Robust mean-variance portfolio selection

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1 Introduction

2 Robust MV formulation

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• **Markowitz** (52, J. Finance) : cornerstone of portfolio allocation theory
  - Maximize expected portfolio return given risk \( \equiv \) portfolio variance
  - Important insight is **well-diversification** : optimal portfolio should contain all available assets! (same conclusion with the “more academic” utility criterion)

• However, in practice, we observe that investors trade only in a small part of available stocks : **under-diversification**
  - Empirical studies : Mitton, Vorkink (07, RFS), Calvet, Campbell, Sodini (08, JPE), Guidolin, Liu (14, JFQA), ...

► How to reconcile theory and practice?
Classical Markowitz (mean-variance) problem in a dynamic setting

- $X^\alpha = (X^\alpha_t)_t$ wealth process with $\alpha = (\alpha_t)$: amount invested in $d$ risky assets at any time $t \in [0, T]$, $T < \infty$ investment horizon, starting from initial capital $x_0$.

- Mean-variance criterion: on $(\Omega, \mathcal{F}, \mathbb{P})$, and given $\lambda > 0$, maximize over $\alpha$

$$\mathbb{E}[X_T^\alpha] - \lambda \text{Var}(X_T^\alpha).$$

- Expectation and variance under $\mathbb{P} \leftrightarrow$ belief/scenario on a probabilistic model for the assets price
- $\lambda > 0 \leftrightarrow$ risk aversion
Optimal MV portfolio in Black-Scholes model

- Multidimensional $BS(b^o, \Sigma^o)$ model: risk-free asset $\equiv 1$, $d$ stocks with
  - $b^o \in \mathbb{R}^d$: vector of stocks return
  - $\Sigma^o \in \mathcal{S}^d_+$: covariance matrix of stocks

- Optimal amount invested in the $d$ stocks (no constraints):
  $$\alpha^*_t = \Lambda_o(X^*_t)(\Sigma^o)^{-1}b^o, \quad 0 \leq t \leq T.$$  
  - $(\Sigma^o)^{-1}b^o$: vector of allocation in the $d$ stocks $\rightarrow$ diversification
  - $\Lambda_o(X^*_t) := (x_0 + \frac{1}{2\lambda}e^{R^o T} - X^*_t) > 0$, weight involving:
    - $R^o := (b^o)^\top(\Sigma^o)^{-1}b^o \in \mathbb{R}$: (square) of risk premium of the $d$ stocks
    - $\lambda$ the risk aversion of the investor

- Refs: Li, XY. Zhou (00), Andersson-Djehiche (11), Fisher-Livieri (16), Pham, Wei (16).
Model risk

- Classical portfolio selection assumed a perfect knowledge of the price dynamics. In particular, within BS model, key inputs of optimal strategy:
  - Expected rate of return $b^o$
  - Covariance matrix $\Sigma^o$

- In practice:
  - Investors are ambiguous about the true model (Knight uncertainty)
  - Inaccuracy in parameters estimation:
    - expected return (drift)
    - correlations: due to asynchronous data and lead-lag effect, especially when $d$ is large, see, e.g., Jagannathan, Ma (03), Ledoit, Wolf (04).

- Our purpose: explore effects of ambiguity (Knightian uncertainty) on portfolio selection and diversification
Literature review for robust portfolio selection

- **Dynamic portfolio optimization under Knightian uncertainty**
  Hansen, Sargent (01), Chen, Epstein (02), Talay, Zheng (02), Gundel (05), Denis, Kervarec (07), Schied (11), Epstein, Ji (11), Tevzadze et al. (12), Jin, XY. Zhou (15), Matoussi, Possamai, C. Z. (12), Kallbald, Obloj, Zariphopoulou (14), Viens et al. (14), Biagini, Pinar (15), Fouque, Sun, Wong (15), Lin, Riedel (15), Neufeld, Nutz (16), Bielecki et al. (16), Ismail, Pham (17), etc
  - Mainly for expected utility criterion.
  - Mainly for ambiguity about drift or (marginal) volatility. Correlation ambiguity is addressed in Fouque, Sun, Wong (15) and Ismail, Pham (17) when \( d = 2 \)
  - Effect on portfolio diversification is not really studied.

