T-duality methods in topological matter

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Topological insulator $\leftrightarrow$ D-brane analogy ($K$-theory charge).

(T)-duality concept from string theory is useful and natural in solid state physics.

Bulk-edge correspondence heuristic: analytic boundary zero modes detect bulk topology $\rightarrow$ useful for formulating new index theorems.

Torsion invariants, antiunitary symmetries, and “super-ness” are extremely important.

Whole zoo of new crystallographic topological T-dualities, involving $K$-theories with graded equivariant twists.

Solid state phys $\rightarrow$ new maths $\rightarrow$ string/M-theory?
Experimentally verified topological matter and torsion

Experiment: Hsieh et al, PRL 103 (2009). Theory (2005+): Fu–Kane–Mele $\mathbb{Z}/2$ invariant (e.g. in $KR$-theory)

Theory: [T+Sato+Gomi, Nucl. Phys. B 923 (2017)], $\mathbb{Z}/2$ monopole and Dirac strings
Why graded symmetry groups?

A *super*, or *graded group* will just be a group $\mathcal{G} \xrightarrow{\phi} \mathbb{Z}_2$. E.g. we generate $\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}_2$ this we walk — L/R switching data.

In crystallography, this is called the *frieze group* $p_{11g}$.

There are 7 frieze groups (2D patterns with 1D translation symmetry). Higher dimension analogue are called *subperiodic groups*, e.g. 75 *rod groups*, 80 *layer groups*. 
Baum–Connes conjecture and (T)-duality

Let $G$ be a discrete (ungraded) group. *Baum–Connes conjecture*:

$$
\mu_G : K_\bullet \left( \mathcal{B}G \right) \xrightarrow{\cong} K_\bullet \left( C^*_r(G) \right).
$$

RHS is hard; here $C^*_r(G)$ is “nonabelian Fourier transform”,
generalising E.g. $C^*_r(\mathbb{Z}) = C(\mathbb{T})$.
LHS is computable with algebraic topology!

No counterexamples! I will use a concrete physics model to
motivate a super-BC conjecture, and relate it to crystallographic T-duality. This is a “good” duality in the following sense:

![Duality diagram](attachment:image.png)
Fourier transform and T-duality

The Fourier transform is the prototypical “good” duality, and involves a Pull-Convolve-Push construction:

\[ L^2(\mathbb{R}) \ni \{ f : x \mapsto f(x) \} \leftrightarrow \{ \hat{f} : p \mapsto \int_{\mathbb{R}} f(x)e^{ipx} \} \in L^2(\hat{\mathbb{R}}) \]

There is a geometric version, called the Fourier–Mukai transform, or topological T-duality:

\[ T : K^\bullet(T) \overset{\cong}{\longrightarrow} K^{\bullet-1}(\mathbb{T}) \]

\[ [E] \mapsto \hat{p}_*[(p^*E) \otimes \mathcal{P}] \]

Here, the “kernel” \( \mathcal{P} \) is the Poincaré line bundle over the “correspondence space” \( T \times \mathbb{T} \).
T-duality and Baum–Connes

T-duality is closely related to BC for the group $\mathbb{Z}$, because there are two different circles associated to $\mathbb{Z}$:

1. the classifying space $T = B\mathbb{Z} = \mathbb{R}/\mathbb{Z}$,
2. Pontryagin dual (irreps) $\mathbb{T} = \text{Hom}(\mathbb{Z}, U(1)) \Rightarrow C^*_r(\mathbb{Z}) \overset{F\text{T}}{\cong} C(\mathbb{T})$.

Then T-duality is BC for $\mathbb{Z}$ with some further identifications,

$$K^\bullet(T) \overset{PD}{\rightarrow} K_{1-\bullet}(T = B\mathbb{Z}) \overset{BC}{\rightarrow} K_{1-\bullet}(C^*_r(\mathbb{Z})) \overset{F\text{T}}{\rightarrow} K^{\bullet-1}(\mathbb{T})$$

Both $T^d$ and $\mathbb{T}^d$ appear naturally in solid state physics, as the unit cell and Brillouin zone respectively!

Crystallography $\Leftrightarrow$ extra finite point group action (with twists).
Why $K_r(C^*_r(G))$ relates to topological phases?

Free electron has Hamiltonian $H = -\nabla^2$ on $L^2(R)$. Euclidean invariance broken to $\mathbb{Z}$ by periodic potential $V$ from crystal atoms.

