Timer Options: Expiry Floats with Realized Variance

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Agenda

1. Product nature and uses of timer options
   - Barrier options in the volatility space: knock-out depends on the discrete realized variance hitting the preset variance budget
   - Perform move-based hedging and avoid volatility misspecification.

2. Analytic price formulas of timer options (two-dimensional Fourier integrals) under 3/2-model of stochastic volatility
   - Decomposition into a portfolio of timerlets
   - Joint characteristic function of log-asset price and integrated variance
3. Numerical pricing of timer options using the Hilbert transform algorithm (generalization of the Fourier transform algorithm to deal with the knock-out feature)

- The Fourier transform of a function multiplied by an indicator function (knock-out feature) is related to the Hilbert transform of the Fourier transform: avoidance of nuisance of moving between the real space and Fourier space to check the knock-out condition

\[
\mathcal{F}\left(1_{(-\infty,b)} \cdot f \right)(\beta) = \frac{1}{2} \hat{f}(\xi) - \frac{i}{2} e^{i\beta b} \mathcal{H}(e^{-i\eta b} \hat{f}(\eta))(\beta).
\]

- Evaluation via the Sinc expansion: exponential decay of the truncation errors

\[
\mathcal{H}f(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{f(y)}{x-y} \, dy = \sum_{l=-\infty}^{\infty} f(lh) \frac{1 - \cos \frac{\pi(x-lh)}{h}}{\pi \frac{x-lh}{h}}, \quad h > 0,
\]

where \(h\) is the fixed discretization step.
**Variance budget**

The investor specifies a maximum bound $T$ on the option life and a target volatility $\sigma_0$ to define a *variance budget* $B = \sigma_0^2 T$.

Let $t_i, i = 0, 1, 2, \ldots, N$, be the monitoring dates. Let $\tau_B$ be the random first hitting time in the tenor of monitoring dates at which the discrete realized variance exceeds the variance budget $B$, namely,

$$\tau_B = \min \left\{ j \left| \sum_{i=1}^{j} \left( \ln \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \geq B \right. \right\} \Delta.$$

Here, $\Delta$ is the uniform time interval between consecutive monitoring dates.

Termination date of a finite-maturity timer option $= \min(\tau_B, T)$, where $T$ is the preset *mandated* expiration date.
Knocked out at $\tau_B$ since the variance budget has been breached. This occurs earlier than $T$. 
Uses of timer options

- Portfolio managers can use the timer put options on an index to hedge sudden market drops (with uncertainty in timing). Compensations from the timer put payoff are received earlier (due to increased volatility) after the incidence of market drop. In the bullish market where the stock price increases, the realized volatility decreases thus giving longer life of the time put.

- The implied volatility is often higher than the realized volatility. If one feels the implied volatility in the market is too high currently, then one can capture the volatility risk premium by longing a timer call and shorting a vanilla call (higher price due to higher level of implied volatility). The volatility target is set below the current implied volatility and the volatility risk premium is captured by the difference in values of the two call options.
Analytic pricing formula of discretely monitored finite-maturity timer options under the stochastic volatility models

Define the continuous integrated variance to be \( I_t = \int_0^t v_s \, ds \). We use \( I_t \) as a proxy of the discrete realized variance for the monitoring of the first hitting time. We define \( \tau_B \) to be

\[
\tau_B = \min \left\{ j \mid I_{t_j} \geq B \right\} \Delta.
\]

This approximation does not introduce a noticeable error for daily monitored timer options. Note that

\[
C_0(X_0, I_0, V_0) = \mathbb{E}_0[e^{-r(T\wedge\tau_B)} \max(S_{T\wedge\tau_B} - K, 0)]
\]

\[
= \mathbb{E}_0[e^{-rT} \max(S_T - K, 0)1_{\{\tau_B > T\}}]
\]

\[
+ e^{-r\tau_B} \max(S_{\tau_B} - K, 0)1_{\{\tau_B \leq T\}},
\]

where \( K \) is the strike price and \( r \) is the constant interest rate.
Decomposition into a portfolio of timerlets

1. No knock-out occurs prior to $T$: $\{\tau_B > T\} \Leftrightarrow \{I_T < B\}$

   Terminal payoff = $\max(S_T - K, 0) \mathbf{1}_{\{I_T < B\}}$.

2. Knock-out occurs at $t_{j+1}$, $\{\tau_B = t_{j+1}\} \Leftrightarrow \{I_{t_j} < B\} \backslash \{I_{t_{j+1}} < B\}$.
A finite-maturity discrete timer call option can be decomposed into a portfolio of timerlets:

\[
C_0 = \mathbb{E}_0[ e^{-rT} \max(S_T - K, 0) 1_{\{I_T \leq B\}} ] \\
+ \mathbb{E}_0 \left[ \sum_{j=0}^{N-1} e^{-r t_{j+1}} \left( \max(S_{t_{j+1}} - K, 0) 1_{\{I_{t_j} < B\}} \\
- \max(S_{t_{j+1}} - K, 0) 1_{\{I_{t_{j+1}} < B\}} \right) \right].
\]

The challenge is the modeling of the joint processes of \{\{S_{t_{j+1}}, I_{t_j}\}\} (at two time levels \(t_j\) and \(t_{j+1}\)) and \{\{S_{t_{j+1}}, I_{t_{j+1}}\}\} under a stochastic volatility model.

