Non-Concave Utility Maximization without the Concavification Principle

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Motivation (1)

- The classical expected utility maximization model (e.g. CRRA/CARA utility) is a concave optimization problem.

- In contrast, many investment objectives are related to non-concave utility, e.g.
Motivation (2)

- Previous literature: ignore portfolio constraints (such as no-short-sale, no borrowing, etc)
- Shortage: Unrealistic high leverage

![Graphs](image-url)

**Figure:** Time to Maturity $\frac{1}{12}$
Motivation (3)

- Ignore portfolio constraints $\Rightarrow$ concavification principle $\Rightarrow$ concave value function
- With and without portfolio constraints: an example with S-shaped utility

![Value Function Graph]

**Figure:** Time to Maturity $\frac{1}{12}$
Our focus

- Non-concave portfolio optimization without the concavification principle
  - general leverage constraints (e.g. no short sale, no borrowing)
  - discontinuous utility

- Concavification principle does not apply
  - numerical methods: scheme? convergence?

- Joint impact of portfolio constraints and non-concave utility on optimal policy
Our findings

- General findings
  - The value function is not necessarily globally concave before maturity
  - Investors are not myopic w.r.t. portfolio constraints in the sense that they may take risker leverage ratios in anticipation of portfolio constraints
  - Investors may short sale stock despite positive risk premium for higher volatility
    The intuition is that in concave utility case, the volatility is a burden which is need to be offset by the higher return. However, in convex utility case, the volatility is actually a resource

- Those model-specific findings in literature remain valid to some extent.
Theoretical contribution

- We prove the comparison principle even in the presence of portfolio constraints and discontinuous utility
  - The value function is the unique discontinuous viscosity solution to the associated HJB equation
  - Convergence of the standard monotone finite difference scheme
The model

- A riskfree bond $B$ with interest rate 0 and a risky stock $S$
  \[ dS_t/S_t = \mu dt + \sigma dB_t \]

- $W^\pi(t)$: the self-financing wealth process
  \[ dW^\pi(t) = \mu \pi(t) W^\pi(t) dt + \sigma \pi(t) W^\pi(t) dB_t, \]
  where $\pi(t)$: the proportion of wealth in stock

\[
\begin{align*}
V(t, w) &= \sup_{d \leq \pi \leq u, \ W^\pi(T) \geq 0 } E \left[ U(W^\pi(T)) \middle| W(t) = w \right],
\end{align*}
\]

where
- $d \leq \pi \leq u$: portfolio constraints, if $d = 0$, no short-selling; if $u = 1$, no borrowing
- $U(\cdot)$: utility function, not necessarily concave or continuous
Examples of non-concave utilities: Goal-reaching problem

A fund manager maximizes the probability of beating some benchmark:

\[ U(W_T) = 1_{\{W_T \geq H\}}, \]

where \( H \) is the target level
Examples of non-concave utilities: S-shaped utility

- Tversky and Kahneman (1979, Econometrica)’s S-Shaped utility:

\[
U(W_T) = \begin{cases} 
(W_T - W_0)^p & \text{for } W_T > W_0 \\
-\lambda(W_0 - W_T)^p & \text{for } W_T \leq W_0 
\end{cases}
\]

- \( W_0 \): the initial wealth, distinguishing the gain and loss
- \( 0 < p < 1 \): the degree of risk aversion, e.g. \( p = 0.88 \)
- \( \lambda > 1 \): pain from loss > pleasure from gain, e.g. \( \lambda = 2.25 \)
- The utility is convex for loss \( W_T < W_0 \) and concave for gain \( W_T > W_0 \)
Examples of non-concave utilities: Option compensation

- Carpenter (2000, JF): A risk averse manager compensated with a call option over the fund he controls

\[ U(W_T) = (m \max\{W_T - K, 0\} + C)^p \]

- \( 0 < p < 1 \): the risk aversion degree
- \( K > 0 \): the strike price of the option
- \( m \): the number of options
- \( C > 0 \): the constant compensation
Consider the HJB equation:

$$\frac{\partial V}{\partial t} + \sup_{d \leq \pi_t \leq u} \left\{ \frac{1}{2} \pi_t^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial w^2} + \pi_t w \mu \frac{\partial V}{\partial w} \right\} = 0, \quad (1)$$

with the boundary condition

$$V(t, 0) = U(0) \quad (2)$$

and an asymptotic condition at maturity:
(i) if $[d, u]$ is unbounded, it degenerates to the standard case

$$\lim_{(t, \zeta) \to (T-, w)} V(t, \zeta) = \hat{U}(w), \quad (3)$$

where $\hat{U}$ is the concave envelope of $U$

(ii) if $[d, u]$ is bounded, (discontinuity!)

