DISCREPANCY BOUNDS and RELATED QUESTIONS.

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Equidistribution:
Arithmetic, Computational and Probabilistic Aspects

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A sequence \((x_n)\) is **uniformly distributed** in \([0, 1]\) iff

\[
\text{for any interval } I \subset [0, 1] : \lim_{N \to \infty} \frac{\#\{n \leq N : x_n \in I\}}{N} = |I|
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Discrepancy of a sequence

For a sequence \(\omega = (\omega_n)_{n=1}^{\infty}\) and an interval \(I \subset [0, 1]\) consider the quantity
\[
\Delta_{N,I} = \#\{\omega_n : \omega_n \in I; n \leq N\} - N|I|.
\]
Define
\[
D^*_N = \sup_{I \subset [0,1]} |\Delta_{N,I}|.
\]
A sequence \((x_n)\) is uniformly distributed in \([0, 1]\) iff

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**Discrepancy of a sequence**

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Define

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D^*_N = \sup_{I \subset [0,1]} |\Delta_{N,I}|.
\]

A sequence \((\omega_n)_{n=1}^{\infty}\) is u.d. in \([0, 1]\) if and only if

\[
\lim_{N \to \infty} \frac{D^*_N}{N} = 0.
\]
Discrepancy function

Consider a set $\mathcal{P}_N \subset [0, 1]^d$ consisting of $N$ points:

Define the discrepancy function of the set $\mathcal{P}_N$ as

$$D_N(x) = \#\{\mathcal{P}_N \cap [0, x)\} - N x_1 x_2 \ldots x_d$$

Extremal discrepancy (star-discrepancy):

$$\|D_N\|_\infty = \sup_{x \in [0, 1]^d} |D_N(x)|.$$

$L^p$ discrepancy:

$$\|D_N\|_p = \left( \int_{[0, 1]^d} |D_N(x)|^p \, dx \right)^{1/p}.$$
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Theorem (K. Roth, 1954)

The following are equivalent:

(i) For every $\omega = (\omega_n)_{n=1}^{\infty} \subset [0, 1]$,

$$D^*_N(\omega) \gtrsim f(N)$$

for infinitely many values of $N$.

(ii) For any distribution $\mathcal{P}_N \subset [0, 1]^2$ of $N$ points,

$$\|D_N\|_\infty \gtrsim f(N)$$
Equivalence of two formulations

Theorem (K. Roth, 1954)

The following are equivalent:

(i) For every $\omega = (\omega_n)_{n=1}^{\infty} \subset [0, 1]^{d-1}$,
\[
D_N^*(\omega) \gtrsim f(N)
\]
for infinitely many values of $N$.

(ii) For any distribution $\mathcal{P}_N \subset [0, 1]^d$ of $N$ points,
\[
\|D_N\|_{\infty} \gtrsim f(N)
\]
Roth’s Theorem

Klaus Roth, October 29, 1925 – November 10, 2015

Theorem (ROTH, K. F. On irregularities of distribution, Mathematika 1 (1954), 73–79.)

There exists \( C_d \geq 0 \) such that for any \( N \)-point set \( \mathcal{P}_N \subset [0, 1]^d \)

\[ \| D_N \|_2 \geq C_d (\log N)^{\frac{d-1}{2}}. \]
Roth’s Theorem

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According to Roth himself, this was his favorite result.

- William Chen (private communication)
- Kenneth Stolarsky (private communication)
- Ben Green (comment on Terry Tao’s blog)

Ben Green:
I did meet Roth, in Inverness around 7 years ago. I asked him what his favourite proof (among his results was) and he said the lower bound for the $L^2$ discrepancy of point sets with respect to axis parallel boxes. It is a very elegant argument, nicely described in Bernard Chazelle’s book “Discrepancy Theory”. Later in his career he became quite interested in the “Heilbronn triangle problem”, which came up in conversation the other day: given $n$ points in the unit square, what’s the smallest area of triangle they are guaranteed to span. I believe that $n^{-2+o(1)}$ is conjectured, and that Roth was the first to improve on the trivial bound $O(1/n)$. Subsequently bounds of the form $O(n^{-1-c})$ were obtained.
Theorem (ROTH, K. F. On irregularities of distribution, Mathematika 1 (1954), 73–79.)

