Optimal make-take fees for market making regulation

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Introduction

Exchanges in competition

- With the fragmentation of financial markets, exchanges are nowadays in competition.
- Traditional international exchanges are now challenged by alternative trading venues.
- Consequently, they have to find innovative ways to attract liquidity on their platforms.
- A possible solution: using a make-take fees system, that is charging in an asymmetric way liquidity provision and liquidity consumption.
Introduction

A controversial topic

- Make-take fees policies are seen as a major facilitating factor to the emergence of a new type of market makers aiming at collecting fee rebates: the high frequency traders.

- As stated by the Securities and Exchanges commission: “Highly automated exchange systems and liquidity rebates have helped establish a business model for a new type of professional liquidity provider that is distinct from the more traditional exchange specialist and over-the-counter market maker.”
Introduction

HFT market makers

The concern with high frequency traders becoming the new liquidity providers is two-fold.

- Their presence implies that slower traders no longer have access to the limit order book, or only in unfavorable situations when high frequency traders do not wish to support liquidity.
- They tend to leave the market in time of stress.
Introduction

Our aim

- Providing a quantitative and operational answer to the question of relevant make-take fees.
- We take the position of an exchange (or of the regulator) wishing to attract liquidity. The exchange is looking for the best make-take fees policy to offer to market makers in order to maximize its utility.
- In other words, it aims at designing an optimal contract with the (unique) market marker to create an incentive to increase liquidity.
- Principal/agent type approach : the wealth of the principal (exchange) depends on the agent's (market maker) effort (essentially his spread), but the principal cannot directly control the effort.
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The market maker

Market maker’s controls

- The market maker has a view on the efficient price (midprice) of the asset

\[ S_t = S_0 + \sigma W_t, \]

where \( \sigma \) is the price volatility.

- He fixes the ask and bid prices

\[ P^a_t = S_t + \delta^a_t, \quad P^b_t = S_t - \delta^b_t. \]
The order flow

Arrival of market orders

- We model the arrival of buy (resp. sell) market orders by a point process \((N^a_t)_{t \geq 0}\) (resp. \((N^b_t)_{t \geq 0}\)) with intensity \((\lambda^a_t)_{t \geq 0}\) (resp. \((\lambda^b_t)_{t \geq 0}\)).
- The inventory of the market maker \(Q_t = N^b_t - N^a_t\).
- We consider a threshold inventory \(\bar{q}\) above which the market maker stops quoting on the ask or bid side.
- From financial economics arguments:

\[
\lambda^a_t = \lambda(\delta^a_t)1_{\{Q_t > -\bar{q}\}}, \quad \lambda^b_t = \lambda(\delta^b_t)1_{\{Q_t < \bar{q}\}},
\]

where \(\lambda(x) = Ae^{-k(x+c)/\sigma}\).
Martingale processes

Equivalent probabilities

The market maker controls the spread $\delta = (\delta^a, \delta^b)$. We define the associated probability $\mathbb{P}^\delta$ such that

$$\tilde{N}^a,\delta_t = N^a_t - \int_0^t \lambda(\delta^a_s)1\{Q_s > \bar{q}\} ds$$

and

$$\tilde{N}^b,\delta_t = N^b_t - \int_0^t \lambda(\delta^b_s)1\{Q_s < \bar{q}\} ds$$

are martingales.
The market maker viewpoint

The profit and loss of the market maker

- We consider a final time horizon $T > 0$.
- The cash flow of the market maker

$$X^\delta_t = \int_0^t P^a_u dN^a_u - \int_0^t P^b_u dN^b_u.$$

- The inventory risk of the market maker is $Q_t S_t$.
- For a given contract $\xi$ given by the exchange, seen as an $\mathcal{F}_T$ measurable random variable, the market maker chooses his spread $\delta$ by maximizing his utility.
The market maker optimization problem

Under the exchange incentive policy $\xi$, the market maker solves now

$$V_{MM}(\xi) = \sup_{\delta} \mathbb{E}^\delta \left[ -\exp \left( -\gamma (X^{\delta}_T + Q_TS_T + \xi) \right) \right].$$

- We obtain an optimal response given by $\hat{\delta}_t(\xi) = (\hat{\delta}_a^t(\xi), \hat{\delta}_b^t(\xi))$.
- We will only consider contracts such that $V_{MM}(\xi)$ is above a threshold utility value $R$:

$$C = \{ \xi \mathcal{F}_T\text{-measurable such that } V_{MM}(\xi) > R \}$$

+ integrability conditions.
- For $\xi = 0$, well studied problem since Avellaneda and Stoikov.
The exchange viewpoint

We assume that the exchange

- Earns $c > 0$ for each market order occurring in its platform.
- Pays the incentive policy $\xi$ to the market maker.

