Duality for almost-sure hedging with price impact

B. Bouchard

CEREMADE, Dauphine-PSL University

Based on works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich) and Y. Zou (ex Dauphine-PSL) + more recent developments with P. Cardialaguet (Dauphine-PSL) and X. Tan (Dauphine-PSL)
Problem formulation and motivation
FBSDE with second order impact

Given $x \in C([0, T])$, find $y \in \mathbb{R}$ and $(g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

$$X = x \wedge 0 + \int_0^T \sigma_t(X, g_t) dW_t$$

$$Y = y + \int_0^T g_t dX_t + \mathcal{B}$$

$$V = \Xi(X) - \int_0^T F_t(X, g_t) dt - \int_0^T Y_t dX_t, \quad (adapted)$$

(possibly weak formulation)
FBSDE with second order impact

Given $x \in C([0, T])$, find $y \in \mathbb{R}$ and $(g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

$$X = x \wedge 0 + \int_0^T \sigma_t(X, g_t) dW_t$$

$$Y = y + \int_0^T g_t dX_t + \mathcal{B}$$

$$V = \Xi(X) - \int_0^T F_t(X, g_t) dt - \int_0^T Y_t dX_t, \quad (adapted)$$

(possibly weak formulation)

**Interpretation:**

- $X$ : stock price,
- $Y$ : number of stocks in the portfolio,
- $V$ : cash value of the portfolio (at the current stock price),
- $F(\cdot, g)$ and $\sigma(\cdot, g)$ : liquidity cost and price impact.
Example

Linear impact rule and covered options: buying $\Delta_t$ stocks leads to
- a permanent price move of $X_{t-} \rightarrow X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$. 

\[ dX_t = f_t(X_t)\, dW_t \]
**Example**

**Linear impact rule and covered options**: buying $\Delta_t$ stocks leads to
- a permanent price move of $X_{t^-} \to X_t = X_{t^-} + f_t(X_{t^-}) \Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t^-} + X_t)$.

When no trading, the stock evolves according to
\[ dX_t = \sigma_t(X_t) dW_t. \]
Example

Linear impact rule and covered options: buying $\Delta_t$ stocks leads to

- a permanent price move of $X_{t-} \rightarrow X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$.

When no trading, the stock evolves according to

$$dX_t = \sigma_t^0(X_t)dW_t.$$ 

Consider rebalancing at times $t^n_i$:

$$X^n = X_0 + \int_0^t \sigma^0(X^n_t)dW_t + \sum_{i=1}^n 1_{[t^n_i, T]} f(X^n_{t^n_i-})\Delta^n_{t^n_i},$$

$$Y^n := \sum_{i=0}^{n-1} 1_{[t^n_i, t^n_{i+1}]} \left( \int_0^t g_t dX^n_t + \int_0^t b_t dt \right), \quad \Delta^n_{t^n_i} = Y^n_{t^n_i} - Y^n_{t^n_{i-1}}$$

$$V^n = V_0 + \sum_{i=1}^n 1_{[t^n_i, T]} \frac{1}{2}(\Delta^n_{t^n_i})^2 f(X^n_{t^n_i-}) + \int_0^T Y^n_{t-} dX^n_t,$$

where

$$V^n = \text{cash part} + Y^n X^n = \text{“portfolio value”}.$$
Example

Linear impact rule and covered options: buying $\Delta_t$ stocks leads to

- a permanent price move of $X_{t-} \to X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$.

When no trading, the stock evolves according to

$$dX_t = \sigma_t^\circ(X_t)dW_t.$$ 

$\Rightarrow$ Let $t_{i+1}^n - t_i^n \to 0$:

$$X = x^0 + \int_0^{t^n} \sigma_t^\circ(X_t)dW_t + \int_0^{t^n} f_t(X_t)dY_t + \int_0^{t^n} g_t(f_t'X_t)(X_t)dt$$

$$Y = y + \int_0^{t^n} g_t dX_t + \int_0^{t^n} b_t dt$$

$$V = V_0 + \int_0^{t^n} \frac{1}{2} g_t^2 f_t(X_t)dt + \int_0^{t^n} Y_t dX_t.$$
Example

Linear impact rule and covered options: buying $\Delta_t$ stocks leads to

- a permanent price move of $X_{t-} \rightarrow X_t = X_{t-} + f_t(X_{t-})\Delta_t$,
- an average buying cost of $\frac{1}{2}(X_{t-} + X_t)$.

