Default Contagion with Domino Effect

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The Model
Many recent researches have put attention on the **systematic risk and contagion** since the Asian banking crisis of the late 90s, and the more recent banking crisis of 2007-2008, e.g., find out the role of integration and diversification to systematic risk ... 

Most of them used directed graphs (network) to model interdependence.

The present paper introduces a structural framework—**first passage time approach**—to model dependent defaults, with a particular interest in their contagion in a very larger market.

We give an explicit form of **contagion probabilities** and discuss some asymptotic behaviour.
Let $X^i_t$ denote the firm value process of the $i$-th company, for $i = 1, 2, \cdots, n$ with $n \geq 2$. Define “default time” by

$$
\tau^i := \inf\{s \geq 0 : X^i_s < k^i\},
$$

where $k^i \in \mathbb{R}$ is a exogenously given default level for the $i$-th company.
We assume that $X \equiv (X^1, \cdots, X^n)$ solves the following equation;

$$X^i_t = x^i - \sum_{j \neq i} C_{i,j} 1_{\{T < \tau^i\}} + \int_0^{t \wedge \tau^i} (\sigma^i(X_s) dW^i_s + \mu^i(X_s) ds) \quad (1)$$

for $i = 1, 2, \cdots, n$, where $W^i, i = 1, \cdots, n$ are independent Brownian motions, and for $i, j = 1, \cdots, n$, $C_{i,j}$ are non-negative constants, $\sigma^i, \mu^i$, each defined on $\mathbb{R}^n$, are smooth function with at most linear growth.
In a more concise way of saying,

- each component is a diffusion process on each interval from a default time to next one,
- the default of \( i \)-th company brings about a jump \( C_{ij} \) to \( j \)-th company,
- which may causes the default of \( j \)-th company.
- The \( i \)-th default may also affects the dynamics of the \( j \)-th firm value process in terms of its growth rate or the volatility.
Define the first contagion time by

$$\tau(1) := \min\{\tau_i : i = 1, \cdots, n\}.$$  

the $j$-th contagion time is defined recursively by

$$\tau(j) := \inf\{\tau_k : \tau_k > \tau(j-1)\}, \quad j = 2, 3, \cdots, n,$$

with the convention that $\inf\emptyset = \infty$. 


To price credit derivatives such as CDO or CDS, we need to know the joint distribution of

\[(\tau(1), \cdots, \tau(n), d(\tau(1)), \cdots, d(\tau(n))),\]

where

\[d(\tau(k)) = \{i \in \{1, \cdots, n\} : \tau_i = \tau(k)\}, \ k = 1, \cdots, n,\]

with the convention that \(\tau(k) = \infty\) if \(d(\tau(k)) = \emptyset\).
Fundamental Lemma
The first key idea is that we regard \((\tau(i), X_{\tau(i)})\) as (something like) a renewal process.

- We shall have a formula of the joint density of
  \[
  (d(\tau(1)), \tau(1), X_{\tau(1)})
  \]
  conditioned by the starting point \(X_0\).
- Here we understand \(X_{\tau(1)+t}, t \geq 0\) to be an \(R^{d(\tau(1))_c}\)-valued process; we are only interested in the survived companies.
- Then, by replacing \(\{1, \cdots, n\}\) with \(d(\tau(1))_c\) and \(X_0\) with \(X_{\tau(1)}\), we obtain the joint distribution of
  \[
  (d(\tau(2)), \tau(2), X_{\tau(2)})
  \]
  conditioned by \(X_{\tau(1)}\), thanks to Markov property of \(X\).
- We can repeat this procedure to get the desired joint distribution.
We obtain the joint distribution of \((d(\tau(1)), \tau(1), X_{\tau(1)})\) in the following different levels:

- In most general case, it is given by a integration by the harmonic measure of \(X_{\tau(1)-}\) (before the “artificial” jumps) of “contagion domain”.
- To get a computable form, we rely on the independence and a representation by disjoint rectangles.
- By imposing further ”symmetries”, the probabilities can be described in terms of (symmetric) polynomials, which might be understood as stochastic integrable system.
- In such cases, (I believe) phase transition can be explicitly characterized.
To take into account that we work on a “renewal” setting described as above, from now on we let the index set of $X$ be arbitrary finite subset. In order to specify, if needed, dependence on the initial index set, we put superscript $I$ to the previously defined notations; $\tau^I(1)$, $d^I$, and so on. We then concentrate on the study of the joint distribution of

$$(d^I(\tau^I(1)), \tau^I(1), X^I_{\tau^I(1)}).$$ (2)$$
The event \( \{d^l(\tau^l(1)) = J\} \) is rephrased as the event that \( X_{\tau^l(1)} \) hit a set. In fact let \( I := \{i_1, \cdots, i_{\#I}\} \) and for a permutation \( \sigma \) over \( I \), or equivalently, \( \sigma \in S_I \), we put

\[
D_{I,\sigma} := \left\{ (x_{i_1}, \cdots, x_{i_{\#I}}) \in \mathbb{R}^I : x_{i_{\sigma(1)}} = K_{i_{\sigma(1)}}, x_{i_{\sigma(2)}} \in [K_{i_{\sigma(2)}}, K_{i_{\sigma(2)}} + C_{i_{\sigma(1)}, i_{\sigma(2)}}], \right. \\
\left. \cdots, x_{i_{\sigma(\#I)}} \in [K_{i_{\sigma(\#I)}}, K_{i_{\sigma(\#I)}} + \sum_{j=1}^{\#I-1} C_{i_{\sigma(j)}, i_{\sigma(\#I)}}] \right\}.
\]
Then, we have the following

**Lemma**

*For \( \emptyset \neq J \subset I \), we have that*

\[
\{ d^I(\tau^I(1)) = J \} = \left\{ \chi^I_{\tau^I(1)-} \in \bigcup_{\sigma \in \mathcal{G}_J} D_{J,\sigma} \times \prod_{i \in I \setminus J} (K^i + \sum_{j \in J} C_{j,i}, \infty) \right\}.
\]

So by measuring the domain by the joint distribution of \((X_{\tau(1)-}, \tau(1)-)\) we get what we want.
We put

\[ D'_I := \bigcup_{\sigma \in \mathcal{G}_I} D_{I,\sigma}, \]

and for non-empty \( J \subsetneq I \), we put

\[ D'_J := D'_J \times A'_{I \setminus J}, \quad (3) \]

where, for nonempty \( J_1 \) and possibly empty \( J_2 \),

\[ A'_{J_2} := \prod_{i \in J_1} (K^i + \sum_{j \in J_2} C_{j,i}, \infty). \quad (4) \]
Let $\emptyset \neq J \subsetneq I$, and $Q^I(x, \cdot, S)$ be the harmonic measure of the process $X^I$ on the boundary

$$\partial \prod_{i \in I} (K^i, \infty) = \bigcup_{i \in I} \left( \{K^i\} \times \prod_{j \neq i} (K^j, \infty) \right) =: D^I.$$ 

More precisely, we set

$$Q^I(x, A, S) = P(X^I_{\tau^I(1)-} \in A, \tau^I(1) \in S | X^I_0 = x)$$

for $x \in \prod_{i \in I} (K^i, \infty)$, $A \in \mathcal{B}(D^I)$ and $S \in \mathcal{B}[0, \infty]$. 
Lemma

For $A \in \mathcal{B}(D^l)$ and $S \in \mathcal{B}[0, \infty]$, we have that

$$P(d^l(\tau^l(1)) = I, X^l_{\tau^l(1)} \in A, \tau^l(1) \in S | X^l_0 = x) = Q^l(x, D^l_j \times s^l_j(A), S).$$

(5)

Here $s^l_j$ is a shift on $\mathbb{R}^l$ defined by

$$s^l_j((x^i, i \in J_1)) = ((x^i + \sum_{j \in J_2} C_{j,i})).$$
Specification 1. Independence
Specication 1. Independence during BAU

We assume that

- \( \sigma^i_j(x) \) in (1), for \( x = (x^1, \cdots, x^n) \in \mathbb{R}^n \), is of the form \( \delta_{i,j} \sigma^i(x^i) \) for some smooth function \( \sigma^i : \mathbb{R} \to \mathbb{R} \) where \( \delta \) here is the Kronecker’s delta, and

- \( \mu^i(x) \) in (1), for \( x = (x^1, \cdots, x^n) \in \mathbb{R}^n \), is of the form \( \mu^i(x^i) \) for some smooth function \( \mu^i : \mathbb{R} \to \mathbb{R} \).