- **Ambiguity and portfolio diversification in a one-period model**:
  - **Drift ambiguity** : Uppal, Wang (03), Boyle, Garlappi, Uppal, Wang (12)
  - **Correlation ambiguity** : Liu and Zeng (17).
Our main contributions

- Framework for continuous-time mean-variance portfolio selection under ambiguity both on expected rate of return and correlation matrix for $d \geq 2$

  - Separation principle for solving the differential game problem ↔ robust portfolio selection:
    - Reduction to parametric infimum computation of the risk premium function

  - Quantify explicitly the diversification effects in terms of ambiguity parameters. Roughly, our findings:
    - Do not trade in stocks with large expected return ambiguity
    - Do not invest in stocks with high level of ambiguity about correlation
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   - Robust MV problem

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Introduction

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 Canonical framework

• Canonical space $\Omega = C([0, T], \mathbb{R}^d)$: continuous paths of $d$ stocks
  $\rightarrow B = (B_t)_t$ canonical process, $\mathbb{F} = (\mathcal{F}_t)_t$ canonical filtration

• Marginal volatilities $\sigma_i > 0$ of each asset $i = 1, \ldots, d$ are assumed to be known constants, e.g., via quadratic variation estimation:
  $\rightarrow$ Marginal volatility matrix

$$\mathcal{S} := \begin{pmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & . \\ \vdots & \vdots & \ddots & \vdots \\ 0 & . & \ldots & \sigma_d \end{pmatrix}.$$
Ambiguity parametrization about drift and correlation

- Parameter convex set: \( \Theta \subset \mathbb{R}^d \times \mathbb{C}^d_{>+} \), where \( \mathbb{C}^d_{>+} \) is the subset of all elements \( \rho = (\rho_{ij})_{1 \leq i < j \leq d} \in [-1, 1]^{d(d-1)/2} \) s.t. the correlation matrix 

\[
C(\rho) := \begin{pmatrix}
1 & \rho_{12} & \ldots & \rho_{1d} \\
\rho_{12} & 1 & \ldots & . \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{1d} & . & \ldots & 1
\end{pmatrix}
\]

is positive definite, i.e., \( \in \mathbb{S}^d_{>+} \).

→ Prior covariance and volatility matrix:

\[
\Sigma(\rho) := \mathcal{G} C(\rho) \mathcal{G} \in \mathbb{S}^d_{>+}, \quad \sigma(\rho) := \Sigma^{1/2}(\rho).
\]

→ Prior risk premium function:

\[
R(\theta) := b^\top \Sigma^{-1}(\rho) b \in \mathbb{R}, \quad \theta = (b, \rho) \in \Theta.
\]
Examples

- **Product set**: $\Theta = \Delta \times \Gamma$, where $\Delta$ is a compact convex set of $\mathbb{R}^d$, e.g., in rectangular form $\Delta = \prod_{i=1}^{d} [b_i, \bar{b}_i]$, and $\Gamma$ is a convex set of $\mathbb{C}^d_{> +}$.

- **Ellipsoidal set**:

  $$\Theta = \{(b, \rho) \in \mathbb{R}^d \times \Gamma : \|\sigma(\rho)^{-1}(b - \hat{b})\|_2 \leq \delta\},$$

- $\hat{b}$ known constant: a priori estimation on expected rate of return
- $\delta \geq 0$: level of ambiguity around $\hat{b}$, e.g., due to estimation error (CLT)

**Remark.** $\delta = 0$: known drift, $\Gamma$ singleton: known correlation, $\Gamma = \mathbb{C}^d_{> +}$: total ambiguity about correlation

- In this talk, we focus on the ellipsoidal set case.
Prior beliefs

• A priori random processes for the unknown drift and correlation:

\begin{align*}
\mathcal{N}_\Theta & : \text{set of } \mathbb{F}\text{-adapted processes } \theta = (\theta_t)_t = (b_t, \rho_t)_t \text{ valued in the ambiguity set } \Theta \\
\iff & \text{Prior probability measures:}
\end{align*}

\[ P^\Theta = \{ P^\theta : \theta \in \mathcal{N}_\Theta \} \]

