Optimal liquidation in spite of increasing prices
How is optimal execution affected by price trends?

Peter Frentrup
(based on joint work with Dirk Becherer and Todor Bilarev)

Singapore – 18 March 2019
How to execute large trade in face of trending prices?

- Problem: How to execute/liquidate a position of \( \theta \) risky assets until a given finite time \( T < \infty \) optimally?

- Question: How would optimal trade execution be affected if prices are expected to rise/to decline?

- Example: liquidate 1 asset with increasing/decreasing prices or no price trend:
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- Example: liquidate 1 asset with increasing/decreasing prices or no price trend:
Positive asset prices with transient price impact

- Liquidate $\theta$ assets by selling/buying continuously or in blocks: bounded variation càdlàg strategy $\Theta_t$, $t \in [0, T]$ with $\Theta_0^- = \theta$, $\Theta_T = 0$.

- Unaffected price: $\bar{S} = e^{\mu t} \mathcal{E}(\sigma W)_t$, $\mu \in \mathbb{R}$.

- Affected price: $S_t := f(Y_t)\bar{S}_t$ for price impact process

  \[
dY_t = -h(Y_t) dt + d\Theta_t, \quad Y_0^- = y,
\]

  resilience function $h(0) = 0$, $h' > 0$, e.g. $h(y) = \beta y$, $\beta > 0$.

  impact function $f, f' > 0$, e.g. $f(y) = e^{\lambda y}$, $\lambda = f'/f > 0$ const.

- Maximize expected trading gains $\mathbb{E}[L_T(\Theta)]$,

  \[
  L_T(\Theta) := -\int_0^T f(Y_t)\bar{S}_t d\Theta_t^c
  \]

- Like Obizhaeva/Wang (2013), but for multiplicative and more general transient price impact.
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- Maximize expected trading gains $\mathbb{E}[L_T(\Theta)]$,

$$L_T(\Theta) := -\int_0^T f(Y_t)\overline{S}_t \, d\Theta_t^c - \sum_{0 \leq t \leq T, \Delta \Theta_t \neq 0} \overline{S}_t \int_0^{\Delta \Theta_t} f(Y_{t^-} + x) \, dx.$$

- Like Obizhaeva/Wang (2013), but for multiplicative and more general transient price impact.
A three-dimensional free boundary problem

Value function \( v(\tau, y, \theta) := \sup_{\Theta} \mathbb{E}[L_{\tau}(\Theta) \mid Y_0 = y, \Theta_0 = \theta] \).

State space \((\tau, y, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2\)
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Martingale optimality principle:
State space \((\tau, y, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2\) should separate into open regions \(B\) (buying)
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State space \((\tau, y, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2\) should separate into open regions \(B\) (buying) and \(S\) (selling) with **free contact** boundary surface \(\mathcal{I} = \bar{B} \cap \bar{S}\) s.t. variational HJB holds:

\[
\begin{align*}
\nu_y + \nu_\theta - f(y) &= 0 & \text{everywhere}, \\
\nu_\tau + h(y)V_y - \mu \nu &= 0 & \text{in } B \cup S, \\
\nu_\tau + h(y)V_y - \mu \nu &= 0 & \text{on } \mathcal{I},
\end{align*}
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with boundary condition \(\nu(0, y, \theta) = \int_{y-\theta}^y f(x) \, dx\).
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with boundary condition \(v(0, y, \theta) = \int_{y-\theta}^y f(x) \, dx\).  (2)

Ansatz:  \( v \in C^1 \), \( \forall (\tau, y, \theta) \in B \cup S \ \exists! d \in \mathbb{R} \setminus \{0\} : (\tau, y + d, \theta + d) \in I \),

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B = \{d > 0\}, \ S = \{d < 0\}.
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B = \{d > 0\}, \quad S = \{d < 0\}.
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Optional projection: deterministic strategies are optimal

\[ \nu(\tau, y, \theta) = \sup_{\Theta} \mathbb{E}_{y, \theta} \left[ - \int_0^\tau e^{\mu t} f(Y_t) \, d\Theta_t^c - \sum_{0 \leq t \leq \tau} \int_0^{\Delta \Theta_t} e^{\mu t} f(Y_{t-} + x) \, dx \right] \]
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\]