The log return \(\log \frac{S_{t_{j+1}}}{S_{t_j}}\) over \((t_j, t_{j+1})\) has dependence on the stochastic process of the instantaneous variance.
Stochastic volatility model

Consider the stochastic volatility model specified as follows:

\[
\frac{dS_t}{S_t} = (r - q)dt + \sqrt{v_t}(\rho dW_t + \sqrt{1 - \rho^2} dW_t^v),
\]
\[
dv_t = \alpha(v_t)dt + \beta(v_t)dW_t^v,
\]

where \( \rho \) is the correlation coefficient between the asset price process \( S_t \) and instantaneous variance process \( v_t \), \( W_t \) and \( W_t^v \) are two independent Brownian motions.

Usually, \( \beta(v_t) \) assumes the form of a power function. The common choices of the power are 3/2 and 1/2.

- For the 3/2-model, we choose
  \[
  \alpha(v_t) = v_t(\theta_t - \kappa v_t) \quad \text{and} \quad \beta(v_t) = \epsilon v_t^{3/2}.
  \]

- For the Heston 1/2-model, we choose
  \[
  \alpha(v_t) = \lambda(\theta_t - v_t) \quad \text{and} \quad \beta(v_t) = \eta v_t^{1/2}.
  \]
Analytic evaluation in the Fourier domain

Write log asset price $X_t = \ln S_t$ and integrated variance $I_t = \int_0^t v_s \, ds$, where $I_t$ is used as a proxy for the discrete realized variance used in the barrier condition in the timer option.

Pricing of the timerlets involves the joint process of $S_t$ and $I_t$ (may or may not be at the same time point).

Let $x$ stands for $\ln S_{t_{j+1}}$ and $y$ stands for $I_{t_j}$ or $I_{t_{j+1}}$. The Fourier transform $\hat{F}(\omega, \eta)$ of the terminal payoff $(S_{t_{j+1}} - K, 0)1\{I_{t_j} < B\}$ and $(S_{t_{j+1}} - K, 0)1\{I_{t_{j+1}} < B\}$ admit the same analytic representation

$$
\hat{F}(\omega, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega x - i\eta y} (e^x - K) + 1\{y < B\} \, dx \, dy = \frac{K e^{-i\eta B}}{(i\omega + \omega^2)i\eta}.
$$

We take $\omega = \omega_R + i\omega_I$ and $\eta = \eta_R + i\eta_I$, where the damping factors are chosen such that $\omega_I < -1$ and $\eta_I < 0$, to ensure the existence of the two-dimensional Fourier transform.
Fourier integral representation of the price formula

By the Parseval Theorem, the finite-maturity discrete timer option price admits the following analytic formula in terms of a two-dimensional Fourier integral:

\[
C_0 = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-rT} \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{tN} + i\eta I_{tN}}] \, d\omega_R d\eta_R \\
+ \sum_{j=0}^{N-1} \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-rt_{j+1}} \left\{ \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j+1}}}] - \hat{F}(\omega, \eta) \mathbb{E}_0[e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j+1}}}] \right\} \, d\omega_R d\eta_R.
\]

The challenging tasks involve the determination of the joint characteristic function of log-asset price and integrated variance:

\[
E_0 \left[ e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j+1}}} \right] \text{ and } E_0 \left[ e^{i\omega X_{t_{j+1}} + i\eta I_{t_{j}}} \right]
\]

under the relevant stochastic volatility model.
Partial Fourier transform of the triple joint density function

Let $G(t, x, y, v; t', x', y', v')$ be the joint transition density of the triple $(X, I, V)$ from state $(x, y, v)$ at time $t$ to state $(x', y', v')$ at a later time $t'$. The joint transition density $G$ satisfies the following three-dimensional Kolmogorov backward equation:

$$
-\frac{\partial G}{\partial t} = \left( r - q - \frac{v}{2} \right) \frac{\partial G}{\partial x} + \frac{v}{2} \frac{\partial^2 G}{\partial x^2} + \frac{v}{2} \frac{\partial G}{\partial y} + \alpha(v) \frac{\partial G}{\partial v} + \beta(v)^2 \frac{\partial^2 G}{\partial v^2} + \rho \sqrt{v} \beta(v) \frac{\partial^2 G}{\partial x \partial v},
$$

with the terminal condition:

$$
G(t', x, y, v; t', x', y', v') = \delta(x - x')\delta(y - y')\delta(v - v'),
$$

where $\delta(\cdot)$ is the Dirac delta function.
We define the generalized partial Fourier transform of $G$ by $\tilde{G}$ as follows:

$$\tilde{G}(t, x, y, v; t', \omega, \eta, v') = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{i\omega x' + i\eta y'} G(t, x, y, v; t', x', y', v') \, dy' dx',$$

where the transform variables $\omega$ and $\eta$ are complex variables.

