Continued Fraction Normals and Subsequence selections - a combinatorial approach

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1 Motivation

2 Subselection along A. P.
Every irrational real number in $r \in [0, 1]$ has a unique continued fraction expansion of the form

$$r = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

where, for every $i \in \mathbb{N}^+$, we have $a_i \in \mathbb{N}^+$. We denote this by $[0; a_1, a_2, a_3, \ldots]$. 
Let $b \geq 2$ be an integer. Denote the set $\{0, 1, \ldots, b - 1\}$ by $\Sigma_b$, and the set of finite strings drawn from this alphabet by $\Sigma_b^*$. For a finite string $w$, let $|w|$ denote its length.
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**Definition**

A real $r$ with base-$b$ expansion $.r_1 r_2 \ldots$ is said to be *normal in base $b$*, if for every $w \in \Sigma^*_b$, we have

$$\lim_{n \to \infty} \frac{|\{1 \leq i < n - |w| + 1 \mid r_i \ldots r_{i + |w| - 1} = w\}|}{n - |w| + 1} = \frac{1}{b^{|w|}}.$$
Sliding Block matching

\[
\begin{array}{cccc}
  r_1 & r_2 & r_3 & \ldots \\
  w_1 & w_2 \\
\end{array}
\]
Sliding Block matching

\[
\begin{array}{cccc}
  r_1 & r_2 & r_3 & \cdots \\
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\end{array}
\]
Denote the set of finite strings of positive integers by $\mathbb{N}^*_b$. For continued fractions, we consider the Gauss measure as the invariant measure.

**Definition**

For a Borel set $A \subseteq [0, 1]$, the *Gauss measure* of $A$ is defined by

$$\gamma(A) = \frac{1}{\ln 2} \int_A \frac{1}{1 + x} \, dx.$$

The left-shift transformation on continued fractions is ergodic wrt $\gamma$. 
An irrational \( r \) with continued fraction expansion \([0; a_1, a_2, \ldots]\) is said to be **continued fraction normal**, if for every \( w \in \mathbb{N}^* \),

\[
\lim_{{n \to \infty}} \frac{\left| \left\{ 1 \leq i < n - |w| + 1 \mid a_i \ldots a_{i+|w|-1} = w \right\} \right|}{n - |w| + 1} = \gamma(C_w),
\]

where \( C_w \) is the cylinder set

\[
\{ r \in [0, 1] - \mathbb{Q} \mid w \text{ is a prefix of the continued fraction expansion of } r \}.
\]
Let $[0; a_1, a_2, \ldots]$ be a continued fraction normal, and let $(m, m + d, m + 2d, \ldots)$, $m \geq 1$, $d \geq 2$ be an arithmetic progression of integers.

**Question:**
Is $[0; a_m, a_{m+d}, a_{m+2d}, \ldots]$ a continued fraction normal?
Subselections along Arithmetic Progression

Base-$b$ normality is preserved when we select a subsequence along an arithmetic progression. [Wall, 1949]
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Continued fraction normality is *not*!

**Theorem ([Heersink and Vandehey, 2016])**

*For any* $[0; a_1, a_2, \ldots]$ *continued fraction normal, and any* $(m, m + d, m + 2d, \ldots)$, *the continued fraction* $[0; a_m, a_{m+d}, a_{m+2d}, \ldots]$ *is not normal.*

The proof uses ergodic-theoretic techniques used in a result by Vandehey [Vandehey, 2017].
Proofs of Wall’s Result

Wall’s result on arithmetic progressions has different proofs using

1. Weyl’s criterion
2. Automata Theoretic
3. Combinatorial (?)
A combinatorial approach

The key ingredients of the proof in [Heersink and Vandehey, 2016]:

$$\lim_{n \to \infty} \frac{|\{0 \leq i \leq N \mid T^{im} r \land T^{im+d} r \in C_{[0;1]}\}|}{N}$$

$$= \sum_{a_1, a_2, \ldots, a_n \in \mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]).$$
A combinatorial approach

The key ingredients of the proof in [Heersink and Vandehey, 2016]:

1. \[
\lim_{n \to \infty} \frac{|\{0 \leq i \leq N \mid T^{im} r \land T^{im+d} r \in C_{[0;1]}\}|}{N} = \sum_{a_1, a_2, \ldots, a_n \in \mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]}).
\]

2. For any \( n \geq 1 \), \[
\sum_{a_1, a_2, \ldots, a_n \in \mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]}) > \gamma(C_{[0;1,1]}).
\]
Our approach:

1. Inequality by induction
A combinatorial approach

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2. If a sequence is continued fraction normal, then the disjoint block frequencies also behave normally.
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1. Inequality by induction
2. If a sequence is continued fraction normal, then the disjoint block frequencies also behave normally.

Key obstacle: loss of compactness, countably infinite alphabet!
Illustrative case for Step 1

Consider A.P.s with common difference 2.

Lemma

$$\sum_{a \in \mathbb{N}^+} \gamma(C_{[0;1,a,1]}) > \gamma(C_{[0;1,1]}).$$
Illustrative case for Step 1

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Lemma

\[
\sum_{a \in \mathbb{N}^+} \gamma(C_{[0;1,a,1]}) > \gamma(C_{[0;1,1]}).
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Proof Strategy: Show that the \textit{Lebesgue measure} of \(C_{[0;1,a,1]}\) is greater than that of \(C_{[0;1,1,a]}\).
Illustrative case for Step 1

Consider A.P.s with common difference 2.

**Lemma**

\[ \sum_{a \in \mathbb{N}^+} \gamma(C_{[0;1,a,1]}) > \gamma(C_{[0;1,1]}). \]

**Proof Strategy:** Show that the *Lebesgue measure* of $C_{[0;1,a,1]}$ is greater than that of $C_{[0;1,1,a]}$.

Use the standard continued fraction recurrence for denominators of the extremities of the cylinders to show:

\[
\text{denom}([0;1,1,a]) \times \text{denom}([0;1,1,(a + 1)] > \text{denom}([0;1,a,1]) \times \text{denom}([0;1,a,2]).
\]
Step 2: loss of compactness

Sliding block frequencies normal $\Rightarrow$ disjoint block frequencies normal.
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A consequence of the Piateskii-Shapiro Theorem. Follows the proof in Kuipers and Niederreiter, with a careful application of Helley Selection.
Tying things up

A. P.
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Thank You!
Using the concavity of the cumulative distribution function of Gauss measure, we can reduce the general inequality to an algebraic inequality involving denominators.