• \( B \) is a semimartingale on \((\Omega, \mathcal{F}, P^\theta)\) with characteristics \((b_t, \Sigma(\rho_t))_t \rightarrow \) positive price process \( S \) driven by

\begin{align*}
\begin{aligned}
dS_t &= \text{diag}(S_t)dB_t, \quad 0 \leq t \leq T, \quad P^\Theta - \text{q.s.} \\
&= \text{diag}(S_t)(b_tdt + \sigma(\rho_t)dW^\theta_t), \quad P^\theta - \text{a.s., } \forall \theta \in \mathcal{N}_\Theta.
\end{aligned}
\end{align*}
Robust MV problem

- Set $A$ of portfolio strategies: $\mathbb{F}$-adapted processes $\alpha$ s.t.
  $\sup_{P\theta \in \mathcal{P}\theta} E_{\theta} \left[ \int_0^T |\alpha_t^\top b_t| dt + \int_0^T \alpha_t^\top \Sigma(\rho_t) \alpha_t dt \right] < \infty$.

  $\rightarrow$ (Sel-financed) wealth process $X^\alpha$:
  
  $dX^\alpha_t = \alpha_t^\top \text{diag}(S_t)^{-1} dS_t = \alpha_t^\top dB_t, \quad 0 \leq t \leq T, \quad X^\alpha_0 = x_0, \quad \mathcal{P}\theta - q.s.$

- Robust problem:
  - Worst-case mean-variance functional
  $J_{wc}(\alpha) = \inf_{P\theta \in \mathcal{P}\theta} \left( E_{\theta} \left[ X^\alpha_T \right] - \lambda \text{Var}_{\theta}(X^\alpha_T) \right), \quad \alpha \in A.$

  A (non standard) differential game problem: McKV control problem under model uncertainty
  
  $V_0 = \sup_{\alpha \in A} J_{wc}(\alpha) = \sup_{\alpha \in A} \inf_{\theta \in \mathcal{V}\theta} J(\alpha, \theta)$

- **Goal**: find $\alpha^* \in A$ s.t. $V_0 = J_{wc}(\alpha^*)$ (optimal robust MV strategy), and possibly $\theta^* \in \mathcal{V}\theta$ s.t. $V_0 = J(\alpha^*, \theta^*)$ (worst-case scenario).
**Separation principle**

**Theorem.**

- Let $\theta^* = (b^*, \rho^*) \in \Theta$ achieving the minimum (when it exists) of the prior risk premium function:

$$R(\theta^*) = \min_{\theta \in \Theta} R(\theta), \quad R(\theta) := b^T \Sigma^{-1}(\rho) b.$$ 

- **Optimal robust MV strategy** $\equiv$ **optimal strategy in** $BS(b^*, \Sigma(\rho^*))$:

$$\alpha^*_t = \Lambda_{\theta^*}(X_t^*) \Sigma^{-1}(\rho^*) b^*, \quad 0 \leq t \leq T, \ P^\Theta - q.s.$$ 

with $\Lambda_{\theta^*}(X_t^*) = (x_0 + \frac{1}{2\lambda} e^{R(\theta^*) T} - X_t^*) > 0, \quad 0 \leq t \leq T, \ P^\Theta$-q.s

Moreover,

$$V_0 = J_{wc}(\alpha^*) = J(\alpha^*, \theta^*)$$

$$= \sup_{\alpha \in \mathcal{A}} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta) = \inf_{\theta \in \mathcal{V}_\Theta} \sup_{\alpha \in \mathcal{A}} J(\alpha, \theta).$$

**Remark.** Similar result for expected utility criterion
Sketch of proof (I)

- **Weak optimality principle**:

Search for a family of processes \( \{ V_t^{\alpha, \theta}, t \in [0, T], \alpha \in A, \theta \in \mathcal{V}_\Theta \} \) in the form:

\[
V_t^{\alpha, \theta} = \nu(t, X_t^\alpha, \mathbb{E}_\theta[X_t^\alpha]),
\]

for some function \( \nu(t, x, \bar{x}), t \in [0, T], x, \bar{x} \in \mathbb{R} \), s.t.