The partial transform $\tilde{G}$ solves the three-dimensional Kolmogorov equation with the terminal condition:

$$\tilde{G}(t', x, y, v; t', \omega, \eta, v') = e^{i\omega x + i\eta y} \delta(v - v').$$
Note that $\tilde{G}$ admits the following solution form:

$$
\tilde{G}(t, x, y, v; t', \omega, \eta, v') = e^{i\omega x + i\eta y} g(t, v; t', \omega, \eta, v'),
$$

where $g$ satisfies the following one-dimensional partial differential equation:

$$
-\frac{\partial g}{\partial t} = \left[ i\omega \left( r - q - \frac{v}{2} \right) - \frac{\omega^2 v}{2} + i\eta v \right] g
$$

$$
+ \left[ \alpha(v) + i\omega \rho \sqrt{v} \beta(v) \right] \frac{\partial g}{\partial v} + \frac{\beta(v)^2}{2} \frac{\partial^2 g}{\partial v^2},
$$

with the terminal condition:

$$
g(t', v; t', \omega, \eta, v') = \delta(v - v').
$$

The double generalized Fourier transform on the log-asset and integrated variance pair reduces the three-dimensional governing equation to a one-dimensional equation.
Conditional characteristic functions

The conditional characteristic function of \((X_{t_j}, I_{t_j})\) is found by integrating with respect to \(v'\) from 0 to \(\infty\)

\[
\mathbb{E}_0[e^{i\omega X_{t_j} + i\eta I_{t_j}}] = \int_0^\infty \tilde{G}(t_0, X_{t_0}, I_{t_0}, v_0; t_j, X_{t_j}, I_{t_j}, v') \, dv' \\
= e^{i\omega X_0 + i\eta I_0} \int_0^\infty g(t_0, v_0; t_j, \omega, \eta, v') \, dv' \\
= e^{i\omega X_0 + i\eta I_0} h(t_0, v_0; t_j, \omega, \eta)
\]

The expectation calculation

\[
\mathbb{E}_0[e^{-r_{t_j+1}} \max(S_{t_{j+1}} - K, 0) \mathbf{1}_{\{I_{t_j} < B\}}]
\]

requires the joint conditional characteristic function of \(I_t\) at \(t_j\) and \(S_t\) at \(t_{j+1}\).
Iterated expectation

Working backward in time from $t_{j+1}$ to $t_j$, we compute $E_{t_j}[e^{i\omega X_{t_j+1}}]$; and from $t_j$ to $t_0$, we compute $E_0[e^{i\omega X_{t_j}+i\eta I_{t_j}}]$. This is done by setting $\eta = 0$ in $h(t_j, v'; t_{j+1}, \omega, \eta)$ and integrating $g(t_0, v_0; t_j, \omega, \eta, v') h(t_j, v'; t_{j+1}, \omega, 0)$ over $v'$ from 0 to $\infty$.

By the two-step expectation calculation, we obtain

$$E_0[e^{i\omega X_{t_j+1}+i\eta I_{t_j}}] = e^{i\omega X_0+i\eta I_0} \int_0^{\infty} g(t_0, v_0; t_j, \omega, \eta, v') h(t_j, v'; t_{j+1}, \omega, 0) \, dv'.$$

Here, $v'$ is the dummy variable for the instantaneous variance $v_{t_j}$.
Analytic expressions for $g$ and $h$ under the $3/2$-model

For the $3/2$-model, we manage to obtain

$$g(t, v; t', v') = e^{a(t'-t)} \frac{A_t}{C_t} \exp \left( -\frac{A_t v + v'}{C_t vv'} \right) (v')^{-2} \left( \frac{A_t v}{v'} \right)^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c} \left( \frac{2}{C_t} \sqrt{\frac{A_t}{vv'}} \right),$$

where $I_{2c}$ is the modified Bessel function of order $2c$,

$$a = i\omega (r - q), \quad \tilde{\kappa} = \kappa - i\omega \rho \varepsilon, \quad A_t = e^{\int_t^{t'} \theta_s ds},$$

$$C_t = \frac{\varepsilon^2}{2} \int_t^{t'} e^{\int_s^{t'} \theta_{s'} ds'} ds, \quad c = \sqrt{\left( \frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2} \right)^2 + \frac{i\omega + \omega^2 - 2i\eta}{\varepsilon^2}}.$$

$$h(t, v; t', \omega, \eta) = \int_0^\infty g(t, v_t; t', \omega, \eta, v') dv'$$

$$= e^{a(t'-t)} \frac{\Gamma(\bar{\beta} - \bar{\alpha})}{\Gamma(\bar{\beta})} \left( \frac{1}{C_t v} \right)^{\bar{\alpha}} M \left( \bar{\alpha}, \bar{\beta}, -\frac{1}{C_t v} \right),$$

where $\bar{\alpha} = -\frac{1}{2} - \frac{\tilde{\kappa}}{\varepsilon^2} + c$, $\bar{\beta} = 1 + 2c$, $\Gamma$ is the gamma function, $M$ is the confluent hypergeometric function of the first kind.
Proof of the formulas for $g$ and $h$