Optimal strategy should consist of

1) initial block buy/sale \( \Delta \Theta_0 = \bar{\theta}(T) - \theta \),
2) continuous trading in rates \( d\Theta_t = -\bar{\theta}'(T - t) \, dt \),
3) final block buy/sale \( \Delta \Theta_T = -\Theta_{T-} = -\bar{\theta}(0) \)

à la Obizhaeva/Wang, generalized to more general transient impact, positive prices, multiplicative impact.
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Search for optimal \( \bar{\theta} \in C^1 \):
\( \Theta_t = \bar{\theta}(T - t) \), \( Y_t = \bar{y}(T - t) \), for \( t \in [0, T) \),
\[
\bar{y}'(\tau) = h(\bar{y}(\tau)) + \bar{\theta}'(\tau),
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terminal position \( \bar{\theta}(0) = \Theta_{T-} = g(Y_{T-}) = g(\bar{y}(0)) \).
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Gains for candidate optimal strategy

In terms of \( \bar{y} \):

Maximize

\[
J(\bar{y}) = \underbrace{F(Y_0) - F(\bar{y}(T))}_{\text{initial block}} + \underbrace{e^{\mu T}(F(y) - F(y - g(y)))}_{\text{terminal block}} \bigg|_{y=\bar{y}(0)} \\
+ \underbrace{e^{\mu T} \int_0^T e^{-\mu \tau} f(\bar{y}(\tau))(\bar{y}'(\tau) - h(\bar{y}(\tau))) \, d\tau}_{\text{trading in rates}}
\]

for \( F(y) = \int_0^y f(x) \, dx \) subject to the isoperimetric condition

\[
\Theta_0 - Y_0 \equiv K(\bar{y}) := g(\bar{y}(0)) + \int_0^T (\bar{y}'(\tau) - h(\bar{y}(\tau))) \, d\tau - \bar{y}(T).
\]

Goal: find intermediate impact \( \bar{y}(\tau) \) and terminal position \( g(y) \).
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J(\bar{y}) = F(Y_0 -) - F(\bar{y}(T)) + \left. e^{\mu T} (F(y) - F(y - g(y))) \right|_{y=\bar{y}(0)}
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\Theta_0 - Y_0 = K(\bar{y}) := \left. g(\bar{y}(0)) \right|_{\bar{y}(0)=\theta(0)} + \int_0^T \left. (\bar{y}' - h(\bar{y})) \right|_{\bar{y}(0)=\theta'(0)} \, d\tau - \bar{y}(T).
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In terms of $\bar{y}$:

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$$+ \int_0^T e^{\mu \tau} f(\bar{y}(\tau)) (\bar{y}'(\tau) - h(\bar{y}(\tau))) \, d\tau$$

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and terminal position $g(y)$. 

Calculus of variations

- Maximize $J(\bar{y})$ subject to $K(\bar{y}) = (\text{const})$ over $\bar{y} \in C^1([0, T])$ with $\bar{y}(T) = Y_0$

- Equivalent problem: $\max_{\bar{y}} (J(\bar{y}) + m_T K(\bar{y}))$ with (unknown) Lagrange multiplier $m_T \in \mathbb{R}$.

- Taylor approximation: $(J + m_T K)(\bar{y} + z)$

  $$= (J + m_T K)(\bar{y}) + \delta (J + m_T K)(\bar{y})[z] + \delta^2 (J + m_T K)(\bar{y})[z] + O(\|z\|_{W^{1,\infty}}^3)$$

  \(\text{first variation}\)

  \(\text{second variation}\)

where $\|z\|_{W^{1,\infty}} := \|z\|_\infty \lor \|z'\|_\infty$. 
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Peter Frentrup (HU Berlin)

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Calculus of variations – candidate solution

Necessary condition \( \delta (J + m_T K)(\bar{y})[z] = 0 \ \forall z \in C^1 \) with \( z(T) = 0 \), gives

