Optimal Dynamic Risk Sharing under the Time-Consistent Mean-Variance Criterion\cite{1}

Bin Li  
Department of Statistics and Actuarial Science  
University of Waterloo, Canada

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Introduction and motivation

Problem setting and main results
Classical setting of risk sharing (risk exchange)

- Consider $n$ agents with initial endowments $(X_1, \ldots, X_n)$ and risk preferences $(J_1, \ldots, J_n)$.

Objective function

$$\max_{(Y_1, \ldots, Y_n)} \sum_{i=1}^{n} \alpha_i J_i(Y_i),$$

subject to $\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i$ and $J_i(Y_i) \geq J_i(X_i)$ for all $i$.

1. $\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} X_i$ means a pure exchange economy.

2. $J_i(Y_i) \geq J_i(X_i)$ is called the individual rational condition.

3. Maximizing $\sum_{i=1}^{n} \alpha_i J_i(Y_i)$ leads to a Pareto optimal strategy $(Y^*_1, \ldots, Y^*_n)$, i.e., whenever an allocation $(Y_1, \ldots, Y_n)$ satisfies $J_i(Y_i) \geq J_i(Y^*_i)$ for all $i$, we have $J_i(Y_i) = J_i(Y^*_i)$ for all $i$. 
Pioneering works: Karl Borch and Kenneth Arrow in 60’s and 70’s.

Chateauneuf-Dana-Tallon (’00): Choquet expected utilities

Carlier-Dana (’08): rank dependent expected utilities

Jouini-Schachermayer-Touzi (’08): law-invariant monetary utilities

Ludkovski-Young (’09): distortion risk measures

Ravanelli-Svindland (’14): law invariant robust utilities

Mastrogiacomo-Gianin (’15): quasiconvex risk measures

Embrechts-Liu-Wang (’18): two-parameter class of quantile-based risk measures
Main focus of existing works is to study the existence of Pareto-optimal risk sharing strategies under various risk preferences.

Closed-form expressions of optimal risk sharing strategies are rare (unless $n = 2$, i.e., two agents).
**Motivation**: How to insure an extreme risk (e.g., catastrophe), which may cause significant loss, with a comprehensive coverage?

A single insurer may only be able to provide a policy with a limited coverage.

A “naive” solution is to invite more insurers to undertake this risk cooperatively.

How to find an optimal risk bearing form?
Our framework

Our framework has a few major differences with the standard risk sharing problem:

1. We consider an exogenous dynamic risk modelled by a Lévy process \( \{U(t)\}_{0 \leq t \leq T} \) with monotone increasing paths, where \( U(t) \) means the aggregate loss at time \( t \).

2. We allow \( n \) insurers to cooperatively undertake this risk, and aim to solve Pareto-optimal risk bearing strategies \( (Y_1^*(t), \ldots, Y_n^*(t))_{0 \leq t \leq T} \).

3. The risk preference \( J_i \) of each insurer follows the time-consistent mean-variance criterion.

In general, our objective is

\[
\begin{aligned}
\max_{(Y_1, \ldots, Y_n)} & \sum_{i=1}^{n} J_i(Y_i(\cdot)), \\
s.t. & \sum_{i=1}^{n} Y_i(t) \leq U(t) \text{ and } J_i(Y_i(\cdot)) \geq J_i(0). 
\end{aligned}
\]
Outline

- Introduction and motivation
- Problem setting and main results
Problem setting: exogenous risk

Consider an exogenous risk with aggregate loss modelled by an increasing Lévy process

\[ U(t) = \int_0^t \int_0^\infty y N(ds, dy), \quad t \in [0, T]. \]

- \( N(ds, dy) \) is a Poisson random measure, representing the random number of losses of size \((y, y + dy)\) that occur within the time period \((s, s + ds)\).
- \( \nu(dy) \) is the Lévy measure representing the expected number of losses of size \((y, y + dy)\) within a unit time interval.
- \( \mathbb{E}[N(ds, dy)] = \nu(dy)ds \).
- In particular, if \( U \) is a compound Poisson process, \( \nu(dy) = \lambda F(dy) \) where \( \lambda \) is the Poisson intensity and \( F \) is the cdf of jumps.
The $i$-th insurer’s controlled surplus process under the risk bearing strategy $l_i$ follows

$$dX^i_t = (1 + \theta_i) \int_0^\infty l_i(t, y) \nu(\text{d}y) \text{d}t - \int_0^\infty l_i(t, y) \text{N}(\text{d}t, \text{d}y).$$