Let \( \tilde{X} \) be a Business As Usual process given as

\[
\tilde{X}_t^i = x^i + \int_0^t (\sigma^i(\tilde{X}_s^i) dW^i_s + \mu^i(\tilde{X}_s^i) ds),
\]

and \( \tilde{\tau}_i \) be its default time:

\[
\tilde{\tau}_i := \inf\{s > 0 : \tilde{X}_s^i \leq K_i \}.
\]
We assume that

\[ P\{\tilde{\tau}_i < \infty \}|\tilde{X}_0^i = x^i\} = 1 \]

for any \( x^i \in (K^i, \infty) \) and the distribution of \((\tilde{\tau}_i, \tilde{X}^i)\) has the following densities:

\[ p_i(x^i, s) = \frac{P(\tilde{\tau}_i \in ds|\tilde{X}_i^i = x^i)}{ds} , \]

and

\[ q_i(x^i, y^i, s) = \frac{P(\tilde{\tau}_i > s, \tilde{X}_s^i \in dy_i|\tilde{X}_i^i = x^i)}{dy_i} , \]

for \( x^i, y^i \in [K^i, \infty) \) and \( s > 0 \).
The “harmonic measure”, the distribution of \( X_{\tau(1)-} \) can be obtained by the following

**Lemma**

*For* \( A \in \mathcal{B}(D^l) \) *and* \( S \in \mathcal{B}(0, \infty) \),

\[
Q^l(x, A, S) = \int_S \sum_{i \in l} p_i(x^i, s) ds \int_A \delta_{K^i}(dy_i) \prod_{j \in l \setminus \{i\}} q_j(x^j, y_j, s) dy_j, \tag{6}
\]

*where* \( \delta_* \) *is the Dirac delta at* \(*\).
Rectangle Representation of Contagion Domain

Given the product form (6) of the harmonic measure, we want to express the contagion domain by a disjoint union of rectangles. Since we have (3) and (4), the following lemma does this job.

Lemma

Let $J_1, \ldots, J_k$ be such that $\emptyset \neq J_k \subsetneq \cdots \subsetneq J_1$, and set

$$H_{J_1, \ldots, J_k} := D^J_{J_k} \times A^J_{J_k} \times \cdots \times A^J_{J_2}.$$ 

Then it holds that, for any measure $\mu$ on $\mathcal{B}(D^J)$,

$$\mu(D^J) = \mu(H_I) + \sum_{\emptyset \neq J_1 \subsetneq I} (-1) \mu(H_I, J_1) + \sum_{J_1 \subsetneq I} \sum_{\emptyset \neq J_2 \subsetneq J_1} (-1)^2 \mu(H_I, J_1, J_2)$$

$$+ \cdots + \sum_{J_1 \subsetneq I} \cdots \sum_{\emptyset \neq J_{#I-1} \subsetneq J_{#I-2}} (-1)^{#I-1} \mu(H_I, J_1, \ldots, J_{#I-1}).$$

(7)
Let $J_1$ and $J_2$ be finite disjoint subsets of $\mathbb{N}$. For such $J_1$ and $J_2$, we denote

$$g_{J_2}^{J_1}(x^{J_1}, A, s) := \int_{\prod_{i \in J_1}[k^i, \infty) \cap A} \prod_{i \in J_1} q_i(x^i, y_i + \sum_{j \in J_2} C_{j, i}, s) \, dy_i,$$

$$g_{J_2}^{J_1}(x^{J_1}, s) := g_{J_2}^{J_1}(x^{J_1}, R^{J_1}, s)$$

$$= \int_{A_{J_2}} \prod_{i \in J_1} q_i(x^i, y_i, s) \, dy_i$$

and

$$g^{J_1}(x^{J_1}, s) = P(d^{J_1}(\tau^{J_1}(1)) = J_1, \tau^{J_1}(1) \in ds | \chi_0^{J_1} = x^{J_1})/ds,$$