The profit and loss of the exchange is

$$c (N^a_T + N^b_T) - \xi.$$ 

The exchange problem

The exchange designs the contract $\xi$ by solving

$$V_E = \sup_{\xi \in C} \mathbb{E}^{\hat{\delta}}(\xi) \left[ - \exp \left( -\eta (c (N^a_T + N^b_T) - \xi) \right) \right],$$

where $\eta$ is the risk aversion of the exchange.
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Solving the market maker problem for a given contract

Dynamic programming principle

- We fix $\xi$ and compute the best response of the market maker.
- Let $\tau$ be a stopping time with values in $[t, T]$ and $\mu \in A_\tau$, where $A_\tau$ denotes the restriction of the set of admissible controls $A$ to controls on $[\tau, T]$.
- Let $J_T(\tau, \mu) = \mathbb{E}_\tau^{\mu} \left[ - e^{-\gamma \int_\tau^T (\mu^a u dN^a_u + \mu^b u dN^b_u + Q_u dS_u) \right] e^{-\gamma \xi}$ and

$$V_\tau = \text{ess sup}_{\mu \in A_\tau} J_T(\tau, \mu).$$

- Dynamic programming principle:

$$V_t = \text{ess sup}_{\delta \in A} \mathbb{E}_t^{\delta} \left[ - e^{-\gamma \int_t^\tau (\delta^a u dN^a_u + \delta^b u dN^b_u + Q_u dS_u) \right] V_\tau \right].$$
Solving the market maker problem for a given contract

A convenient super-martingale

Let

$$U^\delta_t = V_t e^{-\gamma \int_0^t \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u}.$$ 

$$U^\delta_0 = V_0$$ and

$$U^\delta_T = -e^{-\gamma \left( \int_0^T \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u + \xi \right)}.$$ 

From the DPP, we get that $U^\delta_t$ is a $\mathbb{P}^\delta-$super-martingale. We want to find the optimal controls $(\delta^a, \delta^b)$ turning it into a martingale.

To do so we find a suitable representation of $U^\delta_t$. 

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Solving the market maker problem for a given contract

Doob-Meyer and martingale representation

- **Doob-Meyer**: \( U_t^\delta = M_t^\delta - A_t^\delta \), where \( M_t^\delta \) is a \( \mathbb{P}^\delta \)–martingale and \( A_t^\delta \) is an integrable non-decreasing predictable process starting at zero.

- **Martingale representation theorem**: There exists a predictable process \( \tilde{Z}^\delta = (\tilde{Z}^\delta, S, \tilde{Z}^\delta, a, \tilde{Z}^\delta, b) \) such that \( M_t^\delta \) can be represented as

\[
V_0 + \int_0^t \tilde{Z}_r^\delta . d\chi_r - \int_0^t \tilde{Z}_r^{\delta, a} \lambda(\delta_r^a)1\{Q_r > -\bar{q}\} \, dr - \int_0^t \tilde{Z}_r^{\delta, b} \lambda(\delta_r^b)1\{Q_r < \bar{q}\} \, dr,
\]

with \( \chi = (S, N^a, N^b) \).
Solving the market maker problem for a given contract

Reducing the class of contracts

- Let $Y$ be the process defined by $V_t = -e^{-\gamma Y_t}$.
- $Y_T = \xi$ and using Ito’s formula together with the previous result and the martingale property of $U_t$ for the optimal controls we get

$$dY_t = Z^a_t dN^a_t + Z^b_t dN^b_t + Z^S_t dS_t - H(Z_t, Q_t)dt,$$

for an explicit function $H$ and where the $Z^i$ do not depend on $\delta$.
- Any contract $\xi$ can be (uniquely) represented under the preceding form! We can restrict ourselves to such contracts.
- Natural financial interpretation of the contracts:
  - The exchange rewards the market maker by $Z^a$ (resp. $Z^b$) for each buy (resp. sell) market order.
  - The exchange participates to the market/inventory risk of the market maker by taking $-Z^S$ of his share.
  - The market maker pays a continuous coupon $H(Z_t, Q_t)dt$. 
Solving the market maker problem for a given contract

