Implementing Portfolio Liquidation Models

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Based on joint work with many people

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Outline

- portfolio liquidation models
  - general stochastic models
  - deterministic models
- empirical implementation
  - linking market impact to market microstructure
  - empirical liquidation strategies
Portfolio liquidation
almost all trading nowadays takes place in limit order markets
  ▶ limit order book: list of prices and available liquidity
  ▶ limited liquidity available at each price level

models of optimal portfolio liquidation:
  ▶ unaffected benchmark price
  ▶ execution price: benchmark price + impact from trading
  ▶ cost of trading: book value - revenues (+risk)
  ▶ liquidation constraint: singular control problem
Forms of market impact

- **instantaneous impact**
  - current trade does not affect future trades
  - pure liquidity cost; immediate recovery

- **permanent impact**
  - current trade affects all future trades
  - generates a drift of the benchmark price/midquote

- **persistent impact**
  - impact of current trade on future trades decays over time
  - generates a mean-reverting drift of the benchmark price/midquote
General stochastic models
Stochastic models

Consider an order to sell $X$ shares by time $T$. The portfolio process is

$$X_t = X - \int_0^t \xi_s \, ds;$$

the liquidation constraint is $X_T = 0$. The transaction price process is

$$\tilde{S}_t = S_t - \eta t \xi_t - Y_t - \int_0^t \lambda_s \xi_s \, ds$$

where

$$Y_t = \int_0^t \left\{ -\rho Y_s + \gamma_t \xi_s \right\} \, ds.$$

denotes a mean-reverting “spread” or “midquote” or ...

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Stochastic models

The *liquidation cost* is defined as

\[ C = \text{book value} - \text{revenue} \]

\[ = S_0 X - \int_0^T \tilde{S}_t \xi_t \, dt = S_0 X - \int_0^T \tilde{S}_t \, dX_t \]

\[ = S_0 X - \int_0^T \xi_t \, dS_t + \int_0^T \lambda_t \xi_t X_t \, dt + \int_0^T \eta_t \xi_t^2 \, dt + \int_0^T Y_t \xi_t \, dt \]

Taking expectations, doing partial integration, adding a risk term:

\[ E \left[ \int_0^T \left( \lambda_t \xi_t X_t + \eta_t \xi_t^2 + Y_t \xi_t + \kappa_t X_t^2 \right) \, dt \right] \rightarrow \min_{\xi} \quad \text{s.t. } X_T = 0. \]

The impact terms award, the risk term penalises slow liquidation.
Theorem (Graewe, H. & Sere (2018), H. & Xia (2018), ...) 

Suppose there is only instantaneous market impact and that

\[ \eta_t \equiv \eta(Z_t); \quad \kappa_t \equiv \kappa(Z_t) \]

for some Itô diffusion \( Z \). Under standard assumptions,

\[ V(t, z, x) = v(t, z)x^2, \quad \xi^*(t, z, x) = 2v(t, z)x \]

where \( v \) is the unique continuous viscosity/classical/\( \pi \)-strong solution in

\[ C_{poly}([0, T^-] \times \mathbb{R}^d) \]

to a singular terminal value problem of the form

\[
\begin{cases}
-\partial_t v - \mathcal{L}v - F = 0, & \text{on } [0, T) \times \mathbb{R}^d, \\
\lim_{t \to T} v(t, z) = +\infty & \text{locally uniformly on } \mathbb{R}^d.
\end{cases}
\]
Theorem (Graewe & H. (2017), H. & Xia (2018))

Suppose there is only instantaneous and persistent impact

\[ \eta_t \equiv \eta; \quad \gamma_t \equiv \gamma; \quad (\rho_t), (\kappa_t) \text{ adapted processes.} \]

Then,

\[ \xi^*_t = \frac{A_t - \gamma B_t}{\eta} X_t - \frac{\gamma C_t - B_t + 1}{\eta} Y_t \]

where \((A, B, C)\) is the unique solution to a coupled (matrix-valued) BS(R)DE system with singular terminal condition

\[ (A_t, B_t, C_t) \to (\infty, 1, 0) \quad \text{as } t \to T. \]
Deterministic models
Constant coefficients and risk neutrality