When no trading, the stock evolves according to

$$dX_t = \sigma_t(X_t)dW_t.$$  

$\Rightarrow$ Let $t^n_{i+1} - t^n_i \rightarrow 0$:

$$X = x_0 + \int_0^{t^n} \frac{\sigma_t(X)}{1 - f_t(X_t)g_t} dW_t + \int_0^{t^n} (\cdots) dt$$

$$Y = y + \int_0^{t^n} g_t dX_t + \int_0^{t^n} b_t dt$$

$$V = V_0 + \int_0^{t^n} \frac{1}{2} g_t^2 f_t(X_t) dt + \int_0^{t^n} Y_t dX_t.$$
Example

Linear impact rule and resilience

\[ X = X_0 + \int_0^\cdot \sigma_s^\circ(X_s)dW_s + R \]
\[ R = R_0 + \int_0^\cdot f_s(X_s)dY_s + \int_0^\cdot (g_s(f_s'\sigma_s^\circ)(X_s) - \rho R_s)ds \]
\[ Y = y + \int_0^\cdot g_t dX_t + \int_0^\cdot b_t dt \]
\[ V = \Xi(X) - \int_0^T \frac{1}{2} g_t^2 f_t(X_t) dt - \int_0^T Y_t dX_t. \]
Example

Linear impact rule and resilience

\[ X = X_0 + \int_0^\cdot \sigma_s^\circ(X_s)\,dW_s + R \]
\[ R = R_0 + \int_0^\cdot f_s(X_s)\,dY_s + \int_0^\cdot (g_s(f'_s\sigma_s^\circ)(X_s) - \rho R_s)\,ds \]
\[ Y = y + \int_0^\cdot g_t\,dX_t + \int_0^\cdot b_t\,dt \]
\[ V = \Xi(X) - \int_0^T \frac{1}{2} g_t^2 f_t(X_t)\,dt - \int_0^T Y_t\,dX_t. \]

Resilience does not play any role... we omit it.
PDE point of view

B. Bouchard, G. Loeper, and Y. Zou.
Almost-sure hedging with permanent price impact.

B. Bouchard, G. Loeper, and Y. Zou.
Hedging of covered options with linear market impact and gamma constraint.

Second order stochastic target problems with generalized market impact.

G. Loeper,
Option Pricing with Market Impact and Non-Linear Black and Scholes Equations,
*arXiv:1301.6252v3*
Markovian setting

Given \( x \in \mathbb{R} \), find \( y \in \mathbb{R} \) and \( \phi := (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2 \) such that

\[
X = x + \int_0^t \sigma_t(X_t, g_t) dW_t
\]

\[
Y = y + \int_0^t g_t dX_t + \mathcal{B}
\]

\[
V = \Xi(X_T) - \int_0^T F_t(X_t, g_t) dt - \int_0^T Y_t dX_t, \quad (\text{adapted})
\]
Markovian setting

Given $x \in \mathbb{R}$, find $y \in \mathbb{R}$ and $\phi := (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

$$X = x + \int_0^\cdot \sigma_t(X_t, g_t) dW_t$$

$$Y = y + \int_0^\cdot g_t dX_t + \mathcal{B}$$

$$V = \Xi(X_T) - \int_0^T F_t(X_t, g_t) dt - \int_0^T Y_t dX_t, \quad (adapted)$$

Assume a solution $V = v(\cdot, X)$ exists, then $dV = dv(\cdot, X)$ and therefore:

- $Y = \nabla_x v(\cdot, X)$,
- $F(X, g) = \partial_t v(\cdot, X) + \frac{1}{2} \sigma^2(X, g) \nabla_{xx} v(\cdot, X)$
Markovian setting