Write $\tilde{g} = ge^{a(t-t')}$. and define

$$\tilde{g}(t, v; t', \omega, \eta, v') = \frac{u^\alpha}{(u')^{\alpha-2}} f(t, u; t', \omega, \eta, u').$$

then the governing equation for $f$ becomes

$$-\frac{\partial f}{\partial t} = \frac{\varepsilon^2 u \partial^2 f}{2 \partial u^2} + \left[\varepsilon^2(\alpha + 1) + \kappa - \theta tu\right] \frac{\partial f}{\partial u} - \alpha \theta f$$

$$+ \left[\frac{\varepsilon^2}{2} (\alpha^2 + \alpha) + \kappa \alpha - \frac{i\omega + \omega^2}{2} + i\eta \right] \frac{f}{u},$$

subject to

$$f(t', u; t', \omega, \eta, u') = \delta(u - u').$$
Apparently, if we choose $\alpha = \alpha(\omega, \eta)$ such that

$$\frac{\epsilon^2}{2}(\alpha^2 + \alpha) + \tilde{\kappa}\alpha - \frac{i\omega + \omega^2}{2} + i\eta = 0,$$

then the governing equation of $f$ becomes

$$-\frac{\partial f}{\partial t} = \frac{\epsilon^2 u \partial^2 f}{2 \partial u^2} + [\epsilon^2(\alpha + 1) + \tilde{\kappa} - \theta_t u] \frac{\partial f}{\partial u} - \alpha \theta_t f,$$

where all the coefficients are affine in $u$. It follows that $\alpha$ can take two values

$$\alpha = -\left(\frac{1}{2} + \frac{\tilde{\kappa}}{\epsilon^2}\right) \pm c,$$

where

$$c \equiv c(\omega, \eta) = \sqrt{\left(\frac{1}{2} + \frac{\tilde{\kappa}}{\epsilon^2}\right)^2 + \frac{i\omega + \omega^2 - 2i\eta}{\epsilon^2}}.$$
We write the Laplace transform of $f$ with respect to $u'$ as follows

$$\hat{f}(t,u;t',\omega,\eta,\xi) = \int_0^\infty e^{-\xi u'} f(t,u;t',\omega,\eta,u') \, du'.$$

Then, $\hat{f}$ admits the following exponential affine solution:

$$\hat{f}(t,u;t',\omega,\eta,\xi) = \exp(B(t,\xi)u + D(t,\xi)),$$

where $B(t,\xi)$ and $D(t,\xi)$ are parameter functions determined by the following Riccati system of ODEs:

$$-\frac{\partial B}{\partial t} = \frac{\varepsilon^2}{2} B^2 - \theta_t B,$$

$$-\frac{\partial D}{\partial t} = [\varepsilon^2(\alpha + 1) + \tilde{\kappa}]B - \alpha \theta_t,$$

with boundary conditions $B(t',\xi) = -\xi$ and $D(t',\xi) = 0$. It can be found that

$$B(t,\xi) = -\frac{\xi}{A_t + C_t \xi},$$

where

$$A_t = e^{\int_t^{t'} \theta_s \, ds}, \quad C_t = \frac{\varepsilon^2}{2} \int_t^{t'} e^{\int_s^{t'} \theta_\tau \, d\tau} \, ds.$$
We obtain

\[ D(t, \xi) = -2 \left[ \alpha + 1 + \frac{\tilde{\kappa}}{\varepsilon^2} \right] \ln (A_t + C_t \xi) + \left[ \alpha + 2 + \frac{2\tilde{\kappa}}{\varepsilon^2} \right] \int_t^{t'} \theta_s \, ds. \]

Next, we take the inverse Laplace transform of

\[ \hat{f}(t, u; t', \omega, \eta, \xi) = A_t^{\alpha+2} e^{\frac{2\tilde{\kappa}}{\varepsilon^2} t} \exp \left( -\frac{\xi u}{A_t + C_t \xi} \right) (A_t + C_t \xi)^{-2\alpha-2-\frac{2\tilde{\kappa}}{\varepsilon^2}} \]

to obtain

\[ f(t, u; t', \omega, \eta, u') = \frac{A_t^{3/2+c+\frac{\kappa}{\varepsilon^2}}}{2\pi i C_t} \int_{\tau-i\infty}^{\tau+i\infty} e^{rac{u'(p-A_t)}{C_t}} p^{-2c-1} e^{-\frac{u(p-A_t)}{C_tp}} \, dp \]

\[ = \frac{A_t^{3/2+c+\frac{\kappa}{\varepsilon^2}}}{2\pi i C_t} \int_{\tau-i\infty}^{\tau+i\infty} e^{-\frac{u+A_t u'}{C_t}} \frac{u'_p}{e^{C_t p}} p^{-2c-1} e^{-\frac{uA_t}{C_tp}} \, dp \]

\[ = \frac{A_t^{3/2-c+\frac{\kappa}{\varepsilon^2}}}{C_t} e^{-\frac{u+A_t u'}{C_t}} (\frac{A_t u'}{u})^c I_{2c} \left( \frac{2}{C_t} \sqrt{A_t uu'} \right). \]

