FROM HOTELLING TO NAKAMOTO: THE ECONOMIC MEANING OF BITCOIN MINING

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jointly with
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AGENDA

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THE MODEL

CALIBRATION

QUANTITATIVE ANALYSIS

CONCLUSION
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Basic ingredients: (a) Users, and (b) Miners
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What is the economic meaning of Bitcoin mining?
Mining Business

Mining business consists of

- **Revenue:**
  - Block Bitcoin rewards *(deterministic, exogenously given by the system, and are vanished after 2140)*
  - Transaction fees in Bitcoin *(stochastic, endogenously determined by the system)*

- **Cost:**
  - Running costs *(e.g., mining machines, electricity, etc.)*
  - Liquidation costs
  - Others

- **Risk/Uncertainty:**
  - Mining lottery *(strong competition, low chance)*
  - Exchange rate *(Bitcoin/USD, extremely volatile due to adoption, policy uncertainty etc.)*
Block rewards: Deterministic, Exogenous, and Scarce

Scarcity $\Rightarrow$ Bitcoin is an exhaustible resource!
Transaction fees: Stochastic, Endogenous, and Unlimited
Key incentive to miners after the end of block rewards
**Stylized Facts: Exchange Rate & Average Transaction Fee Rate**

**Figure:** The dynamics of average transaction fee rate and Bitcoin price from 2013 to 2018.

Average fee rate at $t = \frac{\text{Total transaction fees at } t}{\text{Processed transaction volume at } t}$. 

**Stylized Facts: Miner’s Inventory**

**Figure:** Miner’s inventory proportional to the supply from 2012 to 2015 (Athey et al. 2016).

Propotional inventory $= \frac{\text{Miners’ aggregate inventory at time } t}{\text{Cumulative Bitcoin supply at time } t}$.
Our Main Results

- We build a partial equilibrium model from miners’ perspective by extending the classical Hotelling (1931) model with inventory and feedback supply.
- We calibrate our model to the empirical data from 2013 to 2018.
- Our model has many interesting implications including
  - A high (low) trading volume leads to a high (low) transaction fee rate.
  - High jump risk forces miners to sell their holding of Bitcoin in an early stage even when Bitcoin price is quite low.
**Literature Review**

- **Model on transaction fees:**
  
<table>
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<th>Easley, O’Hara, and Basu (2019, JFE)</th>
<th>One period</th>
<th>Nash equilibrium of users’ fee paying strategy</th>
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<td>Our model</td>
<td>Continuous time dynamic model</td>
<td>Transaction fees from miner’s perspective incorporating declined block rewards and miners’ inventory</td>
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- **Resource models:** Hotelling (1931, JPE); Levhari and Pindyck (1981, QJE); Pindyck (2001);

- **Bitcoin as currency:** Athey et al. (2016); Gandal and Halaburda (2015); Halaburda and Sarvary (2016); Bolt et al. (2016); Jermann (2018).

- **Others:** Cong, He, and Li (2018); Dixon (1980); Bass (2004).
A Resource Production Model

Originating from Hotelling (1931, JPE), resource mining problem can be written in general as:

$$\sup_{Q_u \geq 0} \mathbb{E}_t \left[ \int_t^\infty e^{-\beta(u-t)} \left( \text{Rev}(Q_u) - \text{Cost}(Q_u) \right) du \right]$$

where $\beta > 0$ is a discount factor, and

- $\text{Rev}(Q_u) = P_u Q_u$
- $\text{Cost}(Q_u) = \lambda_1 P_u Q_u^2$ [liquidation] + $\lambda_2 P_u Q_u^2 / H_u$ [utility] + $c$ [running]
- $Q$: Miner’s selling rate
- $P$: Bitcoin Price
- $H$: Holding Inventory
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- $Q$ : Miner’s selling rate
- $P$ : Bitcoin Price $P_t = \theta_p X_t$, where
  $$dX_t = \mu(\xi_t, X_t)dt + \sigma(\xi_t, X_t)dW_t - (1 - Z)X_t dJ_t$$
- $H$ : Holding Inventory
  $$dH_u = \{(b_u \text{[block]} + I_u \text{[transaction]}) \pi \text{[probability]} - Q_u\} du$$
Bitcoin price satisfies an inverse demand function.