Consider the state dynamics

\[ X_t = x - \int_0^t \xi_s ds \]
\[ Y_t = -\rho \int_0^t Y_s ds + \gamma \int_0^t \xi_s ds \]

as well as the following cost terms:

- instantaneous impact: \( H_t = \eta \int_0^t \xi_s^2 ds \)
- permanent impact: \( G_t = \lambda \int_0^t \xi_s ds \)
- persistent impact: \( Y_t = \gamma \int_0^t \xi_s e^{-\rho(t-s)} ds \)
- total cost:

\[ C = H_T + \int_0^T \xi_s Y_s ds + \int_0^T \xi_s G_s ds \]
Constant coefficients and risk neutrality

- the cost from *permanent* impact is independent of the strategy
- the Euler-Lagrange ansatz yields:

\[
\frac{\gamma}{2\eta} \int_0^T y' e^{-\rho|t-s|} ds + y' = C
\]

- this is a Wiener-Hopf integral equation of the second kind
Constant coefficients and risk neutrality

- only *instantaneous* impact (Almgren-Chris model)

\[ X_t^* = x - \frac{t}{T} x \]

- only *persistent* impact (Obizhaeva-Wang model)

\[ X_t^* = x - \frac{x}{\rho T + 2} \left( H_0(t) + \rho t + H_T(t) \right) \]

- *instantaneous and persistent* impact (Graewe-H model)

\[ X_t^* = x - x \left( \frac{a + bt + c \sinh(k(t - \frac{T}{2}))}{2a + bT} \right) \]

for constants \( a, b, c, k \) depending on the impact parameters
Interpolating between AC and OW

Figure: Optimal portfolio processes in the GH model
Microstructure and market impact
Microstructure and market impact

- **permanent impact**: drift added to the fundamental price process after each trade
  - information asymmetries
  - order imbalances
- **temporary impact**: expectation of future permanent impact, due to the persistence of trade flows
  - herding effects
  - splitting effects
  - mathematical model: Hawkes processes
- **instantaneous impact**: market makers’ demand for carrying additional inventory (offer curve)
Permanent impact (added drift)

- let
  \[ \delta_t = \#\text{buy} - \#\text{sell} \text{ MOs since last price change} \]
- let the probability \( p_t \) of a mid-price up movement be
  \[ p_t = f(g(\delta_t)), \quad g(\delta) = B_0 + B_1\delta, \quad f(x) = \frac{1}{1 + e^{-x}} \]
- mid-price process is a martingale if
  \[ \delta_t = \bar{\delta} = \frac{f^{-1}(\frac{1}{2}) - B_0}{B_1} \]
- we define the permanent impact and permanent impact factor as
  \[ \Lambda := f(g(\bar{\delta} + 1)) \bar{Z}; \quad \lambda := f(g(\bar{\delta} + 1)) \frac{\bar{Z}_t}{L} \]
Temporary impact (expected future perm. impact)

- we assume that MO arrivals follow a Hawkes process $N$ with intensity

$$I_t^m = \mu^m + A \int_0^t e^{-B(t-t_i)} dN_s^m$$

- adding our market order placement dynamics, the intensity becomes

$$I_t^m = \mu^m + \frac{x_0}{T} + A \int_0^t e^{-B(t-t_i)} dN_s^m$$

- the expected number of additional orders is

$$x_0 \frac{P}{1-P}, \quad P = \frac{A}{B}$$
Temporary impact (expected future perm. impact)

- equating this with the total impact in a continuous time model:
  \[
  \frac{x_0 \gamma}{T} \int_0^T e^{-\rho(T-s)} ds = \lambda x_0 \frac{P}{1 - P}
  \]

- assuming \( \gamma = \lambda \), using a Taylor approximation of order two,
  \[
  \rho \approx \frac{2}{T} \frac{1 - 2P}{1 - P}, \quad P < 0.5
  \]

