\). The problem is to find \((u_1^*, u_2^*) \in \mathcal{U}_1 \times \mathcal{U}_2\) s.t.

\[
\begin{cases}
J_1(u_1^*, u_2^*) = \inf_{u_1 \in \mathcal{U}_1} J_1(u_1, u_2^*), \\
J_2(u_1^*, u_2^*) = \inf_{u_2 \in \mathcal{U}_2} J_2(u_1^*, u_2).
\end{cases}
\]

It is a discrete-time nonzero-sum stochastic game. We call it Problem (NZSG). Such \((u_1^*, u_2^*)\) is called an equilibrium point.
Similar assumptions are supposed on the noises, the controls and the coefficients as in the previous section.

Let us define

\[ H_i = \langle b, p \rangle + \text{tr}[\sigma^T q] + l_i, \quad i = 1, 2. \]

Introduce the following adjoint equations for \( i = 1, 2 \):

\[
\begin{align*}
P_{i,k} &= \mathbb{E} \left\{ H_{ix}(k + 1, X_{k+1}^*, u_{1,k+1}^*, u_{2,k+1}^*, P_{i,k+1}, Q_{i,k+1}) | \mathcal{F}_k \right\}, \\
Q_{i,k} &= \mathbb{E} \left\{ H_{ix}(k + 1, X_{k+1}^*, u_{1,k+1}^*, u_{2,k+1}^*, P_{i,k+1}, Q_{i,k+1}) W_k^T | \mathcal{F}_k \right\}, \\
k &= 0, 1, \ldots, N - 2, \\
P_{i,N-1} &= \mathbb{E} [h_{ix}(X_N^*) | \mathcal{F}_{N-1}], \\
Q_{i,N-1} &= \mathbb{E} [h_{ix}(X_N^*) W_{N-1}^T | \mathcal{F}_{N-1}].
\]
Note that Problem (NZSG) consists of two discrete-time stochastic optimal control problems of Problem (DTC)’s type.

**Theorem 3.** Suppose \((u_1^*, u_2^*)\) is an equilibrium point of Problem (NZSG). Then it holds that \(\forall v_{i,k} \in U_{i,k}, k \in \mathcal{T}, i = 1, 2,\)

\[
\langle H_{iu_i} (k, X^*_k, u_{1,k}^*, u_{2,k}^*, P_{i,k}, Q_{i,k}), v_{i,k} - u_{i,k}^* \rangle \geq 0. \quad (17)
\]

**Theorem 4.** Let \((u_1^*, u_2^*)\) be a pair of admissible controls for Problem (NZSG). Suppose that \(H_1(k, \cdot, \cdot, u_{2,k}^*, P_{1,k}, Q_{1,k}),\)

\(H_2(k, \cdot, u_{1,k}^*, \cdot, P_{2,k}, Q_{2,k}), h_1(\cdot)\) and \(h_2(\cdot)\) are convex for each \(k \in \mathcal{T}.\) Then \((u_1^*, u_2^*)\) is an equilibrium point if it satisfies (17).
Next let us consider a discrete-time zero-sum stochastic game. We still use the system (15) where $u_1$ and $u_2$ are the controls of two players. The discrete-time zero-sum stochastic game, Problem (ZSG), is to find $(u^*_1, u^*_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J(u^*_1, u_2) \leq J(u^*_1, u^*_2) \leq J(u_1, u^*_2), \quad \forall (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2, \quad (18)$$

where

$$J(u_1, u_2) = \mathbb{E} \left\{ \sum_{k=0}^{N-1} l(k, X_k, u_{1,k}, u_{2,k}) + h(X_N) \right\}.$$

Such $(u^*_1, u^*_2)$ is called a saddle point.
By transforms, **Problem (ZSG)** could be turned to a **nonzero-sum stochastic game** of **Problem (NZSG)**’s type. In fact, by defining

\[ l_1 = l, \; h_1 = h, \; J_1 = J, \; l_2 = -l, \; h_2 = -h, \; J_2 = -J, \]