There exists $C_d \geq 0$ such that for any $N$-point set $P_N \subset [0, 1]^d$

$$\|D_N\|_2 \geq C_d (\log N)^{\frac{d-1}{2}}.$$ 

- 4 papers by Roth (On irregularities of distribution. I–IV)
- 10 papers by W.M. Schmidt (On irregularities of distribution. I–X)
- 2 by J. Beck (Note on irregularities of distribution. I–II)
- 4 by W. W. L. Chen (On irregularities of distribution. I–IV)
- 2 by Beck and Chen (Note on irregularities of distribution. I–II)
- a book by Beck and Chen, “Irregularities of distribution”.

Dmitriy Bilyk | Discrepancy bounds & related questions
Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0, 1]^d$ with $\#\mathcal{P}_N = N$:

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$
Roth’s theorem, extensions and sharpness

Theorem (Roth, 1954 \( (p = 2) \); Schmidt, 1977 \( (1 < p < 2) \))

The following estimate holds for all \( \mathcal{P}_N \subset [0, 1]^d \) with \#\( \mathcal{P}_N = N \):

\[
\| D_N \|_p \gtrsim (\log N)^{\frac{d-1}{2}}
\]

Theorem (Davenport, 1956 \( (d = 2, p = 2) \); Roth, 1979 \( (d \geq 3, p = 2) \); Chen, 1982 \( (p > 2, d \geq 3) \); Chen, Skriganov, 2000’s)

There exist sets \( \mathcal{P}_N \subset [0, 1]^d \) with

\[
\| D_N \|_p \lesssim (\log N)^{\frac{d-1}{2}}
\]
Roth’s orthogonal function method

- Dyadic intervals in $[0, 1]$:

$$\mathcal{D} = \left\{ I = \left[ \frac{m}{2^n}, \frac{m + 1}{2^n} \right) : m, n \in \mathbb{Z}, n \geq 0, 0 \leq m < 2^n \right\}.$$
Roth’s orthogonal function method

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- $L^\infty$ normalized Haar function on a dyadic Interval $I$:
  \[ h_I = -1_{I_{left}} + 1_{I_{right}} \]

$\mathbf{h}_I$
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\[ h_I = -1_{I_{\text{left}}} + 1_{I_{\text{right}}} \]

- Orthogonality:

\[ \langle h_{I'}, h_{I''} \rangle = \int_0^1 h_{I'}(x) \cdot h_{I''}(x) \, dx = 0, \quad I', I'' \in \mathcal{D}, I' \neq I''. \]
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  \]

- $f \in L^2([0, 1])$ can be written as
  \[
f = \sum_{I \in \mathcal{D}^*} \frac{\langle f, h_I \rangle}{|I|} h_I
  \]
Roth’s orthogonal function method

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- For a dyadic rectangle $R = I_1 \times \cdots \times I_d \subset [0, 1]^2$
  \[ h_R(x) := h_{I_1}(x_1) \cdot \cdots \cdot h_{I_d}(x_d) \]
Roth’s orthogonal function method

- $L^\infty$ normalized Haar function on a dyadic Interval $I$:
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- $f \in L^2([0, 1]^d)$: \[ f = \sum_{R \in \mathcal{D}^d} \frac{\langle f, h_R \rangle}{|R|} h_R \]
Roth’s orthogonal function method