- if we only consider first generation offsprings,
  \[
  \rho \approx \frac{2}{T} (P^{-1} - 1)
  \]
Instantaneous impact (limit order arrivals)

- orders are added/cancelled at Poison rates
- order sizes are random
- mid-price shift implies a shift in the queues
Instantaneous impact (limit order arrivals)

- liquidity is scattered throughout the book ("many holes")
- regression equation
  \[ O_i = p_0 + \eta E_i + \epsilon_i \]
  where
  - \( O_i \) are the price offsets (differences between price levels with liquidity)
  - \( p_0 \) is the minimum spread
  - \( E_i \) is the aggregated average liquidity

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>arrival</td>
<td>0.875</td>
<td>0.254</td>
<td>0.156</td>
<td>0.098</td>
</tr>
<tr>
<td>cancellation</td>
<td>0.507</td>
<td>0.130</td>
<td>0.079</td>
<td>0.084</td>
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<tr>
<td>aggregate shares</td>
<td>95</td>
<td>190</td>
<td>290</td>
<td>356</td>
</tr>
<tr>
<td>offsets</td>
<td>42.8</td>
<td>61.6</td>
<td>76.3</td>
<td>89.7</td>
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</tbody>
</table>
Figure: Instantaneous impact factor for AMZN.
A two-layer order book model

- order layer:
  - limit orders + cancellation (Poisson arrivals)
  - market orders (Hawkes arrivals)
    - originating from the market
    - originating from us

- price layer (Poisson arrivals with rate $\mu^P$)
A two-layer order book model

- we calibrate the model
- simulate the LOB with and without our strategy
- compute the cost of liquidation for different models
Figure: A sample path of midprice shift due to our trading activity.
Calibration
### Table: Event counts per level: AMZN

<table>
<thead>
<tr>
<th>Level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Submission</td>
<td>39,767</td>
<td>11,937</td>
<td>7,603</td>
<td>4,953</td>
<td>3,395</td>
<td>2,804</td>
</tr>
<tr>
<td>Cancellation</td>
<td>30,987</td>
<td>10,392</td>
<td>6,453</td>
<td>4,176</td>
<td>2,846</td>
<td>2,363</td>
</tr>
<tr>
<td>Execution</td>
<td>8,775</td>
<td>1,537</td>
<td>1,142</td>
<td>772</td>
<td>548</td>
<td>439</td>
</tr>
</tbody>
</table>
Impact factors: AMZN, 5% of ADV (Φ = 20 sec.)

- **price change parameters:**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\mu^P$</th>
<th>$B_0$</th>
<th>$B_1$</th>
<th>$\bar{L}$</th>
<th>$\bar{Z}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mid price change</td>
<td>2.38</td>
<td>0.033</td>
<td>0.894</td>
<td>33.807</td>
<td>10.797</td>
</tr>
</tbody>
</table>

- **market order parameters:**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\mu^m$</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell orders</td>
<td>0.167</td>
<td>8.375</td>
<td>18.53</td>
</tr>
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</table>

- **impact factors:**

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\rho$</th>
<th>$\eta$</th>
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<tbody>
<tr>
<td>0.0162 bps</td>
<td>0.0034 %/second</td>
<td>0.013 bps</td>
</tr>
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</table>
Optimal strategies

Figure: Optimal liquidation strategies for Amazon (+6%)
Optimal strategies

Figure: Optimal liquidation strategies for McDonald’s (+6%)
Optimal strategies

Figure: Optimal liquidation strategies for Ivesco (+10%)
Optimal strategies

Figure: Optimal liquidation strategies for Intel (+17%)
Figure: Optimal liquidation strategies for HP (+20%)
Cost comparison

Figure: Model performance: AC vs. GH; for AMZN we save 1/3 spread
Conclusion

- models of optimal portfolio liquidation with continuous trading
  - abstract existence and uniqueness of solutions results
  - closed form solutions for models with constant coefficients
- we compared the performance of models:
  - only instantaneous impact (AC model)
  - instantaneous and persistent impact (GH model)
- GH outperforms AC in most cases
Thank you!