Expressed in terms of \( v \) and \( v' \), \( g \) is found to be

\[ g(t, v; t', \omega, \eta, v') = e^{a(t'-t)} \frac{A_t}{C_t} \exp \left( -\frac{A_tv + v'}{C_tv v'} \right) \frac{1}{(v')^2} \left( \frac{A_tv}{v'} \right)^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c} \left( \frac{2}{C_t} \sqrt{A_t} \right). \]
\[ h(t, v; t', \omega, \eta) = \int_0^\infty g(t, v; t', \omega, \eta, v') \, dv' \]
\[ = e^{\alpha(t'-t)} \frac{A_t}{C_t} \int_0^\infty e^{-\frac{u + A_t u'}{C_t}} \left( \frac{A_t u'}{u} \right)^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}} I_{2c} \left( \frac{2}{C_t} \sqrt{A_t u u'} \right) \, du' \]
\[ = \frac{e^{\alpha(t'-t)} - \frac{u}{C_t}}{C_t u^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}}} \int_0^\infty e^{-\frac{z}{C_t} z^{\frac{1}{2} + \frac{\tilde{\kappa}}{\varepsilon^2}}} I_{2c} \left( \frac{2 \sqrt{u z}}{C_t} \right) \, dz, \]
where \( z = A_t u' \). Using the equation
\[ \int_0^\infty e^{-st} t^\psi I_\varsigma(\lambda \sqrt{t}) \, dt = \frac{\Gamma(\phi) X^{\varsigma/2}}{\Gamma(\psi) s^{1+\iota} M(\phi, \psi, X)}, \]
where \( \phi = 1 + \iota + \varsigma/2, \psi = 1 + \varsigma, X = \frac{\lambda^2}{4s} \) and \( \Re(\phi, s) > 0 \), we obtain
\[
h(t,v; t', \omega, \eta) = e^{a(t'-t) - \frac{u}{C_t}} \frac{\Gamma(1 - \alpha)}{\Gamma(2c + 1)} \left( \frac{u}{C_t} \right)^{\tilde{\alpha}} M \left( 1 - \alpha, 2c + 1, \frac{u}{C_t} \right)
\]
\[
= e^{a(t'-t)} \frac{\Gamma(\bar{\beta} - \tilde{\alpha})}{\Gamma(\bar{\beta})} \left( \frac{u}{C_t} \right)^{\tilde{\alpha}} M \left( \tilde{\alpha}, \bar{\beta}, -\frac{u}{C_t} \right)
\]
\[
= e^{a(t'-t)} \frac{\Gamma(\bar{\beta} - \tilde{\alpha})}{\Gamma(\bar{\beta})} \left( \frac{1}{C_tv} \right)^{\tilde{\alpha}} M \left( \tilde{\alpha}, \bar{\beta}, -\frac{1}{C_tv} \right).
\]

Here,
\[
\tilde{\alpha} = -\frac{1}{2} - \frac{\tilde{\kappa}}{\varepsilon^2} + c, \quad \text{and} \quad \bar{\beta} = 1 + 2c.
\]

Note that the second equality follows from the identity:
\[
M(a, b, z) = e^z M(b - a, b, -z).
\]
Parameter values in the 3/2 model and finite-maturity discrete timer options

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Comparison of the numerical results for finite-maturity discrete timer call options for varying strike prices $K$ and correlation values $\rho$ obtained from the analytic formula with the benchmark results obtained using the Monte Carlo method (MC) under the 3/2 stochastic volatility model.
Plot of the finite-maturity discrete timer call option prices against variance budget $B$. The discrete timer call option price tends to that of the vanilla European call option (shown in the dashed line) when $B$ is sufficiently large.
Plot of the finite-maturity discrete timer call option price versus maturity (mandated) under two different values of the variance budget. The price sensitivity to maturity can be quite significant for short-lived timer options.
Fourier-space time-stepping algorithm

We define \( \gamma_t = \ln v_t, \ x_t = \ln \frac{S_t}{K} \), where \( K \) is the strike price, \( \Delta \) be the uniform time interval between successive monitoring instants. Let \( V_{t_k}(x_{t_k}, \gamma_{t_k}, I_{t_k}) \) be the option value of the finite-maturity discrete timer call option at monitoring time \( t_k, \ k = 0, 1, \cdots, N \), where \( x_{t_k}, \gamma_{t_k} \) and \( I_{t_k} \) denote the time-\( t_k \) normalized log-asset return, log-variance and realized variance, respectively.

Note that \( V_{t_k} \) is the sum of the continuation value and exercise payoff:

\[
V_{t_k}(x_k, \gamma_k, I_k) = e^{-r\Delta} U_{t_k}(x_k, \gamma_k, I_k)1\{I_k < B\} + K(e^{x_k} - 1)1\{I_k \geq B\},
\]

\( k = 1, 2, \cdots, N - 1 \).

The continuation value conditional on \( \{I_k < B\} \) is given by

\[
U_{t_k}(x_k, \gamma_k, I_k) = E_{t_k}[V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})],
\]
Time-stepping calculations between successive monitoring dates

- Numerical quadrature on $\gamma_{k+1}$.
- Fourier transform algorithm on $x_{k+1}$ and $I_{k+1}$.