New super-martingale representation

- In term of this new representation, we obtain

$$U_\delta^t = M_\delta^t + \gamma \int_0^t U_u^\delta (H(Z_u, q_u) - h(\delta_u, Z_u, q_u)) \, du,$$

where $h$ is explicit and

$$H(z, q) = \sup_{|\delta^a| \lor |\delta^b| \leq \delta_\infty} h(\delta, z, q).$$

- The process $U_\delta^t$ becomes a martingale if and only if $\delta$ is chosen as the maximizer of $h$. 
Let $\xi$ be an admissible contract. The unique optimal spread of the market maker is given by

$$
\hat{\delta}_t^a(\xi) = -Z_t^a + \frac{1}{\gamma} \log(1 + \frac{\sigma \gamma}{k}), \quad \hat{\delta}_t^b(\xi) = -Z_t^b + \frac{1}{\gamma} \log(1 + \frac{\sigma \gamma}{k}).
$$
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Solving the exchange problem

By representing any contract \( \xi = Y_T^{Y_0^\xi, Z^\xi} \), the exchange problem

\[
V_E = \sup_{\xi \in C} \mathbb{E}^{\hat{\delta}(\xi)} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - \xi \right) \right) \right]
\]

is equivalent to

\[
V_E = \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

Reduction to a classical control problem

\[
V_E = \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
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\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]

\[
= \sup_{\xi} \mathbb{E}^{\hat{\delta}(Y_T^{Y_0^\xi, Z^\xi})} \left[ -\exp \left( -\eta \left( c(N_T^a + N_T^b) - Y_T^{Y_0^\xi, Z^\xi} \right) \right) \right]
\]
HJB of the exchange problem

Reduction to a HJB equation

- \( V_E = v(0, Q_0) \) with

\[
\partial_t v(t, Q) + \sup_z h_E(Q, v(t, Q), v(t, Q + 1), v(t, Q - 1), z) = 0
\]

and \( v(T, q) = -1 \), where the function \( h_E \) is explicit.

- The optimal control \( Z^\star \) is obtained so that \( Z^\star_t \) is solution of the maximization problem of

\[
z \mapsto h_E(Q_t, v(t, Q_t), v(t, Q_t + 1), v(t, Q_t - 1), z).
\]

It is explicit in terms of the parameters of \( h_E \).
Reduction of the HJB equation

**Reduction to a linear equation**

If we take \( u(t, Q) = (−v(t, Q))^{−\frac{k}{\sigma \eta}} \), we get

\[
\begin{aligned}
\partial_t u(t, Q) + C_1 Q^2 &- C_2 (u(t, Q - 1)1_{\{Q > -\bar{q}\}} + u(t, Q + 1)1_{\{Q < \bar{q}\}}) = 0, \\
u(T, Q) &= 1,
\end{aligned}
\]

with \( C_1 \) and \( C_2 \) are positive explicit constants.

- Guarantees the existence and uniqueness of \( v \).
- Easy numerical computation of \( u \) and \( v \).
Optimal contract

Theorem

The contract $\xi^*$ that solves the exchange problem is given by

$$\xi^* = Y^* + \int_0^T Z_t^{a^*} dN_t^a + Z_t^{b^*} dN_t^b + Z_t^{S^*} dS_t - H(Z_t^*, Q_t) dt,$$

with

$$Z_t^{a^*} = -\frac{\sigma}{k} \log \left( \frac{u(t, Q_t)}{u(t, Q_t - 1)} \right) + \hat{c},$$

$$Z_t^{b^*} = -\frac{\sigma}{k} \log \left( \frac{u(t, Q_t)}{u(t, Q_t + 1)} \right) + \hat{c}, \quad Z_t^{S^*} = -\frac{\gamma}{\eta + \gamma} Q_t.$$

with $\hat{c} = c + \frac{1}{\eta} \log \left( 1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right)$. 
The quantities

\[- \log \left( \frac{u(t, Q_t)}{u(t, Q_t - 1)} \right) \text{ and } - \log \left( \frac{u(t, Q_t)}{u(T, Q_t + 1)} \right)\]

are roughly proportional respectively to $Q_t$ and $-Q_t$.