Given \(x \in \mathbb{R}\), find \(y \in \mathbb{R}\) and \(\phi := (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2\) such that

\[
X = x + \int_{0}^{\cdot} \sigma_t(X_t, g_t) dW_t
\]

\[
Y = y + \int_{0}^{\cdot} g_t dX_t + \mathcal{B}
\]

\[
V = \Xi(X_T) - \int_{\cdot}^{T} F_t(X_t, g_t) dt - \int_{\cdot}^{T} Y_t dX_t, \quad (adapted)
\]

Assume a solution \(V = v(\cdot, X)\) exists, then \(dV = dv(\cdot, X)\) and therefore:

- \(Y = \nabla_x v(\cdot, X)\),
- \(F(X, g) = \partial_t v(\cdot, X) + \frac{1}{2} \sigma^2(X, g) \nabla_{xx} v(\cdot, X)\)

Moreover, \(Y = \nabla_x v(\cdot, X)\) implies \(dY = d\nabla_x v(\cdot, X)\) and therefore

- \(g = \nabla_{xx} v(\cdot, X)\),
Markovian setting

Given $x \in \mathbb{R}$, find $y \in \mathbb{R}$ and $\phi := (g, \mathcal{B}) \in \mathcal{A}_2 \times \mathcal{B}_2$ such that

$$X = x + \int_0^\cdot \sigma_t(X_t, g_t) dW_t$$

$$Y = y + \int_0^\cdot g_t dX_t + \mathcal{B}$$

$$V = \Xi(X_T) - \int_0^T F_t(X_t, g_t) dt - \int_0^T Y_t dX_t, \quad \text{(adapted)}$$

Assume a solution $V = v(\cdot, X)$ exists, then $dV = dv(\cdot, X)$ and therefore:

- $Y = \nabla_x v(\cdot, X)$,
- $F(X, \nabla_{xx} v(\cdot, X)) = \partial_t v(\cdot, X) + \frac{1}{2} \sigma^2(X, \nabla_{xx} v(\cdot, X)) \nabla_{xx} v(\cdot, X)$

Moreover, $Y = \nabla_x v(\cdot, X)$ implies $dY = d\nabla_x v(\cdot, X)$ and therefore:

- $g = \nabla_{xx} v(\cdot, X)$,
Markovian setting

This leads to the PDE:

\[ 0 = -\partial_t v(\cdot, x) - \frac{1}{2}\sigma^2(x, \nabla_{xx} v(\cdot, x))\nabla_{xx} v(\cdot, x) + F(x, \nabla_{xx} v(\cdot, x)) \]
Markovian setting

This leads to the PDE:

\[ 0 = - \partial_t v(\cdot, x) - \frac{1}{2} \sigma^2(x, \nabla_{xx} v(\cdot, x)) \nabla_{xx} v(\cdot, x) + F(x, \nabla_{xx} v(\cdot, x)) \]
\[ = - \partial_t v(\cdot, x) - \tilde{F}(x, \nabla_{xx} v(\cdot, x)) \]

with

\[ \tilde{F}(x, g) := \frac{1}{2} \sigma^2(x, g) g - F(x, g). \]

and terminal condition

\[ v(T, \cdot) = \Xi. \]
In this case

\[ F(x, g) = \frac{1}{2} \left( \frac{\sigma^\circ(x)g}{1 - f(x)g} \right)^2 f(x) \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}} \]

\[ \bar{F}(x, g) = \frac{1}{2} \frac{\sigma^\circ(x)^2g}{1 - f(x)g} \mathbf{1}_{\{f(x)g < 1\}} + \infty \mathbf{1}_{\{f(x)g \geq 1\}}. \]
In this case

\[
F(x, g) = \frac{1}{2} \left( \frac{\sigma^\circ(x)g}{1 - f(x)g} \right)^2 f(x)1\{f(x)g < 1\} + \infty 1\{f(x)g \geq 1\}
\]

\[
\bar{F}(x, g) = \frac{1}{2} \frac{\sigma^\circ(x)^2 g}{1 - f(x)g} 1\{f(x)g < 1\} + \infty 1\{f(x)g \geq 1\}.
\]

**Gamma constraint:** \(\{\bar{F}(x, g) < \infty\} = \{g < \gamma(x)\}\), where \(\gamma := 1/f\) in the linear case.
Markovian setting - Linear impact case

In this case

\[
F(x, g) = \frac{1}{2} \left( \frac{\sigma^\circ(x)g}{1 - f(x)g} \right)^2 f(x) 1\{f(x)g < 1\} + \infty 1\{f(x)g \geq 1\}
\]

\[
\bar{F}(x, g) = \frac{1}{2} \frac{\sigma^\circ(x)^2 g}{1 - f(x)g} 1\{f(x)g < 1\} + \infty 1\{f(x)g \geq 1\}.
\]

Gamma constraint : \(\{\bar{F}(x, g) < \infty\} = \{g < \gamma(x)\}\), where \(\gamma := 1/f\) in the linear case.