Numerical quadrature rule which involves integration over the density function $p_\gamma(\gamma_{t_{k+1}}|\gamma_{t_k})$: discretization along the dimension of log-variance $\gamma_{t_{k+1}}$ at the discrete nodes $\zeta_j$, $j = 1, 2, \cdots, J$

$$U_{t_k}(x_k, \gamma_k, I_k) \approx \sum_{j=1}^{J} w_j p_\gamma(\zeta_j|\gamma_k) E\left[ V_{t_{k+1}}(x_{k+1}, \gamma_{k+1}, I_{k+1})|F_{t_k}, \gamma_{k+1} = \zeta_j \right],$$

where $w_j$ is the weight at the quadrature node $\zeta_j$, $j = 1, 2, \cdots, J$.

With availability of the joint conditional characteristic function of $x_{k+1}$ and $I_{k+1}$, we apply the Fourier transform method to perform the inner expectation calculations.
Fourier transform algorithm with damping

Let $w = \alpha_1 + i\beta_1$ and $u = \alpha_2 + i\beta_2$, where $\alpha_1$ and $\alpha_2$ are constants. At $\gamma_{tk+1} = \zeta_j$, $x_{tk+1} = x$ and $I_{tk+1} = y$, we define

$$V_{tk+1}^{\alpha_1,\alpha_2}(x, \zeta_j, y) = e^{\alpha_1 x + \alpha_2 y}V_{tk+1}(x, \zeta_j, y).$$

To guarantee that the Fourier transforms are well defined, we need to introduce a proper exponential damping factor. The parameters $\alpha_1, \alpha_2$ are chosen to insure the existence of the generalized two-dimensional Fourier transform of $V_{tk+1}(x, \zeta_j, y)$ as defined by

$$\hat{V}_{tk+1}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\beta_1 x + i\beta_2 y}V_{tk+1}^{\alpha_1,\alpha_2}(x, \zeta_j, y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{wx + uy}V_{tk+1}(x, \zeta_j, y) \, dx \, dy,$$

where $w = \alpha_1 + i\beta_1$ and $u = \alpha_2 + i\beta_2$. 
We may express the inner expectation integral at $\gamma_{t_k+1} = \zeta_j$ by the following two-dimensional inverse Fourier transform

$$
E[V_{t_k+1}(x_{k+1}, \gamma_{k+1}, I_{k+1}) | \mathcal{F}_{t_k}, \gamma_{k+1} = \zeta_j] = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-wx_k-uI_k} \hat{V}_{t_k+1}^{\alpha_1, \alpha_2}(\zeta_j; \beta_1, \beta_2) \Psi(-w, -u; \zeta_j, \gamma_k) \, d\beta_1 \, d\beta_2,
$$

where

$$
\Psi(w, u; \gamma_t, \gamma_s) = E[e^{w(x_t-x_s)+u(I_t-I_s)} | \mathcal{F}_s, \gamma_t], \quad t > s.
$$

We manage to express the two-dimensional moment generating function $\Psi$ in terms of the one-dimensional characteristic function $\Phi$ under the Heston model and the 3/2 stochastic volatility model.
Suppose $S_t$ follows the following stochastic volatility model under $Q$:

$$\frac{dS_t}{S_t} = (r - q)dt + \sqrt{v_t}(\rho \ dW_t^v + \sqrt{1 - \rho^2} \ dW_t)$$

$$d v_t = \alpha(v_t) dt + \beta(v_t) dW_t^v.$$

The asset price process admits the following representation

$$S_t = S_0 e^{(r-q)t + a_t + \sqrt{b_t} \epsilon},$$

where $\epsilon$ is a standard normal variable, $a_t$ and $b_t$ are defined by

$$a_t = \rho[f(v_t) - f(v_0)] - \rho H_t - \frac{I_t}{2} \quad \text{and} \quad b_t = (1 - \rho^2) I_t,$$

with

$$H_t = \int_0^t h(v_s) \ ds \quad \text{and} \quad f(v_t) = \int \frac{\sqrt{v_t}}{\beta(v_t)} \ d\nu_t.$$

Here, $h$ is defined by

$$h(v_t) = \alpha(v_t) f'(v_t) + \frac{1}{2} \beta^2 (v_t) f''(v_t).$$
**Heston model**

For the Heston stochastic volatility model, the dynamics for its variance is defined by

$$dv_t = \lambda (\bar{v} - v_t) \, dt + \eta \sqrt{v_t} \, dW_t.$$ 

In the Heston model, \( \alpha(v_t) = \lambda (\bar{v} - v_t) \) and \( \beta(v_t) = \eta \sqrt{v_t} \). It follows that

$$f(v_t) = \frac{v_t}{\eta} \quad \text{and} \quad h(v_t) = \frac{\lambda (\bar{v} - v_t)}{\eta}.$$ 