for $s > 0$ and $A \in \mathcal{B}(R^{J_1})$. 
Theorem

(i) For a non-empty $J \subsetneq I$, $S \in \mathcal{B}(0, \infty)$, and $A \in \mathcal{B}(\mathbb{R}^{|J|})$,

$$P(d^I(\tau^I(1)) = J, X^{|J|}_{\tau^I(1)} \in A, \tau^I(1) \in S| X_0^I = x^I)$$

$$= \int_S g^I(x^I, s)g^{|J|}_J(x^{|J|}, A, s) \, ds.$$  \hspace{1cm} (8)

(ii) For $s > 0$,

$$g^I(x^I, s)$$

$$= \sum_{m=0}^{#I-1} (-1)^m \sum_{\emptyset \neq I_m \subsetneq \ldots \subsetneq I_1 \subsetneq I_0 := I} \sum_{i \in I_m}$$

$$\left( p_i(x^i, s)g^{|I_m\setminus\{i\}}_{|I_m\setminus\{i\}}(x^{|I_m\setminus\{i\}}, s) \prod_{l=1}^{m} g^{|I_{l-1}\setminus I_{l}}_{I_{l-1}\setminus I_{l}}(x^{|I_{l-1}\setminus I_{l}|}, s) \right).$$  \hspace{1cm} (9)

Here $x^I = (x^i)_{i \in I}$ and $x^{\{i\}}$ is denoted by $x^i$. 

"Transition Probability" in Independent Case
Specification 2. Dealing with Uncertain Parameters
In practice, the parameters: the initial capital $x^i$, the default boundary $K^i$, and the shocks $C_{i,j}$, $j \in I \setminus \{i\}$ (and in fact the firm value process $X^i$), for each $i \in I$, are not directly observable. We need to estimate them. We may assume that they are random. If we do not have any particular prior knowledge, we may well further assume that they are conditionally independent and identical.
The Case with Minimum Prior Knowledge

Specifically, we assume that

- all the firms have the same parameters; $\sigma_i$ and $\mu_i$ and hence $p_i$ and $q_i$, are independent of $i$,
- the parameters $K^i$ initial value $x^i$ and the shocks $C_{i,j}$, $j \in I \setminus \{i\}$ are independent of the wiener process $W = (W^1, \cdots, W^n)$,
- They are also conditionally independent and identically distributed;

$$P(x^i \in dy^i, K^i \in dk^i, C_{i,j} \in dc^{i,j}, i \in I, j \in I \setminus \{i\} | \theta_x, \theta_K, \theta_C)$$

$$= \prod_{i \in I} P(\bar{x} \in dy^i | \theta_x) P(\bar{K} \in dk^i | \theta_K) \prod_{j \in I \setminus \{i\}} P(\bar{C} \in dc^{i,j} | \theta_C), \quad (10)$$

where $\theta_x$, $\theta_K$, and $\theta_C$ are (hyper) parameters independent of $i$, and so the factors in the right-hand-side of (10) are all independent of $i$. 
Denote

\[ p(s) \equiv p(\theta_x, s) := \int p(x, s)P(\bar{x} \in dx | \theta_x), \]

and

\[ w_l(s) \equiv w_l(\theta_x, \theta_C, \theta_K, s) := \int \cdots \int \left( \int_{(k+\sum_{j=1}^{l} c_j, \infty]} q_i(x, y, s) \, dy \right) \]

\[ \times P(\bar{x} \in dx | \theta_x)P(\bar{K} \in dk | \theta_K) \prod_{j=1}^{l} P(\bar{C}_j \in dc_j | \theta_C), \]

for \( l = 0, 1, \ldots \).
Theorem

For $\emptyset \neq J \subset I$, we have that

$$P(d^I(\tau^I(1)) = J) = \int_0^\infty ds \, p(s)(w_{#J}(s))^{#I-#J}$$

$$\sum_{m=0}^{#J-1} \left( -1 \right)^m \sum_{k_0 + \cdots + k_m = #J} \frac{#J!}{(k_0 - 1)! \cdots k_m!} (w_0(s))^{k_0-1} \prod_{l=1}^m \left( w_0 \sum_{j=0}^{l-1} k_j(s) \right)^{k_l},$$

(11)

with the convention that $\prod_{l=1}^0 \cdot = 1$.  