**Problem (ZSG)** is equivalent to finding \((u_1^*, u_2^*)\) such that

\[ J_1(u_1^*, u_2^*) = \inf_{u_1 \in U_1} J_1(u_1, u_2^*), \quad J_2(u_1^*, u_2^*) = \inf_{u_2 \in U_2} J_2(u_1^*, u_2), \]

where

\[ J_i(u_1, u_2) = \mathbb{E} \left\{ \sum_{k=0}^{N-1} l_i(k, X_k, u_{1,k}, u_{2,k}) + h_i(X_N) \right\}, \quad i = 1, 2. \]
Define the Hamiltonian by $H = \langle b, p \rangle + tr \left[ \sigma^\top q \right] + l$, and the adjoint equation by

$$
\begin{align*}
P_k &= \mathbb{E} \left[ H_x(k + 1, X_{k+1}^*, u_{1,k+1}^*, u_{2,k+1}^*, P_{k+1}, Q_{k+1}) \mid \mathcal{F}_k \right], \\
Q_k &= \mathbb{E} \left[ H_x(k + 1, X_{k+1}^*, u_{1,k+1}^*, u_{2,k+1}^*, P_{k+1}, Q_{k+1}) W_k^\top \mid \mathcal{F}_k \right], \\
k &= 0, 1, \cdots, N - 2, \\
P_{N-1} &= \mathbb{E} \left[ h_x(X_N^*) \mid \mathcal{F}_{N-1} \right], \\
Q_{N-1} &= \mathbb{E} \left[ h_x(X_N^*) W_{N-1}^\top \mid \mathcal{F}_{N-1} \right].
\end{align*}
$$

By the necessary MP of Problem (NZSG) [see Theorem 3], we get the following necessary MP for Problem (ZSG).

**Theorem 5.** Suppose $(u_1^*, u_2^*)$ is a saddle point of Problem (ZSG). Then for each $k \in \mathcal{T}$,

$$
\begin{align*}
\left\langle H_{u_1}(k, X_k^*, u_{1,k}^*, u_{2,k}^*, P_k, Q_k), v_{1,k} - u_{1,k}^* \right\rangle &\geq 0, \forall v_{1,k} \in U_{1,k}, \\
\left\langle H_{u_2}(k, X_k^*, u_{1,k}^*, u_{2,k}^*, P_k, Q_k), v_{2,k} - u_{2,k}^* \right\rangle &\leq 0, \forall v_{2,k} \in U_{2,k}.
\end{align*}
$$

(19)
By the sufficient MP of Problem (NZSG) [see Theorem 4], we get the following sufficient MP for Problem (ZSG).

**Theorem 6.** Let \((u_1^*, u_2^*)\) be a pair of admissible controls for Problem (ZSG). Assume that \(H(k, \cdot, \cdot, u_{2,k}^*, P_k, Q_k)\) is convex, \(H(k, \cdot, u_{1,k}^*, \cdot, P_k, Q_k)\) is concave and \(h(x) = Rx + \xi\) with \(R\) being a bounded \(\mathcal{F}_T\)-measurable r.v. and \(\xi\) an integrable \(\mathcal{F}_T\)-measurable r.v.. Then \((u_1^*, u_2^*)\) is a saddle point if (19) holds.
Outline

1. Introduction

2. Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle

3. Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game

4. Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem

5. Conclusions and more extensions
Example 1. Suppose there is a market in which \( m + 1 \) assets are traded discontinuously. The price \( \{ B_k, k \in \mathcal{T} \} \) of one risk-less asset satisfies

\[
B_{k+1} - B_k = r_k B_k, \quad k \in \mathcal{T}; \quad B_0 = b_0. \tag{20}
\]

The prices of \( m \) risk assets, \( \{ S_{i,k}, k \in \mathcal{T} \} \), \( i = 1, 2, \ldots, m \), satisfy

\[
S_{i,k+1} - S_{i,k} = S_{i,k}(\mu_{i,k} + \sigma_{i,k} W_k), \quad k \in \mathcal{T}; \quad S_{i,0} = s_i. \tag{21}
\]