- Main idea:

\[ D_N \approx \sum_{R: |R| \approx \frac{1}{N}} \frac{\langle D_N, h_R \rangle}{|R|} h_R \]
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- How many \( R \) such that \( |R| = 2^{-n} \)? How many shapes?
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- The number of shapes is
  \[ \#\{(r_1, \ldots, r_d) \in \mathbb{Z}_+^d : r_1 + \cdots + r_d = n\} = \binom{n + d - 1}{d - 1} \approx n^{d-1} \]
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- Orthogonality yields
  \[ \| D_N \|_2 \gtrsim n^{\frac{d-1}{2}} \approx (\log N)^{\frac{d-1}{2}} \]
Conjecture

\[ \| D_N \|_\infty \gg (\log N)^{\frac{d-1}{2}} \]
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Theorem (Schmidt, 1972; Halász, 1981)

In dimension \( d = 2 \) we have

\[ \|D_N\|_\infty \gtrsim \log N \]
$L^\infty$ estimates

**Conjecture**

$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$

**Theorem (Schmidt, 1972; Halász, 1981)**

*In dimension $d = 2$ we have $\|D_N\|_\infty \gtrsim \log N$*

**$d = 2$: Lerch, 1904; van der Corput, 1934**

There exist $\mathcal{P}_N \subset [0, 1]^2$ with $\|D_N\|_\infty \approx \log N$
Low discrepancy sets

The van der Corput set with $N = 2^{12}$ points
$(0.x_1x_2...x_n, 0.x_nx_{n-1}...x_2x_1)$, $x_k = 0$ or $1$.
Discrepancy $\approx \log N$
The irrational \((\alpha = \sqrt{2})\) lattice with \(N = 2^{12}\) points
\((n/N, \{n\alpha\}), \ n = 0, 1, ..., N - 1\).
Discrepancy \(\approx \log N\)
Low discrepancy sets

Random set with \( N = 2^{12} \) points

Discrepancy \( \approx \sqrt{N} \)
Conjecture

\[ \|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}} \]

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There exist \( P_N \subset [0, 1]^2 \) with \( \| D_N \|_\infty \approx \log N \)

**\( d \geq 3 \), Halton, Hammersley (1960):**

There exist \( P_N \subset [0, 1]^d \) with \( \| D_N \|_\infty \lesssim (\log N)^{d-1} \)
Conjectures and results

Conjecture 1

\[ \| D_N \|_\infty \gtrsim (\log N)^{d-1} \]
### Conjectures and results

<table>
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<tr>
<th><strong>Conjecture 1</strong></th>
<th>$|D_N|_\infty \gtrsim (\log N)^{d-1}$</th>
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<td><strong>Conjecture 2</strong></td>
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Conjectures and results

Conjecture 1

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Conjecture 2

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- “The Great Open Problem of Discrepancy Theory”
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Conjectures and results

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\[ \| D_N \|_\infty \gtrsim (\log N)^{d/2} \]

“The Great Open Problem of Discrepancy Theory”
(Beck, Chen)

Theorem (DB, M.Lacey, A.Vagharshakyan, 2008)
For \( d \geq 3 \) there exists \( \eta = \eta_d > 0 \) such that
\[ \| D_N \|_\infty \gtrsim (\log N)^{d-1/2} + \eta. \]
Conjectures: $(\log N)^{d-1} \ vs. \ (\log N)^{d/2}$

Conjecture 1: $(\log N)^{d-1}$

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- classical conjecture;

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- typical $\sqrt{\log N}$ difference between expectation and supremum of sums of $N$ subgaussian variables;
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**Conjecture 1: $(\log N)^{d-1}$**
- classical conjecture;
- the exponent $d-1$ is natural;
- the best known constructions (incl. recent lower bounds of M. Levin);

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- Sharp for the linear harmonic analysis model;
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**Conjecture 2:** \((\log N)^{d/2}\)

- recent conjecture;
- typical \(\sqrt{\log N}\) difference between expectation and supremum of sums of \(N\) subgaussian variables;
- Sharp for the linear harmonic analysis model;
- related conjectures in probability and approximation theory;
The small ball inequality

Instead of studying $D_N$ we shall look at

$$\sum_{|R|=2^{-n}} \alpha_R h_R$$
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### Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$n^{d-2} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$
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Instead of studying $D_N$ we shall look at $\sum_{|R|=2^{-n}} \alpha_R h_R$