(i) \( \nu(T, x, \bar{x}) = x - \lambda (x - \bar{x})^2 \)

(ii) \( \mathbb{E}_{\theta^*}[V_T^{\alpha, \theta^*}] \leq V_0^{\alpha, \theta^*} = \nu(0, x_0, x_0) \), for all \( \alpha \in A \), and some \( \theta^* \in \mathcal{V}_\Theta \)

(iii) \( \mathbb{E}_\theta[V_T^{\alpha^*, \theta}] \geq V_0^{\alpha^*, \theta} = \nu(0, x_0, x_0) \), for some \( \alpha^* \in A \), and all \( \theta \in \mathcal{V}_\Theta \)

▷ In this case, we would have:

\[
\nu(0, x_0, x_0) \overset{(ii)}{=} \mathbb{E}_{\theta^*}[V_T^{\alpha, \theta^*}] \overset{(i)}{=} J(\alpha, \theta^*), \quad \forall \alpha \in A
\]

\[
\nu(0, x_0, x_0) \overset{(iii)}{=} \mathbb{E}_\theta[V_T^{\alpha^*, \theta}] \overset{(i)}{=} J(\alpha^*, \theta), \quad \forall \theta \in \mathcal{V}_\Theta
\]

\( (1) + (2) \) and using the fact that \( \sup_{\alpha} \inf_{\theta} J(\alpha, \theta) \leq \inf_{\theta} \sup_{\alpha} J(\alpha, \theta) \implies \)

\[
\nu(0, x_0, x_0) = J(\alpha^*, \theta^*) = \sup_{\alpha \in A} \inf_{\theta \in \mathcal{V}_\Theta} J(\alpha, \theta) = \inf_{\theta \in \mathcal{V}_\Theta} \sup_{\alpha \in A} J(\alpha, \theta) = J_{\text{wc}}(\alpha^*) = V_0.
\]
Sketch of proof (II)

- Construction of the function \( \nu(t, x, \bar{x}) \): we look for \( \nu \) in the form

\[
\nu(t, x, \bar{x}) = K(t)(x - \bar{x})^2 + x + \chi(t), \quad t \in [0, T], x, \bar{x} \in \mathbb{R}.
\]

- Terminal condition (i) is satisfied when \( K(T) = -\lambda \), \( \chi(T) = 0 \).

- Take constant \( \theta^* = (b^*, \rho^*) \in \Theta \), apply Itô’s formula to \( V_{t}^{\alpha, \theta^*} = \nu(t, X_t^{\alpha}, \mathbb{E}_{\theta^*}[X_t^{\alpha}]) \), and take expectation under \( \mathbb{P}^{\theta^*} \): after square completion, we get

\[
\frac{d\mathbb{E}_{\theta^*}[V_{t}^{\alpha, \theta^*}]}{dt} = (\dot{K}(t) - K(t)R(\theta^*))\text{Var}_{\theta^*}(X_t) + \dot{\chi}(t) - \frac{1}{4K(t)}R(\theta^*)
\]

\[+ K(t)\mathbb{E}_{\theta^*}[(\alpha_t - \hat{a}(t, X_t, \mathbb{E}_{\theta^*}[X_t]))^\top\Sigma(\rho^*)(\alpha_t - \hat{a}(t, X_t, \mathbb{E}_{\theta^*}[X_t]))].\]

where \( \hat{a}(t, x, \bar{x}) = -\Sigma^{-1}(\rho^*)b^*(x - \bar{x} + \frac{1}{2}\frac{1}{K(t)}) \). By cancelling the red terms, and using condition (i), we get explicit form for \( K(t) < 0 \), and \( \chi(t) > 0 \), and thus

\[
\frac{d\mathbb{E}_{\theta^*}[V_{t}^{\alpha, \theta^*}]}{dt} \leq 0, \quad \forall \alpha \in \mathcal{A},
\]

which shows in particular condition (ii) : \( \mathbb{E}_{\theta^*}[V_{T}^{\alpha, \theta^*}] \leq V_0^{\alpha, \theta^*} \).
Sketch of proof (III)

- Choose now $\theta^* \in \arg \min_{\theta \in \Theta} R(\theta)$, and set $\alpha_t^* = \hat{a}(t, X_t^*, E_{\theta^*}[X_t^*]), \ t \in [0, T]$.