$$\lim_{(t, \zeta) \to (T-, w)} V(t, \zeta) - U(w-) - 2\Phi(y)(U(w+) - U(w-)) = 0 \quad (4)$$

where

$$y = \frac{0 \wedge (\ln \zeta - \ln w)}{\max\{-d, u\} \sigma \sqrt{T-t}}$$

and $\Phi$ is the CDF of a standard normal random variable
Theoretic analysis of constrained non-concave problem

Theorem (Comparison Principle)

(i) Assume $[d, u]$ is bounded (unbounded). Let $v^*$ and $v_*$ be separately viscosity subsolution and supersolution to (1) with boundary conditions (2) and (4) (with (2) and (3))

Suppose $|v^*|, |v_*| \leq C_1 w^p + C_2$, for some $0 < p < 1$, $C_1, C_2 > 0$

Then $v^* \leq v_*$ for all $w \geq B$ and $0 < t < T$

▶ This theorem covers both the continuous and the discontinuous case

▶ Comparison principle $\Rightarrow \begin{cases} \text{uniqueness of viscosity solution} \\ \text{convergence of numerical schemes} \end{cases}$
Theoretic analysis of constrained non-concave problem

**Theorem**

\[ V(t, w) \] is the unique viscosity solution of the HJB equation to (1) with boundary conditions (2) and (4) (with (2) and (3)) which satisfies

\[ |v| \leq C_1 w^p + C_2, \quad \text{for some } 0 < p < 1, \ C_1, C_2 > 0 \quad (5) \]

**Theorem (Numerical Scheme Convergence)**

The numerical solution of a fully implicit finite difference scheme with upwind treatment for the HJB equation converges to the value function as the discretization size tends to zero.
General findings (1)

Figure: Goal reaching

Figure: S-shaped utility

In general the value function is not globally concave before maturity
Goal reaching problem ($r = 0.07, \mu = 0.15, \sigma = 0.3, T = 1$)

Investors are not myopic with respect to portfolio constraints

Investors may gamble by short-selling (borrowing) stock even with positive (negative) risk premium
General findings (3)

S-shaped utility maximization

\[ (r = 0.03, \mu = 0.07, \sigma = 0.3, p = 0.5, \lambda = 2.25, W_0 = 1, T = 1/12, B = 0.5) \]

Investors are not myopic with respect to portfolio constraints

Investors may gamble by short-selling (borrowing) stock even with positive (negative) risk premium
Goal-reaching problem

Optimal strategy for different $\mu$, while $r = 0.07, \sigma = 0.3, T = 1, B = 0$

- The optimal goal-reaching strategy is no longer equivalent to the replicating strategy of a digital option rather than the assertion in Browne (1999)
S-shaped utility ($W_0 = 1$)

\[ r = 0.03, \mu = 0.07, \sigma = 0.3, p = 0.5, \lambda = 2.25, W_0 = 1, \]
\[ T = 1/12, B = 0.5 \]

- Reduce more stock near reference point: a more conservative strategy compared to Berkelaar, Kouwenberg and Post (2004)
Option compensation

Figure: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $K = 1$, $\alpha = 0.2$, $C = 0.02$, $W_0 = 1$, $T - t = 1/12$, $B = 0.5$

- Convex incentives may reduce stock investment in more scenarios compared to Carpenter (2000, JF)
Conclusion(1)

- Non-concave portfolio optimization without the concavification principle: portfolio constraints, discontinuous utility

- We prove comparison principle by introducing an asymptotic condition at maturity. This implies
  - the convergence of the standard monotone finite difference method
  - uniqueness of discontinuous viscosity solutions to the associate HJB equations
Conclusion (2)

▶ Three general findings
  ▶ The concavification technique no longer applies, and in general the value function is not globally concave before maturity
  ▶ Investors may take action in anticipation of future portfolio constraints being binding
  ▶ Investors may gamble against market trend in the case of underperformance

▶ Those model-specific findings hold to some extent