"Transition Probability" in the Case with Minimum Prior Knowledge
We put $f_k$, $k = 1, 2, \cdots$ the polynomial in the last line of (11); that is,

$$f_k(w_0, w_1, \cdots, w_{k-1}) := \sum_{m=0}^{k-1} (-1)^m \sum_{k_0 + \cdots + k_m = k \atop k_0, \cdots, k_m \in \mathbb{N}} \frac{\#l!}{(k_0 - 1)! \cdots k_m!} (w_0)^{k_0-1} \prod_{l=1}^{m} (w_{\sum_{j=0}^{l-1} k_j})^{k_l}.$$ 

Here, $w_0, w_1 \cdots$ are just indeterminates. Then, we have the following polished expressions.

**Corollary**

Let $\#l = n$ and $k = 1, \cdots, n$. Then,

$$\bar{P}(\#d^l(\tau^l(1)) = k) = \int_0^\infty ds \ p(s) \ \binom{n}{k} (w_k(s))^{n-k} f_k(s) \ ds.$$
Further, the family of polynomials $f_k$, $k = 1, 2, \cdots$ is characterized by the following relation.

**Proposition**

*For any integer $n \geq 2$, we have that*

$$
\sum_{l=1}^{n} \binom{n}{k} w_k^{n-k} f_k(w_0, \cdots, w_{k-1}) = nw_0^{n-1}.
$$
Specification 3. Directed Structure as Prior Knowledge
We suppose that we a priori know that $C_{i,j} = 0$ for all $i > j$. In this case, we have the following remarkable property

**Lemma**

If $C_{i,j} = 0$ for all $i > j$, the set $D_i$ above is reduced to be

$$D_i = \{K^1\} \times (K^2, K^2 + C_{1,2}) \times \left(K^3, K^3 + \sum_{i=1}^{2} C_{i,3}\right) \times \cdots \times \left(K^n, K^n + \sum_{i=1}^{n-1} C_{i,n}\right).$$
“Transition Probability” in Directed Structure

**Theorem**

*We have that*

\[
\bar{P}(\#d^I(\tau^I(1)) = k) = \int_0^\infty p(s) \prod_{i=1}^{k-1} (w_0(s) - w_i(s)) h_{\#l-k,k+1}(w_0(s), \ldots, w_k(s)) ds,
\]

*and*

\[
\bar{P}(\#d^I(\tau^I(1)) \geq k) = \int_0^\infty p(s) \prod_{i=1}^{k-1} (w_0(s) - w_i(s)) \sum_{l=0}^{\#l-k} w_0^l h_{l,k}(w_0(s), \ldots, w_{k-1}(s)) ds,
\]

*where* \( h_{l,m} \) *is the complete symmetric polynomial of degree* \( l \) *in* \( m \) *variables;*

\[
h_{l,m}(x_1, \ldots, x_m) = \sum_{0 \leq a_1, \ldots, a_m \leq l \atop a_1 + \cdots + a_m = l} x_1^{a_1} \cdots x_m^{a_m}.
\]
Large Population Limit
Theorem

Let \( \#I = N \). We suppose \( \lim_{N \to \infty} C_{j,i} N \) exist for any \( i \). Then, for any \( k \),

\[
0 < \liminf_{N \to \infty} \bar{P}(\#d^l(\tau(1)) \geq k) \leq \limsup_{N \to \infty} \bar{P}(\#d^l(\tau(1)) \geq k) < 1.
\]

To prove this, we will make use of the explicit generating function of \( h; \)

\[
\sum_{l=0}^{\infty} t^l h_{l,k}(w_0(s), \ldots, w_{k-1}(s)) = \prod_{j=0}^{k-1} (1 - tw_j)^{-1}
\]
Conclusion
Conclusions

- Provided a systematic risk model based on first passage time approach with domino effect.
- Give some explicit form of contagion probabilities, based on very algebraic computations, which enables some explicit calculations for phase transition.
- In that sense, it is an example of stochastic integrable system.
Thank you for your attention