- $l_i(t, y) \in [0, y]$ represents the risk undertaken by the $i$-th insurer for a loss of size $y$ occurring at time $t$.
- $\theta_i > 0$ is her premium rate.
- WLOG, assume $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$. 
Problem setting: objective function

- $i$-th insurer’s reward function under the strategy $l_i$ is

$$J^i_l(t, x) = \mathbb{E}_{t, x} \left[ X^i_l(T) \right] - \frac{\gamma_i}{2} \text{Var}_{t, x} \left[ X^i_l(T) \right],$$

where $\gamma_i > 0$ is her risk aversion parameter.

- Due to time inconsistency of the mean-variance criterion, our objective is to solve equilibrium Pareto-optimal risk bearing strategies $(l^*_1(t, y), \ldots, l^*_n(t, y))_{0 \leq t \leq T, y > 0}$ for

$$\left\{ \begin{array}{l}
\max_{(l_1, \ldots, l_n)} \sum_{i=1}^{n} J^i_l(t, x), \\
\text{s.t. } \sum_{i=1}^{n} l_i(t, y) \leq y \text{ and } J^i_l(t, x) \geq J^0_i(t, x) = x_i \text{ for all } i
\end{array} \right.$$

- We strategically select time-consistent mean-variance criterion because it can be considered as the SIMPLEST criterion in continuous-time framework.
Time inconsistency: equilibrium strategy

- In **discrete time**, strategies can be determined **backwardly**

  \[ 0 \overset{\text{step T-1}}{\iff} 1 \iff \cdots \iff T - 2 \overset{\text{step 2}}{\iff} T - 1 \overset{\text{step 1}}{\iff} T \]

- In **continuous time**, define a **perturbed strategy** \( l^\varepsilon \) as

  \[
l^\varepsilon(s, y) = \begin{cases} 
  \bar{l}(y), & s \in [t, t + \varepsilon), \ y > 0, \\
  l^*(s, y), & s \in [t + \varepsilon, T], \ y > 0, 
\end{cases}
\]

  where \( \bar{l} : (0, \infty) \to (0, \infty) \). We say \( l^* \) is an equilibrium strategy if for all \( \bar{l} \),

  \[
  \liminf_{\varepsilon \to 0} \frac{J^{l^*}(t, x) - J^{l^\varepsilon}(t, x)}{\varepsilon} \geq 0.
  \]

- Value function \( J^{l^*}(t, x) \) is **LINEAR in** \( x \)!
Figure 1: Equilibrium risk bearing strategies with three insurers
Define an increasing sequence \( \{ a_k \}_{k=1}^{n+1} \) given by

\[
a_1 = 0, \quad a_k = \sum_{j=1}^{k-1} \frac{\theta_j - \theta_k}{\gamma_j} \quad \text{for } k = 2, \ldots, n + 1
\]

where \( \theta_{n+1} := 0 \).

**Theorem**

An equilibrium bearing function for the \( i \)-th insurer is given by

\[
l_i^*(t, y) = \sum_{j=i}^{n} c_{ij} \left[ (y \wedge a_{j+1}) - a_j \right]_+, \quad y > 0,
\]

where \( c_{ij} := \frac{\gamma_i^{-1}}{\gamma_1^{-1} + \cdots + \gamma_j^{-1}} \) for \( j \geq i \).

- When \( n = 1 \), \( l_1^*(t, y) = y \wedge \frac{\theta_1}{\gamma_1} \), which is a stop-loss contract.
Main implications

1. The equilibrium bearing functions are time-homogeneous and of a mixture form of the proportional and the stop-loss strategies.

2. Loss coverage is broken into a set of thresholds \( \{a_j\}_{j=1}^{n+1} \).

3. The loss in the range \([a_j, a_{j+1}]\) is completely undertaken by the first \( j \) insurers in a proportional form. The proportion undertaken by the \( i \)-th insurer \( (i \leq j) \) is \( c_{ij} = \frac{\gamma_i^{-1}}{\gamma_1^{-1} + \ldots + \gamma_j^{-1}} \), which represents her relative degree of risk seeking.

4. The limit of loss coverage is \( a_{n+1} = \sum_{j=1}^{n} \frac{\theta_j}{\gamma_j} \), which can be increased with more insurers participating.
Thank you for your attention!