In general, restrict to \(g\) such that \(g < \gamma(X)\) and the terminal condition \(\Xi\) is replaced by the smallest function above \(\Xi\) satisfying the gamma constraint.
Markovian setting - Convex case

Assume that: \( g \mapsto \tilde{F}(x, g) \) is convex (as in the linear impact case).
Markovian setting - Convex case

Assume that: $g \mapsto \bar{F}(x, g)$ is convex (as in the linear impact case).

Then,

$$0 = - \partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x))$$

$$= \inf_{s \in \mathbb{R}} \left( - \partial_t v(\cdot, x) - \frac{1}{2} s^2 \nabla_{xx} v(\cdot, x) + \bar{F}^*(x, s) \right)$$

where

$$\bar{F}^*(\cdot, s) := \sup_{g < \gamma} \left( \frac{1}{2} s^2 g - \bar{F}(\cdot, g) \right),$$

so that

$$\bar{F}(\cdot, g) := \sup_{s \in \mathbb{R}} \left( \frac{1}{2} s^2 g - \bar{F}^*(\cdot, s) \right).$$
Markovian setting - Convex case (continued)

If \( v \) solves

\[
0 = - \partial_t v(\cdot, x) - \bar{F}(x, \nabla_{xx} v(\cdot, x))
\]

\[
= \inf_{s \in \mathbb{R}} \left( -\partial_t v(\cdot, x) - \frac{1}{2} s^2 \nabla_{xx} v(\cdot, x) + \bar{F}^*(x, s) \right)
\]

then

\[
v(0, x) = \bar{v}(0, x) := \sup_{s \in A_2} \mathbb{E} \left[ \Xi(\bar{X}_T^s) - \int_0^T \bar{F}_t^*(\bar{X}_t^s, s_t) dt \right]
\]

with

\[
\bar{X}_t^s := x + \int_0^t s_t dW_t.
\]

\( \Rightarrow \) Dual formulation!
General Path dependent case
It is non-trivial...

Let us consider a super-solution

\[ V_0 + \int_0^T Y_t dX_t + \int_0^T F_t(X, g_t) dt \geq \Xi(X). \]

Then, \( \exists (g, \mathcal{B}) \) such that

\[ V_0 \geq \mathbb{E} \left[ \Xi(X) - \int_0^T F_t(X, g_t) dt \right] \]
It is non-trivial...

Let us consider a super-solution

\[ V_0 + \int^T Y_t dX_t + \int^T F_t(X, g_t) dt \geq \Xi(X). \]

Then, \( \exists (g, \mathcal{B}) \) such that

\[ V_0 \geq \mathbb{E} \left[ \Xi(X) - \int^T F_t(X, g_t) dt \right] \]

but we do not have

\[ V_0 \geq \sup_G \mathbb{E} \left[ \Xi(X) - \int^T F_t(X, g_t) dt \right]. \]
It is non-trivial...

Let us consider a super-solution

\[ V_0 + \int_0^T Y_t dX_t + \int_0^T F_t(X, g_t) dt \geq \Xi(X). \]

Then, \( \exists (g, \mathcal{B}) \) such that

\[ V_0 \geq \mathbb{E} \left[ \Xi(X) - \int_0^T F_t(X, g_t) dt \right] \]

but we do not have

\[ V_0 \geq \sup_g \mathbb{E} \left[ \Xi(X) - \int_0^T F_t(X, g_t) dt \right]. \]