We now consider an investor whose wealth at time \( k \) is \( X_k \). Denote by \( u_{i,k} \) the wealth invested in the \( i \)-th stock at time \( k \), and call \( u_k = (u_{1,k}, u_{2,k}, \ldots, u_{m,k})^\top \) a portfolio of the investor at time \( k \). Assume moreover that consumption occurs at each time \( k \in \mathcal{T} \), and denote by \( \{ v_k, k \in \mathcal{T} \} \) the consumption sequence. Then

\[
X_{k+1} - X_k = \sum_{i=1}^m \frac{u_{i,k}}{B_k} (B_{k+1} - B_k) + \sum_{i=1}^m \frac{u_{i,k}}{S_{i,k}} (S_{i,k+1} - S_{i,k}) - v_k.
\]
Thus the dynamics of the wealth satisfies

\[
\begin{aligned}
X_{k+1} &= [(r_k + 1)X_k + b_k^\top u_k - v_k] + \sigma_k^\top u_k W_k, \\
X_0 &= x_0,
\end{aligned}
\]

where \(b_k = (\mu_{1,k} - r_k, \mu_{2,k} - r_k, \cdots, \mu_{m,k} - r_k)^\top\) and \(\sigma_k = (\sigma_{1,k}, \sigma_{2,k}, \cdots, \sigma_{m,k})^\top\). Assume that \(r, b, \sigma\) are bounded.

Let \(\mathcal{U}\) be the set of \(u = \{u_k, k \in \mathcal{T}\}\), where each \(u_k\) is an \(\mathcal{F}_k\)-measurable and \(\mathbb{R}^m\)-valued r.v. with necessary integrability. Denote by \(\mathcal{V}\) the collection of all \(v = \{v_k, k \in \mathcal{T}\}\) such that each \(v_k\) is real-valued and \(\mathcal{F}_k\)-measurable, \(v_k \geq a\) with some constant \(a > 0\), and \(v\) has necessary integrability.
The objective of the investor is to find \((u^*, v^*) \in \mathcal{U} \times \mathcal{V}\) such that

\[
J(u^*, v^*) = \inf_{(u,v) \in \mathcal{U} \times \mathcal{V}} J(u, v)
\]

with

\[
J(u, v) = \mathbb{E} \left\{ \sum_{k=0}^{N-1} \frac{1}{2} L_k |u_k - \eta_k|^2 - G_k \left( \frac{v_k}{1 - \delta} \right)^{1-\delta} \right\} - RX_N .
\]

Assume that \(L_k, G_k\) and \(R\) are bounded scalar-valued r.v.'s, \(\eta_k\) is a bounded \(\mathbb{R}^m\)-valued r.v. representing a benchmark, and \(\delta\) is a constant representing the Arrow-Pratt index of the risk aversion. Assume \(0 < \delta < 1, G_k > 0, L_k > c\) and \(R > c\) for some \(c > 0\).

Note that the objective consists of three parts. The first one is a minimization of the difference between the portfolio and a given benchmark, the second is a maximization of a utility of the consumption and the last is a maximization of the terminal wealth.
Let (H1) and (H2) hold true. The Hamiltonian is defined by

\[
H = \left[ (r_k + 1)x + b_k^T u - v \right] p + \sigma_k^T u q + \frac{L_k}{2} |u - \eta_k|^2 - G_k \frac{v^{1-\delta}}{1 - \delta}.
\]

It’s easily seen that \( H \) and \( h(x) = -Rx \) are convex functions of \((x, u, v)\). Applying Theorem 1 shows that the optimal control \((u^*, v^*)\) satisfies

\[
\langle b_k P_k + \sigma_k Q_k + L_k (u^*_k - \eta_k), u - u^*_k \rangle \geq 0, \quad \forall u \in \mathbb{R}^m, \quad (22)
\]
\[
\left[ -P_k - G_k (v^*_k)^{-\delta} \right] (v - v^*_k) \geq 0, \quad \forall v \geq a \quad (23)
\]

for each \( k \), where \( \{(P_k, Q_k), k \in \mathcal{T}\} \) uniquely solves

\[
\begin{cases}
    P_k = \mathbb{E} [(r_{k+1} + 1) P_{k+1} | \mathcal{F}_k], \quad k = 0, 1, \cdots, N - 2, \\
    Q_k = \mathbb{E} [(r_{k+1} + 1) P_{k+1} W_k | \mathcal{F}_k], \quad k = 0, 1, \cdots, N - 2, \\
    P_{N-1} = -\mathbb{E} [R | \mathcal{F}_{N-1}], \quad Q_{N-1} = -\mathbb{E} [RW_{N-1} | \mathcal{F}_{N-1}].
\end{cases} \quad (24)
\]
By Theorem 2, an admissible control \((u^*, v^*)\) satisfying (22) and (23) is indeed an optimal control.