  The key lemma is the following saddle point property: define the function

  \[ H(\theta) := b^T \Sigma^{-1}(\rho^*) \Sigma(\rho) \Sigma^{-1}(\rho^*) b^*, \ \theta = (b, \rho) \in \Theta. \]

  Then, for all $\theta = (b, \rho) \in \Theta$,

  \[ H(b^*, \rho) \leq H(\theta^*) = R(\theta^*) \leq H(b, \rho^*). \]

  ▶ From this saddle-point property, we show that

  \[ E_{\theta}[V_T^{\alpha^*, \theta}] - V_0^{\alpha^*, \theta} = E_{\theta}[X_T^*] - \lambda \text{Var}_{\theta}(X_T^*) - \nu(0, x_0, x_0) \geq 0, \ \forall \theta \in \mathcal{V}_{\Theta}, \]

  i.e., condition (iii).

  Remark. Compared to the proof of condition (ii), it is not true that $t \in [0, T]$

  \[ \rightarrow E[V_t^{\alpha^*, \theta}] \text{ is nondecreasing.} \]
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In the sequel, we focus on

$$\Theta = \{(b, \rho) \in \mathbb{R}^d \times \Gamma : \left\| \sigma(\rho)^{-1}(b - \hat{b}) \right\|_2 \leq \delta \}.$$ 

We define the estimated **Sharpe ratios** (SR):

$$\hat{\beta}_i := \frac{\hat{b}_i}{\sigma_i}, \quad i = 1, \ldots, d,$$

ordered w.l.o.g.

$$|\hat{\beta}_1| \geq |\hat{\beta}_2| \geq \ldots \geq |\hat{\beta}_d|,$$

and define the **SR-proximity** between asset $i$ and $j$:

$$\hat{\varrho}_{ij} := \frac{\hat{\beta}_j}{\hat{\beta}_i} \in [-1, 1], \quad 1 \leq i < j \leq d.$$
Result 1: no risky investment under large ambiguity about expected return

- Suppose that there exists $\rho^* \in \Gamma$ solution to: $\arg \min_{\rho \in \Gamma} R(\hat{b}, \rho)$. Then, $\theta^* = (b^*, \rho^*)$ with

$$b^* = \left(1 - \frac{\delta}{\|\sigma(\rho^*)^{-1}\hat{b}\|_2}\right)1_{\{|\sigma(\rho^*)^{-1}\hat{b}\|_2 > \delta\}} \hat{b},$$

- Consequently: when $\delta \geq \|\sigma^{-1}(\rho^*)\hat{b}\|_2$,

$$\alpha^* \equiv 0.$$

Remark: $\rho^*$ exists when $\Gamma$ is a singleton, compact and also when it is equal to $\mathbb{C}^d_{> +}$ (see next slide).
Result 2: anti-diversification under total ambiguity about correlation

\( \Gamma = \mathbb{C}^d_{>+} \): suppose that |\( \hat{\beta}_1 \)\( < |\hat{\beta}_2| < \ldots < |\hat{\beta}_d| \). Then, \( \rho^* = (\rho^*_{ij})_{1 \leq i < j \leq d} \in \mathbb{C}^d_{>+} \) with

\[
\rho^*_{ij} = \hat{\rho}_{ij} := \frac{\hat{\beta}_j}{\hat{\beta}_i}, \quad 1 \leq i < j \leq d.
\]

Moreover, one invests only in the first asset

\[
\alpha_t^* = \left[ x_0 + \frac{1}{2\lambda} e^{(|\hat{\beta}_1| - \delta)^2} - X_t^* \right] \left( 1 - \frac{\delta}{|\hat{\beta}_1|} \right) \left( \frac{b_1}{\sigma_1^2}, 0, \ldots, 0 \right)^T 1_{|\hat{\beta}_1| > \delta}, \quad 0 \leq t \leq T, \quad \mathcal{P}^{\emptyset} - q.s.,
\]

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Result 3: two-asset model with partial ambiguity on drift and correlation: \( d = 2, \Gamma = [\rho, \bar{\rho}], \) with \(-1 < \rho \leq \bar{\rho} < 1\).