One needs a solution to deduce something:

\[ V_0 = \mathbb{E} \left[ \Xi(X) - \int_0^T F_t(X, g_t) dt \right] \leq \sup_g \mathbb{E} \left[ \Xi(X) - \int_0^T F_t(X, g_t) dt \right]. \]
Dupire derivative of the gain function and calculus of variation

**Assumption**: \( \bar{v}(t, x) \) admits a solution \( \hat{s}[t, x] \) (need weak...) + smoothness assumptions.
Dupire derivative of the gain function and calculus of variation

**Assumption**: \( \bar{v}(t, x) \) admits a solution \( \hat{s}[t, x] \) (need weak...) + smoothness assumptions.

**Result #1**: The gain function

\[
J(t, x; \bar{s}) := \mathbb{E} \left[ \Xi(\bar{X}^{t,x,\bar{s}}) - \int_t^T \bar{F}_r^*(\bar{X}^{t,x,\bar{s}}, \bar{s}_r) dr \right],
\]

\[
\bar{X}^{t,x,\bar{s}} := x + \int_t^\cdot \bar{s}_r d\mathcal{W}_r,
\]

admits a Dupire vertical derivative

\[
\nabla_x J(t, x; \bar{s}) := \mathbb{E} \left[ \mathcal{B}^{x,\bar{s}}_T - \mathcal{B}^{x,\bar{s}}_t \right]
\]

where \( \mathcal{B}^{x,\bar{s}} \) is an adapted BV process.
Dupire derivative of the gain function and calculus of variation (continued)

Result #2: By a simple calculus of variations argument,

$$\partial_s \bar{F}^*(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t, x]$$

where $(m[t, x], \beta[t, x])$ is the element of $\mathbb{R} \times \mathcal{A}_2$ such that

$$m[t, x] + \int_t^T \beta[t, x]_u dW_u = \mathcal{B}^{x, \hat{s}[t,x]}_T - \mathcal{B}^{x, \hat{s}[t,x]}_t.$$
Dupire derivative of the gain function and calculus of variation (continued)

Result #2: By a simple calculus of variations argument,

\[ \partial_s \bar{F}^* (\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t,x] \]

where \((m[t,x], \beta[t,x])\) is the element of \(\mathbb{R} \times A_2\) such that

\[ m[t,x] + \int_t^T \beta[t,x] u dW_u = \mathcal{B}^{x, \hat{s}[t,x]}_T - \mathcal{B}^{x, \hat{s}[t,x]}_t. \]

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) := \mathbb{E} \left[ \mathcal{B}^{x, \hat{s}[t,x]}_T - \mathcal{B}^{x, \hat{s}[t,x]}_T | \mathcal{F} \right], \)
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2**: By a simple calculus of variations argument,

\[ \partial_s \bar{F}(\tilde{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t,x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x] u dW_u = \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]} \]

Since, \(\nabla_x J(\cdot, \tilde{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t,x]) := \mathbb{E}\left[ \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]} | \mathcal{F} \right] \),

\[ \hat{Y}[t, x] := m[t, x] + \int_0^t \beta[t, x] u dW_u - (\mathcal{B}_t^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]}) \]

satisfies

\[ \hat{Y}[t, x] = \nabla_x J(\cdot, \tilde{X}^{t,x,\hat{s}[t,x]}; \hat{s}[t, x]). \]
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2** : By a simple calculus of variations argument,

\[ \partial_s \tilde{F}^*(\tilde{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t, x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x] u d\mathcal{W}_u = \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]} \]

Since, \(\nabla_x J(\cdot, \tilde{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t, x]) := \mathbb{E}\left[ \mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]} | \mathcal{F} \right] \),

\[ \hat{Y}[t, x] := m[t, x] + \int_0^t \beta[t, x] u d\mathcal{W}_u - (\mathcal{B}_T^{x,\hat{s}[t,x]} - \mathcal{B}_t^{x,\hat{s}[t,x]}) \]

satisfies

\[ \hat{Y}[t, x] = \nabla_x J(\cdot, \tilde{X}^{t,x,\hat{s}[t,x]}, \hat{s}[t, x]). \]
Dupire derivative of the gain function and calculus of variation (continued)