- Lagrange multiplier \( m_T \),
- candidate terminal position

\[
g(y) = y - f^{-1}\left( f \frac{h\lambda + h' - \mu}{h'} \right)(y), \quad y > y_\infty,
\]

where \( (h\lambda + h' - \mu)(y_\infty) = 0 \),

- ODE for candidate impact trajectory \( \bar{y}(\tau) \):

\[
\bar{y}' = \mu \left( \frac{f \frac{h\lambda + h' - \mu}{h'}}{f \left( \frac{h\lambda + h' - \mu}{h'} \right)'} \right)(\bar{y}).
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Write \( \bar{y}(\tau; z), \bar{\theta}(\tau; z) \) for the solution with \( \bar{y}(0; z) = z, \ \bar{\theta}(0; z) = g(z) \),

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\bar{\theta}_\tau(\tau; z) = \bar{y}_\tau(\tau; z) - h(\bar{y}(\tau; z)).
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Write $\bar{y}(\tau; z)$, $\bar{\theta}(\tau; z)$ for the solution with $\bar{y}(0; z) = z$, $\bar{\theta}(0; z) = g(z)$,

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Write $\bar{y}(\tau; z), \bar{\theta}(\tau; z)$ for the solution with $\bar{y}(0; z) = z, \bar{\theta}(0; z) = g(z)$,

$$\bar{\theta}_\tau(\tau; z) = \bar{y}_\tau(\tau; z) - h(\bar{y}(\tau; z)).$$
Proof of optimality for this non-convex problem:

1. showing local optimality with 2\textsuperscript{nd} variation;

2. use this to extend to global optimality.

Non-convex problem – proving optimality

Proof of optimality for this non-convex problem:

1. showing local optimality with 2nd variation;
2. use this to extend to global optimality.

Local optimality

**Theorem (strict local maximizer \( \bar{y} \))**

*Under technical conditions*\(^\dagger\) *on impact and resilience functions* \( f \) *and* \( h \), *\( \exists \varepsilon > 0 \) s.t. for all* \( y \in C^1 \) *with* \( y(T) = \bar{y}(T) \), *\( \|y - \bar{y}\|_{W^{1,\infty}} \in (0, \varepsilon) \):*

\[
(J + m_T K)(\bar{y}) > (J + m_T K)(y).
\]

**Proof:** 2\( ^{\text{nd}} \) variation \( \delta^2 (J + m_T K)(\bar{y})[z] < 0 \), *higher order terms are* \( O(\|z\|_{W^{1,\infty}}^3) \).

- **Candidate buy-sell boundary** \( \mathcal{I} = \{ (\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \mid \tau \in [0, \infty), z > y_\infty \} \)
- **Candidate value function** \( V(T, Y_{0-}, \Theta_{0-}) := J(\bar{y}; T, Y_{0-}, \Theta_{0-}) \).

\(^\dagger\) \( f, h \in C^3, f, f' > 0, \lim_{y \to -\infty} f(y) = 0, \)  
\( h' > 0, h(0) = 0, (h\lambda)' > 0, (h\lambda + h')' > 0 \) *where* \( \lambda := f'/f \),  
\( \exists y_\infty : (h\lambda + h' - \mu)(y_\infty) = 0, \)  
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and \( h'' < (h\lambda)'h'/(h\lambda - \mu) \) for \( y > y_0 \).

Satisfied, e.g. for  
\( f(y) = e^{\lambda y}, h(y) = \beta y, \)  
\( \lambda, \beta > 0 \) const.
Local optimality

Theorem (strict local maximizer $\bar{y}$)

Under technical conditions† on impact and resilience functions $f$ and $h$, $\exists \varepsilon > 0$ s.t. for all $y \in C^1$ with $y(T) = \bar{y}(T)$, $\|y - \tilde{y}\|_{W^{1,\infty}} \in (0, \varepsilon)$:

$$(J + m_TK)(\bar{y}) > (J + m_TK)(y).$$

Proof: 2nd variation $\delta^2(J + m_TK)(\bar{y})[z] < 0$, higher order terms are $O(\|z\|_{W^{1,\infty}}^3)$.