Bitcoin price is determined by quantity equation of medium of exchange (Bolt et al. 2016, WP; Fisher 1911; Friedman 1973):

\[ P_t = \theta_p X_t. \]

where the constant \( \theta_p \) is determined by Bitcoin supply and velocity.
Modeling Demand Shock

Demand shock (Bass 2004; Gronwald 2015; Gandal et al. 2018):

\[ dX_t = \mu(\xi_t, X_t)dt + \sigma(\xi_t, X_t)dW_t - (1 - Z)X_t dJ_t \]

where

- \( \xi_t \in \{H, L\} \) represent two transaction states: High-active/Low-active markets, with transition intensities \( \zeta = (\zeta_H, \zeta_L) \).
- \( \mu(\xi_t, X_t) = \kappa_\xi (\nu_\xi - \ln X_t)X_t \), and \( \sigma(\xi_t, X_t) = \sigma_\xi X_t \) denote the adoption term and volatility term respectively in state \( \xi_t \) (Gompertz Model).
- \( J_t \) is a jump process with intensity \( \lambda_J \), and \( 1 - Z \) is the proportional jump size (Weil 1987, QJE).
Miner’s inventory $H_t$ satisfies

$$dH_t = [(b_t + I_t)\pi - Q_t]dt,$$

- $\pi = \frac{\omega}{D \times 2^{32}/600}$ is the probability of successful validations and $D$ is the difficulty level (Hayes 2017).
- $b_t$ is the block reward at $t$ with total supply $\bar{S} = \int_0^\infty b_t dt = \int_0^T b_t dt < \infty$
- $I_t$ is the transaction fees in candidate blocks at $t$.

Note. $D \times 2^{32}/600$ is also called network hash rate.
Modeling Transaction Fees

- Total volume of submitted orders by others:

\[ L_t = \theta(\xi_t)(S_t - H_t) \log(1 + X_t) \text{ with } \theta(\xi_t) \in \{\theta_H, \theta_L\}. \]

- The distribution of orders with different fee rate:

\[ f(\phi), \quad \phi \in (0, \bar{\phi}) \text{ with C.D.F. } F(\phi). \]

- Each time, a fixed number of orders \( G \) will be processed by miners.
**Transaction Fees**

- The miner selects fee threshold $\Phi_t$ to solve

$$\max_{\Phi_t} I_t(\Phi_t) = K(\Phi_t)L_t$$

subject to  $k(\Phi_t)L_t \leq G$,

where $k(\Phi_t) = \int_{\Phi_t}^{\Phi} f(\phi) d\phi$, and $K(\Phi_t) = \int_{\Phi_t}^{\Phi} f(\phi) \phi d\phi$.

- Optimal fee threshold satisfies:

$$\Phi^*_t = \begin{cases} F^{-1}(1 - \frac{G}{L_t}), & \text{if } L_t > G, \\ 0, & \text{if } L_t \leq G, \end{cases}$$

- The miner’s average transaction fee rate:

$$r_t = \frac{K(\Phi^*_t)}{k(\Phi^*_t)}.$$
**Average transaction Fee Rate**

**Proposition**

1. In state $\xi$, for demand level lower than $G/\theta(\xi)$, the average transaction fee rate is constant $K(0)/k(0)$. For demand level higher than $G/\theta(\xi)$, the average transaction fee rate is an increasing function of demand.

2. The above results hold for the market average transaction fee rate (aggregation).
HJB Equation

- Short-run case: $t < T$, there are block rewards.
- In state $\xi$, for $(t, X_t, H_t) = (t, x, h) \in (0, \infty)^2 \times [0, S(t)]$,

\[
\frac{\partial V_\xi}{\partial t} + \mathcal{L}V_\xi + \max_{\{q \geq 0\}} \left\{ (\pi(b_t + K(\phi)L) - q) \frac{\partial V_\xi}{\partial h} + Pq - \lambda_q Pq^2 - c \right\} \\
+ \lambda_J \left[ V_\xi(t, Zx, h) - V_\xi(t, x, h) \right] + \zeta_\xi \left[ V_\tilde{\xi}(t, x, h) - V_\xi(t, x, h) \right] = \beta V_\xi
\]

where

\[
\mathcal{L}V_\xi = \frac{1}{2} \sigma(\xi, x)^2 \frac{\partial^2 V_\xi}{\partial x^2} + \mu(\xi, x) \frac{\partial V_\xi}{\partial x}.
\]