**Result #2:** By a simple calculus of variations argument,

\[ \partial_s \bar{F}^*(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) = \beta[t, x] \]

where \((m[t, x], \beta[t, x])\) is the element of \(\mathbb{R} \times \mathcal{A}_2\) such that

\[ m[t, x] + \int_t^T \beta[t, x] u dW_u = \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} \]

Since, \(\nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) := \mathbb{E} \left[ \mathcal{B}_T^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} \big| \mathcal{F} \right] \),

\[ \hat{Y}[t, x] := m[t, x] + \int_0^t \partial_s \bar{F}^*_u(\bar{X}^{t,x}, \hat{s}[t,x], \hat{s}[t,x]) u dW_u - (\mathcal{B}_x^{x, \hat{s}[t,x]} - \mathcal{B}_t^{x, \hat{s}[t,x]} \big) \]

satisfies

\[ \hat{Y}[t, x] = \nabla_x J(\cdot, \bar{X}^{t,x}, \hat{s}[t,x]; \hat{s}[t,x]). \]
Regularity of the value function

**Assumption**: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).
Regularity of the value function

**Assumption**: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

**Result #3**: Set

$$
\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x_{\land t} + 1_{\{t\}}(y^2 - x_t)) dy^2 dy^1,
$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y)$ is concave.

Recall that:

$$
J(t, x; s) := \mathbb{E} \left[ \Xi(\bar{X}^{t,x,s}) - \int_t^T \bar{F}^*_r(\bar{X}^{t,x,s}, s_r) dr \right],
$$

$$
\bar{X}^{t,x,s} := x_{\land t} + \int_t^s s_r dW_r,
$$
Regularity of the value function

**Assumption:** $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

**Result #3:** Set

$$\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x_t + 1_{\{t\}}(y^2 - x_t)) dy^2 dy^1,$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y)$ is concave.

**Result #4:** $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x] T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}$$
Regularity of the value function

Assumption: $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

Result #3: Set

$$\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \wedge t + 1_{\{t\}} (y^2 - x_t)) dy^2 dy^1,$$
then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}} y)$ is concave.

Result #4: $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}$$
and (Meyer-Tanaka + martingale property)

$$\bar{v}(t', \bar{X}^{t, x, \hat{s}[t, x]}) = \bar{v}(t, x) + \int_t^{t'} \nabla_x \bar{v}(r, \bar{X}^{t, x, \hat{s}[t, x]}) d\bar{X}^{t, x, \hat{s}[t, x]}$$
$$+ \int_t^{t'} \bar{F}^*(r, \bar{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]) dr.$$
Regularity of the value function

**Assumption:** $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

**Result #3:** Set

$$\Gamma(t, x) = \int_0^{x_t} \int_0^{y_1} \gamma_t(x \wedge t + 1\{t\}(y^2 - x_t))dy^2dy^1,$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1\{t\} y)$ is concave.

**Result #4:** $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E} \left[ \hat{\mathcal{B}}[t, x]_T - \hat{\mathcal{B}}[t, x]_t \right], \quad \hat{\mathcal{B}}[t, x] := \mathcal{B}^{x, \hat{s}[t, x]}$$

and (Meyer-Tanaka + martingale property)

$$\bar{v}(t', \bar{X}^{t, x, \hat{s}[t, x]} ) = \bar{v}(t, x) + \int_t^{t'} \nabla_x \bar{v}(r, \bar{X}^{t, x, \hat{s}[t, x]} ) d\bar{X}^{t, x, \hat{s}[t, x]}$$

$$+ \int_t^{t'} \bar{F}^*(r, \bar{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]_r)dr.$$
Regularity of the value function

**Assumption** : $\bar{F}$ is bounded from below (by a map with linear growth in $x$).

**Result #3** : Set

$$\Gamma(t, x) = \int_0^{\gamma_0} \int_0^{\gamma_1} \gamma_t(x \land t + 1_{\{t\}}(y^2 - x_t))dy^2dy^1,$$

then $y \mapsto (\bar{v} - \Gamma)(t, x + 1_{\{t\}}y)$ is concave.