From (22) we derive

\[
  u_k^* = \eta_k - L_k^{-1}(b_k P_k + \sigma_k Q_k). \tag{25}
\]

Since \(H(v) := -P_k v - G_k \frac{v^{1-\delta}}{1-\delta}\) is a differentiable and convex function, (23) implies that \(H(v), v \in [a, +\infty)\), takes its minimum at \(v_k^*\). Note that \(P_k < 0, k \in T\). Then the solution of (23) is

\[
  v_k^* = \begin{cases} 
  \hat{v}_k, & \text{if } \hat{v}_k \geq a, \\
  a, & \text{if } \hat{v}_k < a,
\end{cases} \tag{26}
\]

with \(\hat{v}_k = \left(-\frac{G_k}{P_k}\right)^{\frac{1}{\delta}}\).
It’s easy to see that each $P_k$ is bounded. Thus, $Q_k \in L^\alpha$ and so $Q_k \in L^\beta$ since $\beta \leq \alpha$. This implies that $\{u^*_k\}$ defined by (25) is indeed admissible. Using induction we can get $P_k < -c$ for $k \in \mathcal{T}$, which shows that $\hat{v}_k$ is bounded. Thus, $\{v^*_k\}$ defined by (26) is bounded and so admissible. Consequently, $\{(u^*_k, v^*_k)\}$ defined by (25) and (26) is the optimal strategy of this investment consumption choice problem.

Eq. (24) can be solved explicitly in some special cases. If $r_k, k = 1, 2, \cdots, N - 1$ and $R$ are deterministic constants, and $\mathbb{E}[W_k|\mathcal{F}_k] = 0, k \in \mathcal{T}$, then (24) is uniquely solved by $Q_k \equiv 0$ and

$$P_k = \begin{cases} -R(r_{N-1} + 1)(r_{N-2} + 1) \cdots (r_{k+1} + 1), & \text{if } 0 \leq k \leq N - 2, \\ -R, & \text{if } k = N - 1. \end{cases}$$
Example 2. Let us consider one kind of discrete-time LQ stochastic optimal control problem in the case when $d = 1$, $\mathbb{E}[W_k|\mathcal{F}_k] = 0$ and $U_k = \mathbb{R}^m$ for each $k \in T$.

Let us define

$$b(k, x, v) = A_k x + B_k v + C_k, \quad \sigma(k, x, v) = D_k x + E_k v + F_k,$$

$$l(k, x, v) = \frac{\langle G_k x, x \rangle + \langle L_k v, v \rangle}{2}, \quad h(x) = \langle Rx, x \rangle,$$

where the coefficients are deterministic and bounded matrices. Moreover, $G_k, R \geq 0$ and $L_k > 0$ for each $k$. 
By Theorems 1 and 2, \( u^*_k = -L_k^{-1}(B_k^T P_k + E_k^T Q_k) \) is an optimal control of this LQ problem if it’s admissible, where \{\( (P, Q) \)\} solves
\[
\begin{cases}
  P_k = \mathbb{E} \left\{ \left[ A_{k+1}^T P_{k+1} + D_{k+1}^T Q_{k+1} + G_{k+1} X^*_{k+1} \right] \mid \mathcal{F}_k \right\}, \\
  Q_k = \mathbb{E} \left\{ \left[ A_{k+1}^T P_{k+1} + D_{k+1}^T Q_{k+1} + G_{k+1} X^*_{k+1} \right] W_k \mid \mathcal{F}_k \right\}, \\
  k = 0, 1, \cdots, N - 2, \\
  P_{N-1} = \mathbb{E}[RX^*_N \mid \mathcal{F}_{N-1}], \quad Q_{N-1} = \mathbb{E}[RX^*_N W_{N-1} \mid \mathcal{F}_{N-1}].
\end{cases}
\]