Thus, when the inventory is highly positive, the exchange provides incentives to the market-maker so that it attracts buy market orders and tries to dissuade him to accept more sell market orders, and conversely for a negative inventory.
Comments on the optimal contract

Discussion

- The integral
  \[ \int_0^T Z_u^{S*} dS_u \]
  can be understood as a risk sharing term.

- Indeed, \( \int_0^t Q_u dS_u \) corresponds to the price driven component of the inventory risk \( Q_t S_t \). Hence in the optimal contract, the exchange supports part of this risk so that the market maker maintains reasonable quotes despite some inventory.

- The proportion of risk handled by the platform is \( \frac{\gamma}{\gamma + \gamma_p} \).
Comments on the optimal contract

Discussion

- We see that when acting optimally, the exchange transfers the totality of the taker fee $c$ to the market maker. It is neutral to the value of $c$ (its optimal utility function does not depend on $c$).

- However, $c$ plays an important role in the optimal spread offered by the market maker which is approximately given by

$$-2c - \frac{2}{\gamma_p} \log \left(1 - \frac{\sigma^2 \gamma \gamma_p}{(k + \sigma \gamma)(k + \sigma \gamma_p)}\right) + \frac{2}{\gamma} \log(1 + \frac{\sigma \gamma}{k}).$$

- The exchange may fix in practice the transaction cost $c$ so that the spread is close to one tick by setting

$$c \approx -\frac{1}{2} \text{Tick} - \frac{1}{\gamma_p} \log \left(1 - \frac{\sigma^2 \gamma \gamma_p}{(k + \sigma \gamma)(k + \sigma \gamma_p)}\right) + \frac{1}{\gamma} \log(1 + \frac{\sigma \gamma}{k}).$$

- For $\sigma \gamma/k$ small enough, $c \approx \frac{\sigma}{k} - \frac{1}{2} \text{Tick}$. 
Analyzing the effect of the exchange optimal incentive policy

The benefits of the incentive policy

- We can compute the spread, optimal contract, profit and losses of the market maker and exchange, order flows...
- We compare these quantities to the ones obtained in the case where $\xi = 0$.

\[ T = 600s, \quad \sigma = 0.3\text{Tick}.s^{-1/2}, \quad A = 0.9s^{-1}, \quad k = 0.3s^{-1/2}, \]
\[ \bar{q} = 50 \text{ unities}, \quad \gamma = 0.01\text{Tick}^{-1}, \quad \eta = 1\text{Tick}^{-1}, \quad c = 0.5\text{Tick}. \]
Impact of the incentive policy on the spread

The optimal spread is given by $S_t^* = \delta_t^a + \delta_t^b$ with

$$\delta_t^i = \delta_t^i(\xi^*) = -Z_t^i + \frac{1}{\gamma} \log \left( 1 + \frac{\sigma \gamma}{k} \right), \quad i = a, b.$$ 

**Figure** – Optimal initial spread with/without the exchange incentive policy as a function of the initial inventory $Q_0$. 
Impact of the incentive policy on the spread

**Figure** – Optimal initial ask (left) and bid (right) spread component with/without the exchange incentive policy as a function of the initial inventory $Q_0$. 
Impact of the volatility on the incentive policy

**Figure** – The initial optimal spread difference between both situations with/without incentive policy from the exchange toward the market maker as a decreasing function of the volatility $\sigma$. 
Impact of the incentive policy on the market liquidity

**Figure** – Average order flow on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).
Impact of the incentive policy on the market maker and exchange profit and loss

**Figure** – Average total P&L of the market maker and the exchange on [0, T] with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).
Impact of the incentive policy on trading costs

We consider that there is only one market taker who wants to buy a fixed quantity $Q_{final} = 200$ units. We compute the trading cost in both situations:

$$\int_0^T \delta_s^a dN_s^a.$$

**Figure** – Average trading cost on $[0, T]$ with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).
Conclusion

Benefits of the exchange incentive policy

- Smaller spreads.
- Better market liquidity.
- Increase of the profit and loss of the market maker and the exchange.
- Lower transaction costs.