**Result #4** : $\bar{v}$ admits a continuous vertical Dupire derivative given by

$$\nabla_x \bar{v}(t, x) = \nabla_x J(t, x; \hat{s}[t, x]) := \mathbb{E}\left[\mathbb{B}[t, x]_T - \mathbb{B}[t, x]_t\right], \quad \mathbb{B}[t, x] := \mathbb{B}^{x, \hat{s}[t, x]}$$

and (Meyer-Tanaka + martingale property)

$$\bar{v}(t', \bar{X}^{t, x, \hat{s}[t, x]}) = \bar{v}(t, x) + \int_t^{t'} \hat{Y}[t, x]_r d\bar{X}^{t, x, \hat{s}[t, x]}_r + \int_t^{t'} \bar{F}^*(r, \bar{X}^{t, x, \hat{s}[t, x]}, \hat{s}[t, x]_r)dr.$$
Construction of the hedging strategy

Assumption: \((\partial_g \sigma^2) g = \partial_g \bar{F}\) (satisfied in the linear impact model).
Construction of the hedging strategy

**Assumption**: \((\partial_g \sigma^2)g = \partial_g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\bar{v}(T, \bar{X}^x, \hat{s}[t, x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*_r(\bar{X}^x, \hat{s}[x], \hat{s}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \partial_s \bar{F}^*_t(\bar{X}^x, \hat{s}[x], \hat{s}[x]_t) d\mathcal{W}_t - (\hat{\mathbb{B}}[x] - \hat{\mathbb{B}}[x]_0).
\]
Construction of the hedging strategy

**Assumption:** \((\partial g \sigma^2) g = \partial g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\bar{v}(T, \bar{X}^x, \hat{s}[t, x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*_t(\bar{X}^x, \hat{s}[x], \hat{g}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \partial_s \bar{F}^*_t(\bar{X}^x, \hat{s}[x], \hat{g}[x]_t) d\mathcal{W}_t - (\hat{\mathcal{B}}[x] - \hat{\mathcal{B}}[x]_0).
\]

Under the above assumption, for \(\hat{g}[x] \hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{g}[x])\),

\[
\bar{X}^x, \hat{s}[x] = X^x, \hat{g}[x], \hat{S}[x], \quad \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{g}[x]) = F(X^x, \hat{g}[x], \hat{S}[x], \hat{g}[x]).
\]
Construction of the hedging strategy

**Assumption**: \((\partial_g \sigma^2)g = \partial_g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\bar{v}(T, \bar{X}^x, \hat{s}[t,x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*_r(\bar{X}^x, \hat{s}[x], \hat{s}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \partial_s \bar{F}^*_t(\bar{X}^x, \hat{s}[x], \hat{s}[x]_t) d\mathcal{W}_t - (\bar{B}[x] - \hat{B}[x]_0).
\]

Under the above assumption, for \(\hat{g}[x] \hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x])\),

\[
\bar{X}^x, \hat{s}[x] = X^x, \hat{g}[x], \hat{B}[x], \quad \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]) = F(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x]).
\]
Construction of the hedging strategy

**Assumption**: $(\partial_g \sigma^2)g = \partial_g \bar{F}$ (satisfied in the linear impact model).

Recall that $\bar{v}(T, \cdot) = \Xi$ and that

$$\bar{v}(T, \bar{X}^x, \hat{s}[t, x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T \bar{F}^*_r(\bar{X}^x_r, \hat{s}[x]_r, \hat{s}[x]_r) dr,$$

$$\hat{Y}[x] := m[x] + \int_0^T \hat{g}[x]_t \hat{s}[x]_t dW_t - (\hat{B}[x] - \hat{B}[x]_0).$$

Under the above assumption, for $\hat{g}[x] \hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]),$

$$\bar{X}^x, \hat{s}[x] = X^x, \hat{g}[x], \hat{B}[x], \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]) = F(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x]).$$
Construction of the hedging strategy

**Assumption:** \((\partial_g \sigma^2)g = \partial_g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\bar{v}(T, \vec{X}^x, \hat{\sigma}[t,x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\vec{X}^x_r, \hat{\sigma}[x] + \int_0^T \bar{F}^*_r(\vec{X}^x, \hat{\sigma}[x], \hat{\sigma}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \hat{g}[x]_t \hat{\sigma}[x]_t dW_t - (\hat{\mathcal{B}}[x] - \hat{\mathcal{B}}[x]_0).
\]