Let us give a state-feedback form of \( u^* \) in the simple case when \( D_k = 0 \) and \( E_k = 0 \) for each \( k \in \mathcal{T} \). In this case, we have
\[
u^*_k = -L_k^{-1} B_k^T P_k, \tag{27}
\]
where
\[
\begin{cases}
  P_k = \mathbb{E} \left\{ \left[ A_{k+1}^T P_{k+1} + G_{k+1} X^*_{k+1} \right] \mid \mathcal{F}_k \right\}, \quad k = 0, 1, \cdots, N - 2, \\
  P_{N-1} = \mathbb{E}[RX^*_N \mid \mathcal{F}_{N-1}].
\end{cases}
\]

To conclude, we only need to express \( P_k \) in terms of \( X^*_k \). Since the diffusion term is independent of \( x \) and \( v \), and \{\( Q_k \)\} is not needed, the square-integrability issue disappears in this case.
By Lemma 4, we have

\[
\begin{cases}
  P_k = (A_k^{N-1})^\top R \mathbb{E}[X^*_N | \mathcal{F}_k] + \sum_{j=k+1}^{N-1} (A_k^{j-1})^\top G_j \mathbb{E}[X^*_j | \mathcal{F}_k], \\
  k = 0, \cdots, N-2, \\
  P_{N-1} = R \mathbb{E}[X^*_N | \mathcal{F}_{N-1}]
\end{cases}
\]  \tag{28}

where $A_i^j = A_j A_{j-1} \cdots A_{i+1}$ if $i < j$, and $A_i^i = I_n$ if $i = j$.

The subsequent idea is: $P_k \iff E[X^*_{k+1} | \mathcal{F}_k] \iff X^*_k$.

Let us assume

\[
P_k = \alpha_k \mathbb{E}[X^*_{k+1} | \mathcal{F}_k] + \beta_k, \quad k \in \mathcal{T}, \tag{29}
\]

where $\alpha_k, \beta_k, k \in \mathcal{T}$ are deterministic matrices which are to determined later. Then by (27), we have

\[
u^*_k = -L_k^{-1} B_k^\top (\alpha_k \mathbb{E}[X^*_{k+1} | \mathcal{F}_k] + \beta_k). \tag{30}
\]
By substituting (30) into the state equation and taking conditional expectation we get

$$\Phi_k \mathbb{E}[X_{k+1} | \mathcal{F}_k] = A_k X_k^* + \Psi_k, \quad k \in \mathcal{T},$$

where $\Phi_k = I_n + B_k L_k^{-1} B_k^\top \alpha_k$ and $\Psi_k = C_k - B_k L_k^{-1} B_{1,k}^\top \beta_k$.

Let us assume

$$\Phi_k^{-1} \text{ exists and is bounded, } \forall k \in \mathcal{T}. \quad (31)$$

Then

$$\mathbb{E}[X_{k+1} | \mathcal{F}_k] = \phi_k X_k^* + \psi_k, \quad k \in \mathcal{T} \quad (32)$$

with

$$\phi_k = \Phi_k^{-1} A_k, \quad \psi_k = \Phi_k^{-1} \Psi_k. \quad (33)$$
Consequently, by (30) and (32) we get

$$u_k^* = -L_k^{-1} B_k^T \alpha_k \phi_k X_k^* - L_k^{-1} B_k^T (\alpha_k \psi_k + \beta_k). \quad (34)$$

This is the feedback form. It remains to determine $\{(\alpha_k, \beta_k)\}$.