- Long-run case: $b_t = 0$ for $t \geq T$
- $V_\xi(t, X, H) = V_\xi(T, X, H) := V_\xi^L(X, H)$ for any $t \geq T$. 
In state $\xi$, optimal inventory strategy $q^*_\xi$ satisfies:

$$q^*_\xi = \max \left\{ \frac{h}{2P(\lambda_1 h + \lambda_2)} \left( P - \frac{\partial V_\xi}{\partial h} \right), 0 \right\}$$
In state \( \xi \), optimal inventory strategy \( q^*_\xi \) satisfies:

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q^*_\xi = \max \left\{ \frac{h}{2P(\lambda_1 h + \lambda_2)} \left( P - \frac{\partial V_\xi}{\partial h} \right), 0 \right\}
\]

**Holding / Selling regions:**

- **Selling region:**
  \[
  \{ q^*_\xi > 0 \} = \{ P > \frac{\partial V_\xi}{\partial h} \}
  \]

- **Holding region:**
  \[
  \{ q^*_\xi = 0 \} = \{ P \leq \frac{\partial V_\xi}{\partial h} \}
  \]
Calibration: Data

  - Bitcoin price \( \{ P_t \} \)
  - Difficulty level \( \{ D_t \} \)
  - Miners’ aggregate inventory \( \{ H^A_t \} \) from 2013 - 2015,
  - Market average fee rate: \( \{ r^A_t \} \)
  - Aggregate transaction fees: \( \{ I^A_t \} \)

- Bitcoin prices are informative to parameters \( \Theta_1 = \{ \kappa, \nu, \sigma_H, \sigma_L \} \).
- Miners’ aggregate inventory, average fee rate, and aggregate fee income are informative to parameters \( \Theta_2 = \{ \lambda_1, \lambda_2, \theta_H, \theta_L \} \).
High-active/low-active Market

- Detect the high-active and low-active market by Mempool size (High-active: mempool size > 10 MB).
- Low-active: 2013Q1-2016Q3; High-active: 2016Q4-2017Q4; Low-active: 2018Q1-2018Q4

Note. Red line is the 60-day moving average.
Calibration Method

- **Step 1:** Set $\beta = 0.06; \bar{S} = 1; G = 10; \theta_p = 100; \lambda_J = 57; Z = 0.9$. The $f(\cdot)$ satisfies Beta distribution with parameters $(a, b) = (0.1, 99.9)$.

- **Step 2:** Estimate $\Theta_1 = (\kappa, \nu, \sigma_H, \sigma_L)$ with Bitcoin price data.

- **Step 3:** Given $\Theta_2 = (\lambda_1, \lambda_2, \theta_H, \theta_L)$ and observed Bitcoin price, we can compute the path of demand shock $\{\tilde{X}_t; t = 1, \cdots, T_1\}$. For miners start to mine in year $y \in (2013, \cdots, 2018)$, we can compute the
  - implied transaction fees $\{\tilde{I}_{y,t}; t = 1, \cdots, T_1\}$,
  - implied average fee rate $\{\tilde{r}_{y,t}; t = 1, \cdots, T_1\}$.
  - implied inventory $\{\tilde{H}_{y,t}; t = 1, \cdots, T_2\}$,
Calibration Method

- **Step 3 (continue):** Compute
  - implied aggregate transaction fees \( \{I^A_t; t = 1, \cdots, T_1\} \).
  - implied market average fee rate \( \{r^A_t; t = 1, \cdots, T_1\} \);
  - implied aggregate inventory \( \{H^A_t; t = 1, \cdots, T_2\} \);

We estimate \( \hat{\Theta}_2 \) by minimizing:

\[
\min_{\Theta_2} \frac{1}{T_1} \sum_{t=1}^{T_1} \left\{ w_t^1 (r^A_t - \widehat{r^A}_t)^2 + w_t^2 (I^A_t - \widehat{I^A}_t)^2 \right\} \\
+ \frac{1}{T_2} \sum_{t=1}^{T_2} \left\{ w_t^3 (H^A_t - \widehat{H^A}_t)^2 \right\}
\]

where \( w_t^1, w_t^2, w_t^3 \) are the weight coefficients.
# Summary of Parameters