We can rewrite the normalized log-asset return process as follows

$$x_t = \ln \frac{S_0}{K} + (r - q)t + \frac{\rho}{\eta} (e^{\gamma t} - e^{\gamma_0} - \lambda \bar{v} t)$$

$$+ \left( \frac{\rho \lambda}{\eta} - \frac{1}{2} \right) I_t + \sqrt{(1 - \rho^2)} I_t \epsilon.$$
By the tower rule, we obtain

$$\psi(w, u; \gamma_t, \gamma_s)$$

$$= E\left[E\left[e^{w(x_t-x_s)}|F_s, \gamma_t, I^t - I_s\right]e^{u(I^t - I_s)|F_s, \gamma_t}\right]$$

$$= e^{w\left\{(r-q)(t-s) + \frac{\theta}{\eta}[e^{\gamma_t} - e^{\gamma_s} - \bar{\lambda}\bar{\nu}(t-s)]\right\}}$$

$$\Phi\left(-iw\left(\frac{\rho\lambda}{\eta} - \frac{1}{2}\right) - \frac{1}{2}iw^2(1 - \rho^2) - iu; e^{\gamma_t}, e^{\gamma_s}\right),$$

where

$$\Phi(\xi; \gamma_t, \gamma_s) = E[e^{i\xi\int_s^t v_u \, du}|\gamma_t, \gamma_s]$$

is the conditional characteristic function of the time-integrated log-variance process $\int_s^t v_u \, du$. 
3/2 stochastic volatility model

The variance process evolves according to the following dynamics

$$d v_t = \lambda v_t (\bar{v} - v_t) \, dt + \eta v_t^{3/2} \, dW^v_t.$$  

The use of Itô’s formula gives the corresponding dynamics for $1/v_t$

$$d \left( \frac{1}{v_t} \right) = \lambda \bar{v} \left( \frac{\lambda + \eta^2}{\lambda \bar{v}} - \frac{1}{v_t} \right) \, dt - \frac{\eta}{\sqrt{v_t}} \, dW^v_t.$$  

The reciprocal of the variance process of the 3/2 model follows a mean-reverting square-root process with parameters $(\lambda \bar{v}, \frac{\lambda + \eta^2}{\lambda \bar{v}}, -\eta)$. 
In this case, $\alpha(v_t) = \lambda v_t (\bar{v} - v_t)$ and $\beta(v_t) = \eta v_t^{3/2}$. It follows that

$$f(v_t) = \frac{\ln v_t}{\eta} \quad \text{and} \quad h(v_t) = \frac{\lambda}{\eta} \left[ \bar{v} - \left(1 + \frac{\eta^2}{2\lambda}\right) v_t \right].$$

The normalized log-asset return process can be expressed as in the following form

$$x_t = \ln \frac{S_0}{K} + (r - q)t + \frac{\rho}{\eta} \left[ \gamma_t - \gamma_0 - \lambda \bar{v}t \right] + \left[ \frac{\rho \lambda}{\eta} \left(1 + \frac{\eta^2}{2\lambda}\right) - \frac{1}{2} \right] I_t + \sqrt{(1 - \rho^2)} I_t W_t,$$

Similarly, we have

$$\psi(w, u; \gamma_t, \gamma_s) = e^{w \left\{ (r-q)(t-s) + \frac{\rho}{\eta} [\gamma_t - \gamma_s - \lambda \bar{v}(t-s)] \right\}}$$

$$\Phi \left( -iw \left[ \frac{\rho \lambda}{\eta} \left(1 + \frac{\eta^2}{2\lambda}\right) - \frac{1}{2} \right] - \frac{1}{2} i\omega^2 (1 - \rho^2) - iu; e^{\gamma_t}, e^{\gamma_s} \right).$$
Hilbert transform

Across a monitoring date, we use the fast Hilbert transform method to deal with the barrier feature associated with the accumulated realized variance.

For any $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, we define the Fourier transform $\hat{f}$ by

$$\hat{f} = \mathcal{F}f = \int_{-\infty}^{\infty} e^{i\beta x} f(x) \, dx,$$

and $\hat{f} = \mathcal{F}f \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$.

For any $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, its Hilbert transform is defined by the Cauchy principal value integral

$$\mathcal{H}f(x) = \frac{1}{\pi} PV \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy,$$

and $H\hat{f} \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} = 1$. 
Two key properties of the Hilbert transform

The barrier feature on the value function can be modeled by appending the indicator function \(1_{(-\infty,b)}\) as multiplier.

For any \(b \in \mathbb{R}\), the Fourier transform of a function multiplied by an indicator function \(1_{(-\infty,b)}\) is related to the Hilbert transform of the Fourier transform function by

\[
\mathcal{F}(1_{(-\infty,b)} \cdot f)(\beta) = \frac{1}{2} \hat{f}(\xi) - \frac{i}{2} \frac{e^{i\beta b}}{2} \mathcal{H}(e^{-i\eta b} \hat{f}(\eta))(\beta).
\]

The Hilbert transform can be evaluated based on the Sinc expansion of an analytic function as follows

\[
\mathcal{H}f(x) = \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} \frac{f(y)}{x - y} \, dy = \sum_{l=-\infty}^{\infty} f(lh) \frac{1 - \cos \frac{\pi(x-lh)}{h}}{\pi(x-lh) h}, \quad h > 0,
\]

where \(h\) is the fixed discretization step.
Fast Hilbert transform algorithm

The backward induction procedure in the Fourier domain using the fast Hilbert transform algorithm for pricing finite-maturity discrete timer options can be formulated as follows:

(i) We initiate our time stepping calculations at maturity $t_N$. The generalized Fourier transform of the terminal payoff admits the analytic formula

$$\hat{V}_{t_N}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) = -\frac{K}{(\alpha_1 + i\beta_1)(\alpha_1 + i\beta_1 + 1)(\alpha_2 + i\beta_2)},$$

for $j = 1, 2, \ldots, N$.