1. If \( \hat{\rho}_{12} \in [\rho, \bar{\rho}] \), then \( \rho^* = \hat{\rho}_{12} \), and

\[
\alpha_t^* = \left[ x_0 + \frac{1}{2\lambda} e(|\hat{\beta}_1| - \delta)^2 - X_t^* \right] \left( 1 - \frac{\delta}{|\hat{\beta}_1|} \right) 1\{|\hat{\beta}_1| > \delta\} \left( \frac{\hat{b}_1}{\sigma_1^2} \right),
\]

2. If \( \bar{\rho} < \hat{\rho}_{12} \), then \( \rho^* = \bar{\rho} \), and

\[
\alpha_t^* = \left[ x_0 + \frac{1}{2\lambda} e(\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2^2 - \delta)^2 - X_t^* \right] \left( 1 - \frac{\delta}{\|\sigma(\bar{\rho})^{-1}\hat{b}\|_2} \right) 1\{|\sigma(\bar{\rho})^{-1}\hat{b}\|_2 > \delta\} \Sigma(\bar{\rho})^{-1} \hat{b},
\]

with \( \alpha_{t^1}^* \alpha_{t^2}^* > 0 \) (directional trading), once \( \|\sigma(\bar{\rho})^{-1}\hat{b}\|_2 > \delta \).

3. If \( \rho > \hat{\rho}_{12} \), then \( \rho^* = \rho \), and

\[
\alpha_t^* = \left[ x_0 + \frac{1}{2\lambda} e(\|\sigma(\rho)^{-1}\hat{b}\|_2^2 - \delta)^2 - X_t^* \right] \left( 1 - \frac{\delta}{\|\sigma(\rho)^{-1}\hat{b}\|_2} \right) 1\{|\sigma(\rho)^{-1}\hat{b}\|_2 > \delta\} \Sigma(\rho)^{-1} \hat{b},
\]

with \( \alpha_{t^1}^* \alpha_{t^2}^* < 0 \) (spread trading), once \( \|\sigma(\rho)^{-1}\hat{b}\|_2 > \delta \).
Result 4: three-asset model with partial ambiguity on drift and correlation: \( d = 3, \Gamma = [\rho_{12}, \bar{\rho}_{12}] \times [\rho_{13}, \bar{\rho}_{13}] \times [\rho_{23}, \bar{\rho}_{23}] \subset \mathbb{C}_{> +}^3. \)

**Notations**: define the variance risk ratio

\[
\hat{\kappa}(\rho) = \begin{pmatrix}
\hat{\kappa}_1(\rho) \\
\hat{\kappa}_2(\rho) \\
\hat{\kappa}_3(\rho)
\end{pmatrix} := \Sigma(\rho)^{-1} \hat{b}, \quad \rho \in \Gamma,
\]

and recall

\[
\hat{\varrho}_{ij} := \frac{\hat{\beta}_j}{\hat{\beta}_i} \in [-1, 1], \quad 1 \leq i < j \leq 3.
\]

- There are 11 possible exclusive cases depending on \( \hat{\varrho}_{ij}, \rho_{ij}, \) and \( \bar{\rho}_{ij}, \)

1 \( \leq i < j \leq 3. \)
(A) High-level correlation ambiguity for the second and third assets

If $\hat{\rho}_{12} \in [\underline{\rho}_{12}, \bar{\rho}_{12}]$, $\hat{\rho}_{13} \in [\underline{\rho}_{13}, \bar{\rho}_{13}]$, then $\rho^* = (\hat{\rho}_{12}, \hat{\rho}_{13}, \rho_{23})$ for any $\rho_{23} \in [\underline{\rho}_{23}, \bar{\rho}_{23}]$, and

$$\alpha_t^* = \left[ x_0 + \frac{1}{2\lambda} e(|\hat{\beta}_1| - \delta)^2 T - X_t^* \right] \left( 1 - \frac{\delta}{|\hat{\beta}_1|} \right) 1_{\{|\hat{\beta}_1| > \delta\}} \begin{pmatrix} \frac{\hat{b}_1}{\sigma_1^2} \\ 0 \\ 0 \end{pmatrix}. $$
Under-diversification: no investment in the first asset