By (32), we can use induction to get

$$\mathbb{E}[X_i^* | \mathcal{F}_k] = \phi_{k}^{i-1} \mathbb{E}[X_{k+1}^* | \mathcal{F}_k] + \sum_{j=k+1}^{i-1} \phi_{j}^{i-1} \psi_j, \quad i \geq k + 2, \quad (35)$$

where $\phi_{i}^{j}$ is defined by: $\phi_{i}^{j} = \phi_j \phi_{j-1} \cdots \phi_{i+1}$ if $i < j$, and $\phi_{i}^{i} = I_n$ if $i = j$. That is, we can express $\mathbb{E}[X_i^* | \mathcal{F}_k], i \geq k + 2$, in terms of $\mathbb{E}[X_{k+1}^* | \mathcal{F}_k]$ only.
Taking $i = N$ and $k = N - 2$ in (35) gives

$$E[X_N^*|\mathcal{F}_{N-2}] = \phi_{N-1}E[X_{N-1}^*|\mathcal{F}_{N-2}] + \psi_{N-1}. \tag{36}$$

By (28) we have

$$P_{N-2} = [(A_{N-1})^\top R\phi_{N-1} + G_{N-1}]E[X_{N-1}^*|\mathcal{F}_{N-2}] + (A_{N-1})^\top R\psi_{N-1}.$$ 

Recall from (28) that $\{P_k\}$ satisfies that

$$
\begin{align*}
P_k &= (A_{k}^{N-1})^\top R E[X_N^*|\mathcal{F}_k] + \sum_{j=k+2}^{N-1} (A_{k}^{j-1})^\top G_j E[X_j^*|\mathcal{F}_k] \\
&\quad + G_{k+1} E[X_{k+1}^*|\mathcal{F}_k], \quad k = 0, 1, \ldots, N - 3, \\
\end{align*}

P_{N-2} &= [(A_{N-1})^\top R\phi_{N-1} + G_{N-1}]E[X_{N-1}^*|\mathcal{F}_{N-2}] \\
&\quad + (A_{N-1})^\top R\psi_{N-1}, \\
P_{N-1} &= R E[X_N^*|\mathcal{F}_{N-1}].
$$

\tag{37}
Next, substituting (29) and (35) into (37) and comparing the coefficients leads to

\[
\begin{align*}
\alpha_{N-1} &= R, \quad \beta_{N-1} = 0, \\
\alpha_{N-2} &= G_{N-1} + (A_{N-1})^\top R \phi_{N-1}, \quad \beta_{N-2} = (A_{N-1})^\top R \psi_{N-1}, \\
\alpha_k &= G_{k+1} + \sum_{j=k+2}^{N-1} (A_k^{j-1})^\top G_j \phi_k^{j-1} + (A_k^{N-1})^\top R \phi_k^{N-1}, \\
\beta_k &= \sum_{j=k+2}^{N-1} (A_k^{j-1})^\top G_j \sum_{s=k+1}^{j-1} \phi_s^{j-1} \psi_s + (A_k^{N-1})^\top R \sum_{j=k+1}^{N-1} \phi_j^{N-1} \psi_j, \\
k &= 0, 1, \cdots, N - 3.
\end{align*}
\]

(38)

Note that \(\{(\alpha_k, \beta_k), k \in \mathcal{T}\}\) defined above are bounded.

Finally the conclusion is as follows. Let \(\{(\phi_k, \psi_k)\}\) be defined by (33) and \(\{(\alpha_k, \beta_k), k \in \mathcal{T}\}\) be defined by (38). If (31) holds, then \(u^*\) defined by (34) is the admissible optimal control of this LQ problem.
Outline

1. Introduction

2. Maximum principle
   - Problem formulation and preliminaries
   - Maximum principle

3. Extensions to stochastic games
   - MP for discrete-time nonzero-sum game
   - MP for discrete-time zero-sum game

4. Applications
   - Application to one investment/consumption choice problem
   - Application to one kind of discrete-time LQ problem

5. Conclusions and more extensions
Conclusions.

One kind of discrete-time stochastic optimal control problem is studied. The main result is to establish the necessary as well as sufficient stochastic maximum principle in a rigorous and clear way. A difficulty arising from the lack of necessary integrability of the solution to the adjoint equation is overcome. The results are extended to two kinds of discrete-time stochastic games. Two applications are studied, for which the optimal controls are derived. This research paves a way for further related works.

More extensions.

- general control domain case
- different kinds of control systems, i.e., delayed system, mean-field system, ⋅⋅⋅
- state/control constrained cases
- partial information/ partial observation cases
Thank You!