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- $d = 2$: Talagrand, ’94; Temlyakov, ’95; DB, Feldheim ’15.
The small ball inequality

Instead of studying $D_N$ we shall look at \[ \sum_{|R|=2^{-n}} \alpha_R h_R \]

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- $d = 2$: Talagrand, ’94; Temlyakov, ’95; DB, Feldheim ’15.
- Sharpness: random signs/Gaussians.
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- $\frac{d-1}{2}$ follows from an $L^2$ estimate.
- Connected to probability, approximation, discrepancy.
Instead of studying $D_N$ we shall look at

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- Sharpness: random signs/Gaussians.
- $\frac{d-1}{2}$ follows from an $L^2$ estimate.
- Connected to probability, approximation, discrepancy.
- Known: $\frac{d-1}{2} - \eta(d)$ for $d \geq 3$
  (DB, Lacey, Vagharshakyan, 2008)
Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of $\alpha_R$

$$n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_{\infty} \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$
The small ball conjecture and discrepancy

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$$n^{\frac{1}{2}(d-2)} \left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

**Conjecture 2**

$$\| D_N \|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$
The small ball conjecture and discrepancy

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For dimensions $d \geq 2$, we have for all choices of $\alpha_R$

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Conjecture 2

$$\left\| D_N \right\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

- In both conjectures one gains a square root over the $L^2$ estimate.
Signed Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of $\alpha_R = \pm 1$

$$\left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrapprox n^{\frac{d}{2}}$$

Conjecture 2

$$\|D_N\|_\infty \gtrapprox (\log N)^{\frac{d}{2}}$$

- In both conjectures one gains a square root over the $L^2$ estimate.
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<tr>
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<th>Small Ball inequality (signed)</th>
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<td>$|D_N|_\infty \gtrsim \log N$</td>
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<td>(Schmidt, ’72; Halász, ’81)</td>
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<td><strong>Higher dimensions, $L^2$ bounds</strong></td>
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<tr>
<td>$|D_N|_\infty \gtrsim (\log N)^{d/2}$</td>
<td>$\left| \sum_{</td>
</tr>
<tr>
<td><strong>Higher dimensions, known results</strong></td>
<td></td>
</tr>
<tr>
<td>$|D_N|_\infty \gtrsim (\log N)^{d-1/2} + \eta$</td>
<td>$\left| \sum_{</td>
</tr>
</tbody>
</table>
Small ball inequality \((d=2)\)

For \(d = 2\), we have

\[
\left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|
\]
Small ball inequality (d=2)

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Sidon’s theorem

If a bounded $2\pi$-periodic function $f$ has lacunary Fourier series

$$\sum_{k=1}^{\infty} a_k e^{i n_k x}, \quad n_{k+1}/n_k > \lambda > 1,$$

then

$$\|f\|_\infty \gtrsim \sum_{k=1}^{\infty} |a_k|$$
Riesz product

Small ball inequality (d=2)

For $d = 2$, we have

$$\left\| \sum_{|R|=2^{-n}} \alpha_R h_R \right\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

- Riesz product: $\Psi(x) = \prod_{k=1}^{n} (1 + f_k)$

Sidon’s theorem

If a bounded $2\pi$-periodic function $f$ has lacunary Fourier series

$$\sum_{k=1}^{\infty} a_k e^{in_k x}, \ n_{k+1}/n_k > \lambda > 1, \text{ then}$$