Under the above assumption, for \(\hat{g}[x] \hat{\sigma}[x] := \partial_s \bar{F}^*(\vec{X}^x, \hat{\sigma}[x], \hat{\sigma}[x])\),

\[
\vec{X}^x, \hat{\sigma}[x] = \vec{X}^x, \hat{\sigma}[x], \hat{\mathcal{B}}[x], \quad \bar{F}^*(\vec{X}^x, \hat{\sigma}[x], \hat{\sigma}[x]) = F(\vec{X}^x, \hat{\sigma}[x], \hat{\mathcal{B}}[x], \hat{\sigma}[x])
\]
Construction of the hedging strategy

Assumption : \((\partial_g \sigma^2)g = \partial_g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\Xi(X^x, \hat{g}[x], \hat{B}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r \, d\bar{X}^x_r, \hat{s}[x] + \int_0^T F_r(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x]_r) \, dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \hat{g}[x]_t \, dX^x_t, \hat{s}[x], \hat{B}[x] - (\hat{B}[x] - \hat{B}[x]_0).
\]

Under the above assumption, for \(\hat{g}[x] \hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]).\)

\[
\bar{X}^x, \hat{s}[x] = X^x, \hat{g}[x], \hat{B}[x], \quad \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]) = F(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x]).
\]
Construction of the hedging strategy

**Assumption :** \((\partial_g \sigma^2)g = \partial_g \bar{F}\) (satisfied in the linear impact model).

Recall that \(\bar{v}(T, \cdot) = \Xi\) and that

\[
\Xi(X^x, \hat{g}[x], \hat{B}[x]) = \bar{v}(0, x) + \int_0^T \hat{Y}[x]_r d\bar{X}^x_r, \hat{s}[x] + \int_0^T F_r(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x]_r) dr,
\]

\[
\hat{Y}[x] := m[x] + \int_0^T \hat{g}[x]_t dX^x_t, \hat{s}[x], \hat{B}[x] - (\bar{B}[x] - \hat{B}[x]_0).
\]

Under the above assumption, for \(\hat{g}[x] \hat{s}[x] := \partial_s \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]),\)

\[
\bar{X}^x, \hat{s}[x] = X^x, \hat{g}[x], \hat{B}[x], \bar{F}^*(\bar{X}^x, \hat{s}[x], \hat{s}[x]) = F(X^x, \hat{g}[x], \hat{B}[x], \hat{g}[x])
\]

\Rightarrow \hat{s}[x] provides \((\hat{g}[x], -\hat{B}[x])\) which is the hedging strategy starting from \(V_0 = \bar{v}(0, x)\) and \(Y_0 = \nabla_x \bar{v}(0, x).\)
Conclusion and open question

- **Conclusion**: In a fairly general path-dependent setting, solving the dual problem provides one solution to the BSDE with second order impact (or the hedging problem).

- **Open question**: In the Markovian setting, and under strong smoothness conditions, the minimal super-solution is a solution: the super-hedging price is a hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments? Main issue: the terminal condition \( \mathcal{F}(X) \) depends on the hedging strategy -> standard comparison does not hold.
Conclusion and open question

- **Conclusion**: In a fairly general path-dependent setting, solving the dual problem provides one solution to the BSDE with second order impact (or the hedging problem).

- **Open question**: In the Markovian setting, and under strong smoothness conditions, the minimal super-solution is a solution: the super-hedging price is a hedging price. How to prove this in the path-dependent case by simply using probabilistic arguments?

Main issue: the terminal condition $\Xi(X)$ depends on the hedging strategy $\rightarrow$ standard comparison does not hold.
Thank you!

B. Bouchard, G. Loeper, and Y. Zou.
Almost-sure hedging with permanent price impact.

B. Bouchard, G. Loeper, and Y. Zou.
Hedging of covered options with linear market impact and gamma constraint.

Second order stochastic target problems with generalized market impact.

B. Bouchard, P. Cardaliaguet and X. Tan,
Dual formulation for perfect-hedging with generalized market impact.
Forthcoming.

G. Loeper,
Option Pricing with Market Impact and Non-Linear Black and Scholes Equations,
arXiv :1301.6252v3