- Candidate buy-sell boundary $\mathcal{I} = \{(\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \mid \tau \in [0, \infty), z > y_\infty\}$
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**Theorem (strict local maximizer \( \bar{y} \))**

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Local optimality (II)

\[ I = \{ (\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \mid \tau \in [0, \infty), z > y_\infty \} \]

\[ V(T, Y_{0-}, \Theta_{0-}) := J(\bar{y}; T, Y_{0-}, \Theta_{0-}) \]

Corollary (variational inequality near \( I \))

We have

\[ V_\tau + h(y)V_y - \mu V > 0 \text{ in a neighborhood of } I \]

with equality on \( I \), and \( V_y + V_\theta = f \) everywhere.

Proof: Otherwise, construct a strategy given by \( \hat{y} \), with \( 0 < \| \hat{y} - \bar{y} \|_{W^{1,\infty}} < \varepsilon \) and \( K(\hat{y}) = K(\bar{y}) \) which would give \( J(\hat{y}) \geq J(\bar{y}) \).
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**Corollary (variational inequality near \( \mathcal{I} \))**

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Global optimality

Question: How to conclude from local to global optimality?

- Let $k(d) := (V_\tau + h(y)V_y - \mu V)(\tau; y_b + d, \theta_b + d)$ for fixed $(\tau, y_b, \theta_b) = (\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \in I$.

- Previous corollary: $k(0) = 0$, $k'(0-) \leq 0 \leq k'(0+)$;

- Now, can show analytically the inequalities:
  \[ k'(-d) < 0 < k'(d) \text{ for } d > 0. \]

- Hence, $k(d) > 0$ for $d \in \mathbb{R} \setminus \{0\}$, giving strict global optimality.

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Optimal strategy: dependence on price trend
(comparative statics)

- **no trend** in fundamental price $\bar{S}$ (martingale, Obizhaeva/Wang situation): constant rate of trading in time $(0, T)$.

  - increasing $\bar{S}$: defer asset sales to later times; for strong upwards trend: temporary buying.
  
  - decreasing $\bar{S}$: sell asset at earlier times; for strong downwards trend: go short.
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If there is still some time...
Stochastic liquidity and transaction costs

- Stochasticity in the impact/signal:
  \[
  dY_t = -\beta Y_t \, dt + \sigma \, dB_t + d\Theta_t, \quad Y_{0+} = y,
  \]
  for correlated Brownian motion $B$ with $d[B, W]_t = \rho \, dt$.
  [Becherer/Bilarev/F., FS 2018]

- Bid-ask spread through proportional transaction costs:
  \[
  \text{sell at } S^\text{bid} := f(Y_t) \overline{S}_t, \quad \text{buy at } S^\text{ask} := \kappa f(Y_t) \overline{S}_t,
  \]
  for transaction cost factor $\kappa > 1$.

- Maximize over $\Theta$: càdlàg, adapted, bounded variation, $\geq 0$, until $\tau^\Theta := \inf\{t \geq 0 \mid \Theta_t = 0\}$.
  Infinite time horizon eases analysis: (non-convex) free boundary problem in $\mathbb{R}^2$. 
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Variational (in-)equalities

State space \((y, \theta) \in \mathbb{R} \times \mathbb{R}_+\) should separate into open regions \(B, W, S\) with corresponding HJB variational (in-)equalities... 

- **Ansatz:** there exist free boundary curves \(b, s \in C^1(\mathbb{R}_+)\) s.t. 
  
  \[B = \{y < b(\theta)\}, \quad W = \{b(\theta) < y < s(\theta)\}, \quad S = \{y > s(\theta)\}.\]

- \(V(y, 0) = 0\) for \(y \geq s(0)\),

- \(V(y, \infty) = \tilde{V}(y)\) via corresponding “\(\theta \to \infty\)” infinite fuel limit.
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Open ODE problem for boundary curves

Smooth pasting gives

- Candidate free boundary curves $b(\theta), s(\theta)$ as ODE
  \[(b', s') = \text{function}(b, s),\]

- with asymptotes $b(\infty) = b_\infty, s(\infty) = s_\infty$, and (implicitly) given $s(0)$.

Open questions:

- Existence of ODE solution for $b, s$;

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Thank you!