(U1) High-level correlation ambiguity for the first asset

(i) If \( \bar{\rho}_{23} < \hat{\rho}_{23} \), \( \hat{\kappa}^1(\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) \hat{\kappa}^1(\underline{\rho}_{12}, \underline{\rho}_{13}, \underline{\rho}_{23}) \leq 0 \), then \( \rho^* = (\rho^*_{12}, \rho^*_{13}, \bar{\rho}_{23}) \) with \( (\rho^*_{12}, \rho^*_{13}) \) satisfying \( \hat{\kappa}^1(\rho^*_{12}, \rho^*_{13}, \bar{\rho}_{23}) = 0 \), and

\[ \alpha^2_{t^*} \alpha^3_{t^*} > 0, \quad \alpha^1_{t^*} = 0. \]

(once \( \delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2 \))

(ii) If \( \underline{\rho}_{23} > \hat{\rho}_{23} \), \( \hat{\kappa}^1(\underline{\rho}_{12}, \underline{\rho}_{13}, \rho_{23}) \hat{\kappa}^1(\rho_{12}, \rho_{13}, \rho_{23}) \leq 0 \), then \( \rho^* = (\rho^*_{12}, \rho^*_{13}, \rho_{23}) \) with \( (\rho^*_{12}, \rho^*_{13}) \) satisfying \( \hat{\kappa}^1(\rho^*_{12}, \rho^*_{13}, \rho_{23}) = 0 \), and

\[ \alpha^2_{t^*} \alpha^3_{t^*} < 0, \quad \alpha^1_{t^*} = 0. \]

(once \( \delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2 \))
Well-diversification: small ambiguity about correlation

(W)

(i) If $\hat{\kappa}_1 \hat{\kappa}_2 (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, and $\hat{\kappa}_1 \hat{\kappa}_3 (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, then $\rho^* = (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, and one invests in all the assets (provided that $\delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2$) with global directional trading $(\alpha^1_t, \alpha^2_t, \alpha^3_t > 0, \alpha^1_t, \alpha^3_t > 0)$

(ii) If $\hat{\kappa}_1 \hat{\kappa}_2 (\rho_{12}, \rho_{13}, \rho_{23}) < 0$, and $\hat{\kappa}_1 \hat{\kappa}_3 (\rho_{12}, \rho_{13}, \rho_{23}) < 0$, then $\rho^* = (\rho_{12}, \rho_{13}, \rho_{23})$, and one invests in all the assets (provided that $\delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2$) with spread trading w.r.t. first asset $(\alpha^1_t, \alpha^2_t < 0, \alpha^1_t, \alpha^3_t < 0)$

(iii) If $\hat{\kappa}_1 \hat{\kappa}_2 (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) > 0$, and $\hat{\kappa}_1 \hat{\kappa}_3 (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23}) < 0$, then $\rho^* = (\bar{\rho}_{12}, \bar{\rho}_{13}, \bar{\rho}_{23})$, and one invests in all the assets (provided that $\delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2$) with directional and spread trading w.r.t. first asset $(\alpha^1_t, \alpha^2_t, \alpha^3_t > 0, \alpha^1_t, \alpha^3_t < 0)$

(iv) If $\hat{\kappa}_1 \hat{\kappa}_2 (\rho_{12}, \rho_{13}, \rho_{23}) < 0$, and $\hat{\kappa}_1 \hat{\kappa}_3 (\rho_{12}, \rho_{13}, \rho_{23}) > 0$, then $\rho^* = (\rho_{12}, \rho_{13}, \rho_{23})$, and one invests in all the assets (provided that $\delta < \|\sigma^{-1}(\rho^*)\hat{b}\|_2$) with spread and directional trading w.r.t. first asset $(\alpha^1_t, \alpha^2_t < 0, \alpha^1_t, \alpha^3_t > 0)$
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Conclusion

- Continuous-time model for ambiguity on expected return and correlation matrix of multi-asset
- Separation principle for solving the robust mean-variance portfolio selection problem

➢ Quantitative results documented by empirical studies:
  - No risky investment when expected return is sufficiently ambiguous
  - Anti-diversification when correlations are sufficiently ambiguous
  - Under-diversification: stocks with high correlations are omitted from optimal portfolios
  - Well-diversification when correlation is weakly ambiguous
Thank you for your attention!