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- Riesz product: $P_K(x) = \prod_{k=1}^{K} (1 + \varepsilon_k \cos n_k x)$
<table>
<thead>
<tr>
<th>Discrepancy function</th>
<th>Lacunary Fourier series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_N(x) = #{P_N \cap [0, x]} - Nx_1x_2$</td>
<td>$f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_kx,$</td>
</tr>
<tr>
<td>$|D_N|_2 \gtrsim \sqrt{\log N}$ (Roth, ’54)</td>
<td>$|f|_2 \equiv \sqrt{\sum</td>
</tr>
<tr>
<td>$|D_N|_\infty \gtrsim \log N$ (Schmidt, ’72; Halášz, ’81) Riesz product: $\prod (1 + cf_k)$</td>
<td>$|f|_\infty \gtrsim \sum</td>
</tr>
<tr>
<td>$|D_N|_1 \gtrsim \sqrt{\log N}$ (Halášz, ’81) Riesz product: $\prod (1 + i \cdot \frac{c}{\sqrt{\log N}} f_k)$</td>
<td>$|f|_1 \gtrsim |f|_2$ (Sidon, ’30) Riesz product: $\prod (1 + i \cdot \frac{</td>
</tr>
</tbody>
</table>

Table: Discrepancy function and lacunary Fourier series
Connections between problems

Discrepancy estimates
\[ \|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}} \]

Small Ball Conjecture
\[ n^{\frac{d-2}{2}} \left\| \sum_{|R|=2^{-n}} \alpha_R \hat{h}_R \right\|_\infty \gtrsim 2^{-n} \sum_{R: |R|=2^{-n}} |\alpha_R| \]

Small deviations for the Brownian sheet
\[ -\log \mathbb{P}(\|B\|_{C([0,1]^d)} < \varepsilon) \approx \varepsilon^{-2} (\log 1/\varepsilon)^{2d-1} \]

Metric entropy of MW^2
\[ \log N(\varepsilon, 2, d) \approx \varepsilon^{-1} (\log 1/\varepsilon)^{d-1/2} \]

Kuelbs, Li, ’93
Let $\mathcal{R}_d$ be the family of axis-parallel rectangles. $\mathcal{P}$ is a set of $N$ points.

**Combinatorial discrepancy:**

\[
\text{disc}(N, \mathcal{R}_d) = \sup_{\mathcal{P} \subset [0,1]^d \atop \# \mathcal{P} = N} \inf_{\chi: \mathcal{P} \to \{\pm 1\}} \sup_{R \in \mathcal{R}_d} \left| \sum_{p \in \mathcal{P} \cap R} \chi(p) \right|
\]
Let \( \mathcal{R}_d \) be the family of axis-parallel rectangles
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\]

**Transference**  Sós; Beck; Lovász, Spencer, Vesztergombi; ...

\[
\inf \|D_N\|_{\infty} \lesssim \text{disc}(N, \mathcal{R}_d)
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**Transference** Sós; Beck; Lovász, Spencer, Vesztergombi; ...

$$\inf \|D_N\|_\infty \lesssim \text{disc}(N, \mathcal{R}_d)$$

$$(\log N)^{d-1} \lesssim \text{disc}(N, \mathcal{R}_d) \lesssim (\log N)^{d-\frac{1}{2}}$$
Combinatorial discrepancy: Tusnády’s problem

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**Combinatorial discrepancy:**

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$$

**Transference** Sós; Beck; Lovász, Spencer, Vesztergombi; ...

$$
\inf \|D_N\|_\infty \lesssim \text{disc}(N, \mathcal{R}_d)
$$

$$(\log N)^{d-1} \lesssim \text{disc}(N, \mathcal{R}_d) \lesssim (\log N)^{d-\frac{1}{2}}$$

- Lower bound: Matoušek, Nikolov, 2015
Let $\mathcal{R}_d$ be the family of axis-parallel rectangles.

$\mathcal{P}$ is a set of $N$ points.