Here, the constraints $\alpha_1 < -1$ and $\alpha_2 < 0$ should be observed in order to guarantee the existence of the above generalized Fourier transform.
(ii) For $k = N - 1, N - 2, \cdots, 1$, the numerical approximation of $\hat{V}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2)$ is recursively calculated by computing a sequence of Hilbert transforms

$$
\hat{V}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2) = e^{-r\Delta} \left[ \frac{1}{2} \hat{U}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta_2) - \frac{i}{2} e^{i\beta_2 B} \mathcal{H} \left( e^{-i\beta_2' B} \hat{U}_{t_k}^{\alpha_1, \alpha_2}(\zeta_p; \beta_1, \beta'_2) \right) (\beta_2) \right]
$$

$$
- Ke^{(\alpha_2 + i\beta_2)B} \left( \frac{1}{(\alpha_1 + i\beta_1)(\alpha_1 + i\beta_1 + 1)(\alpha_2 + i\beta_2)} \right), \quad p = 1, 2, \cdots, N.
$$
(iii) We approximate $\hat{U}_{tk}^{\alpha_1,\alpha_2}(\zeta_p; \beta_1, \beta_2)$ using the quadrature rule

$$
\hat{U}_{tk}^{\alpha_1,\alpha_2}(\zeta_p; \beta_1, \beta_2) \approx \sum_{j=1}^{J} w_j \hat{V}_{tk+1}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) \Psi(-w, -u; \zeta_j, \zeta_p).
$$

(iv) For the last step where $k = 0$, the timer call option value is obtained by

$$
V_{t_0}(x_0, \zeta_p, I_0)
\approx \sum_{j=1}^{J} \frac{e^{-r\Delta w_j}}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\alpha_1+i\beta_1)x_0} e^{-(\alpha_2+i\beta_2)I_0} \hat{V}_{t_1}^{\alpha_1,\alpha_2}(\zeta_j; \beta_1, \beta_2) \Psi(-w, -u; \zeta_j, \zeta_p) \, d\beta_1 d\beta_2,
$$

for $p = 1, 2, \cdots, N$. 
The inverse Fourier transform can be evaluated by the following double summations:

\[ g_{h_1,M,h_2,L}(x_1,x_2) = \frac{1}{4\pi^2} \sum_{m=-M}^{M} \sum_{l=-L}^{L} e^{-imh_1x_1}e^{-ilh_2x_2}\hat{g}(mh_1, lh_2)h_1h_2. \]

The trapezoidal sum approximation has been shown to be highly accurate.

**Computational complexity**

The overall computational complexity of the fast Hilbert transform algorithm for pricing finite-maturity discrete timer options is \( O(NMJ^2L\log L) \), where \( N \) is the number of monitoring instants, \( M, J \) and \( L \) are the truncation level parameters in the log-asset dimension, log-variance dimension and realized variance dimension, respectively.
Parameter values in the Heston model and finite-maturity discrete timer options

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<th>$r$</th>
<th>$q$</th>
<th>$B$</th>
<th>$N$</th>
<th>$\lambda$</th>
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Comparison of the numerical results for the finite-maturity discrete timer call options for varying strike prices $K$ and correlation values $\rho$ obtained from the fast Hilbert transform algorithm with the benchmark results obtained using the Monte Carlo method (MC) under the Heston model.
Sensitivity analysis on volatility of variance $\eta$ and correlation coefficient $\rho$ under the Heston model

- The price function may not be a monotonically increasing function of $\eta$.

- When $\rho = -0.5$, the discrete timer call option price firstly increases and then decreases with increasing value of $\eta$.

- When $\rho = 0.5$, the discrete timer call option price is a decreasing function of $\eta$. 
### Sensitivity analysis under the Heston model

<table>
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<th>ρ</th>
<th>η</th>
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<th>( K = 94 )</th>
<th>( K = 98 )</th>
<th>( K = 102 )</th>
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Comparison of the numerical values for finite-maturity discrete timer call option prices with varying values of strike prices, volatility of variance and correlation coefficient under the Heston model.
Conclusion

- By decomposing a finite-maturity timer option into a portfolio of timerlets, we manage to price the timer option based on the explicit representation of the joint characteristic function of log asset price and its integrated variance.

- Our numerical tests on pricing the finite-maturity discrete timer options under the Heston model and 3/2 model demonstrate high level of numerical accuracy and robustness of the analytic formula and the fast Hilbert algorithm for pricing options with exotic barrier feature.