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<th>Parameters</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>Risk-free rate</td>
<td>$\beta$</td>
<td>0.06</td>
</tr>
<tr>
<td>Total supply of Bitcoin</td>
<td>$\bar{S}$</td>
<td>1</td>
</tr>
<tr>
<td>Capacity of blocks per unit of time</td>
<td>$G$</td>
<td>10</td>
</tr>
<tr>
<td>Hash rate per miner (TH/s)</td>
<td>$5.2$</td>
<td></td>
</tr>
<tr>
<td>Coefficient in quantity equation (Billion USD per unit)</td>
<td>$\theta_p$</td>
<td>100</td>
</tr>
<tr>
<td>Upper bound of fee rate</td>
<td>$\phi$</td>
<td>10%</td>
</tr>
<tr>
<td>Beta distribution parameters</td>
<td>$(a, b)$</td>
<td>(0.1, 99.9)</td>
</tr>
<tr>
<td>Adoption speed of Bitcoin</td>
<td>$\kappa$</td>
<td>1.1742</td>
</tr>
<tr>
<td>Log carrying capacity</td>
<td>$\nu$</td>
<td>0.7793</td>
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<td>Volatility of demand shock in high-active market</td>
<td>$\sigma_H$</td>
<td>0.7910</td>
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<td>Volatility of demand shock in low-active market</td>
<td>$\sigma_L$</td>
<td>0.6225</td>
</tr>
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<td>State transition intensity</td>
<td>$(\zeta_H, \zeta_L)$</td>
<td>(0.8, 0.3)</td>
</tr>
<tr>
<td>Jump parameters</td>
<td>$(\lambda_J, Z)$</td>
<td>(57, 0.9)</td>
</tr>
<tr>
<td>parameter in liquidation cost</td>
<td>$\lambda_1$</td>
<td>4.5</td>
</tr>
<tr>
<td>parameter in utility cost in liquidation</td>
<td>$\lambda_2$</td>
<td>0.6</td>
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<tr>
<td>Sensitivity of volume to demand in high-active market</td>
<td>$\theta_H$</td>
<td>251.3</td>
</tr>
<tr>
<td>Sensitivity of volume to demand in low-active market</td>
<td>$\theta_L$</td>
<td>30.6</td>
</tr>
<tr>
<td>Marginal cost of mining (Billion USD per TH/s)</td>
<td>$C_m$</td>
<td>$3.61 \times 10^{-7}$</td>
</tr>
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</table>
Implied Inventory

Note. Proportional inventory.
**Implied Average Fee Rate**

![Graph showing average fee rate over time with two lines: one for implied market average fee rate and one for observed market average fee rate. The x-axis represents years from 2013 to 2018, and the y-axis represents the fee rate in percentage (%).]

- **Implied market average fee rate**
- **Observed market average fee rate**
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Note. Bull market. $t = 2014$. $H_t \in [0, S_t]$, $S_t = 0.5871. n = 1/\pi$. 
Selling boundary in short-run

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Selling boundary in short-run

Short-run: selling and holding regions

- Selling Region
  - $n=10, \lambda_J=0$
  - $n=10, \lambda_J=10$
  - $n=8000, \lambda_J=0$
  - $n=8000, \lambda_J=57$

Note. Bull market. $t = 2014$. $H_t \in [0, S_t], S_t = 0.5871. n = 1/\pi.$
Selling boundary in long-run

Note. Bull market.
**Short-run: Optimal Selling**

Note. Bull market. $t = 2014$. We fix the miner’s holding to be $H_t = 0.1$. $n = 1/\pi$. 
**LONG-RUN: OPTIMAL SELLING**

Note. Bull market. We fix the miner’s holding to be $H_T = 0.1$. $n = 1/\pi$. 

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**Introduction**

**The Model**

**Calibration**

**Quantitative Analysis**

**Conclusion**
Note. $H = 0$. 
Note. Here we assume short-run case is about at $t = 2014$. Left figure shows the average fee rate under different inventory for both long-run and short-run in high-active market, while the right figure shows that in low-active market.
Average fee rate to Capacity

Note. Here we assume short-run case is about at $t = 2014$. Left figure shows the average fee rate under different system capacity for both long-run and short-run in high-active market, while the right figure shows that in low-active market.
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We build a partial equilibrium model from miners’ perspective by extending the classical Hotelling model with inventory and feedback supply, and calibrate our model to the empirical data from 2013 to 2018.

The model can simultaneously generate the dynamics of average transaction fee rate and miners’ inventory holdings consistent with the observed data.

We find trading volume and jump risk are respectively key factors to understand the dynamics of average transaction fee rate and miners’ inventory holdings.
Thanks for your attention!