**Combinatorial discrepancy:**

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\text{disc}(N, \mathcal{R}_d) = \sup_{\mathcal{P} \subseteq [0,1]^d, \#\mathcal{P}=N} \inf_{\chi: \mathcal{P} \rightarrow \{\pm 1\}} \sup_{R \in \mathcal{R}_d} \left| \sum_{p \in \mathcal{P} \cap R} \chi(p) \right|
\]

**Transference** Sós; Beck; Lovász, Spencer, Vesztergombi; ...

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\]

- Lower bound: Matoušek, Nikolov, 2015
- Upper bound: Nikolov, 2017
Some lower and upper bounds in dimension \( d = 2 \)

<table>
<thead>
<tr>
<th>LOWER BOUND</th>
<th>UPPER BOUND</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Axis-parallel rectangles</strong></td>
<td></td>
</tr>
<tr>
<td>( L^\infty )</td>
<td>( \log N )</td>
</tr>
<tr>
<td>( L^2 )</td>
<td>( \log^{1/2} N )</td>
</tr>
<tr>
<td><strong>Rotated rectangles</strong></td>
<td></td>
</tr>
<tr>
<td>( N^{1/4} )</td>
<td>( N^{1/4} \sqrt{\log N} )</td>
</tr>
<tr>
<td><strong>Circles</strong></td>
<td></td>
</tr>
<tr>
<td>( N^{1/4} )</td>
<td>( N^{1/4} \sqrt{\log N} )</td>
</tr>
<tr>
<td><strong>Convex Sets</strong></td>
<td></td>
</tr>
<tr>
<td>( N^{1/3} )</td>
<td>( N^{1/3} \log^4 N )</td>
</tr>
</tbody>
</table>
Higher dimensions: \( d \geq 3 \)

<table>
<thead>
<tr>
<th>LOWER BOUND</th>
<th>UPPER BOUND</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Axis-parallel boxes</strong></td>
<td></td>
</tr>
<tr>
<td>( L^\infty ) ( (\log N)^{\frac{d-1}{2}} + \eta )</td>
<td>( (\log N)^{d-1} )</td>
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<tr>
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<tr>
<td>( N^{\frac{1}{2} - \frac{1}{2d}} )</td>
<td>( N^{\frac{1}{2} - \frac{1}{2d}} \sqrt{\log N} )</td>
</tr>
<tr>
<td><strong>Balls</strong></td>
<td></td>
</tr>
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<td></td>
</tr>
<tr>
<td>( N^{1 - \frac{2}{d+1}} )</td>
<td>( N^{1 - \frac{2}{d+1}} \log^c N )</td>
</tr>
</tbody>
</table>
Geometric discrepancy

- No rotations: discrepancy $\approx \log N$

- All rotations: discrepancy $\approx N^{1/4}$
  (J. Beck, H. Montgomery)

- Partial rotations
  (lacunary sets, sets of small Minkowski dimension, etc)
  DB, X.Ma, C. Spencer, J. Pipher (2016)
Let $\Omega \subset [0, \pi/2)$
Directional discrepancy

- Let $\Omega \subset [0, \pi/2)$
- $A_\Omega = \{ \text{rectangles pointing in directions of } \Omega \}$

$$D_\Omega(N) = \inf_{P_N} \sup_{R \in A_\Omega} |D(P_N, R)|$$
Directional discrepancy

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Theorem (DB, Ma, Pipher, Spencer)

- *Lacunary directions*, e.g. $\Omega = \{2^{-n}\}$: $D_\Omega(N) \preceq \log^3 N$

$Lacunary of order $M$, e.g. $\Omega = \{2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_M}\}$:

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"Superlacunary" sequence, e.g. $\{2^{2^{-n}}\}$

$D_\Omega(N) \preceq \log N \cdot (\log \log N)^2$

$\Omega$ has small upper Minkowski dimension $0 \leq d \ll 1$

$D_\Omega(N) \preceq N^{d+1} + \epsilon$
Directional discrepancy

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- **Lacunary directions**, e.g. $\Omega = \{2^{-n}\}$: $D_\Omega(N) \lesssim \log^3 N$
- **Lacunary of order M**, e.g. $\Omega = \{2^{-n_1} + 2^{-n_2} + \ldots + 2^{-n_M}\}$: $D_\Omega(N) \lesssim \log^{M+2} N$
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- $\Omega$ has small upper Minkowski dimension $0 \leq d \ll 1$

  $$D_{\Omega}(N) \lesssim N^{d + 1 + \varepsilon}$$