Think of $L^2(R) \cong L^2(\mathcal{F}) \otimes \ell^2_{\text{reg}}(\mathbb{Z})$, where $\mathcal{F} = R/\mathbb{Z}$ is the position space “unit cell”.

In the Bloch–Floquet transform, $\ell^2_{\text{reg}}(\mathbb{Z})$ part is Fourier transformed to $L^2(\mathbb{T})$ where $\mathbb{T}$ is momentum space “Brillouin zone”.

$H = -\nabla^2 + V$ decomposes into Bloch Hamiltonians $H_k$ acting on $k$-quasiperiodic functions (Bloch waves)

$$\mathcal{E}_k = \{\psi : \psi(x+1) = \psi(x)e^{ik}\} \sim L^2(\mathcal{F}; \mathcal{E}_k), \quad k \in [0, 2\pi) \cong \mathbb{T}.$$  

Schrödinger’s equation for $H_k$ on compact $\mathcal{F} \rightarrow$ discrete spectrum.
Abelian Bloch–Floquet transform

← For $E < E_{\text{Fermi}}$ get Hermitian vector eigenbundle $\mathcal{E}_{\text{Fermi}} \to \mathbb{T}$.

If $\mathcal{E}_{\text{Fermi}}$ is trivialisable, take (cts/smooth) O.N. frame $\{\phi_i\}_{i=1,...,n}$. Inverse transforms of $\phi_i$ are Wannier wavefunctions $w_i \in L^2(\mathbb{R})$ with orthonormal translates, and decay condition.

“Atomic limit”: $L^2(\mathcal{E}_{\text{Fermi}}) \cong \ell^2_{\text{reg}}(\mathbb{Z}) \otimes \mathbb{C}^n$ via localised basis of wavefunctions.
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For \( \mathcal{G} = \mathbb{Z}^2 \), the first Chern class obstructs trivialisation of \( \mathcal{E}_{\text{Fermi}} \), so no “atomic limit”. [Brouder, Panati, Monaco,..., 2007\textsuperscript{+}]
\(K_0(C^*_r(G))\) as obstruction to “atomic limit”

[Theorem: Ludewig+T ’18/19] Let \(G\) (nonabelian generally), act on Riemannian \(X\) and \(S \subset L^2(X)\) be a spectral subspace of a \(G\)-invariant Hamiltonian. Under mild assumptions, the subspace \(S_0\) of “good wavefunctions” in \(S\) form a f.g.p. module for \(C^*_r(G)\).

\(K_0(C^*_r(G))\) measures failure of \(S_0\) to be a free module (i.e. generated by translates of a good wavefunction) \(\Rightarrow\) obstruction to existence of “atomic limit” description.

Bulk-topology-imposed tails should be visible analytically at a suitable boundary as zero modes \(\Leftrightarrow\) index theorem

Remark: For \(G\) crystallographic, \(K_\bullet(C^*_r(G))\) classifies “twisted equivariant matter” [Freed–Moore ’13]; modelled over \(\mathbb{T}^d\).
Example of bulk-boundary correspondence (BBC)

For $G = \mathbb{Z}$, $K_1(C^*_r(\mathbb{Z})) \cong K^{-1}(\mathbb{T}) = \mathbb{Z}$ characterises another type of (relative) topological phase, which is instructive for BBC.

Consider $\ell^2_{\text{reg}}(\mathbb{Z}) \oplus \ell^2_{\text{reg}}(\mathbb{Z})$, sublattice operator $S = 1_A \oplus -1_B$.

\[
\begin{array}{cccccc}
& B & & B & & B \\
\ldots & & A & & A & & A \\
& | & n = -1 & | & n = 0 & | & n = 1 \\
& | & | & | & | & | & \\
\end{array}
\]

A *supersymmetric* Hamiltonian $H = H^*$ commutes with $\mathbb{Z}$-translations, but $HS = -SH$. Thus $H$ exchanges $A \leftrightarrow B$.

E.g. “Dimers” $H_{\text{blue}}$ (intracell) and $H_{\text{red}}$ (intercell)

General: $HS = -SH \iff H(k) = \begin{pmatrix} 0 & U(k) \\ (U(k))^* & 0 \end{pmatrix}$, $U(k) \in \mathbb{C}$
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SSH model

Gap condition: \( 0 \notin \text{spec}(H) \iff U : \mathbb{T} \to \text{GL}(1) \).
Wind\((U) \iff \text{gapped top. phases of } \mathbb{Z}-\text{invariant, supersymmetric } H! \)

\[ H_{\text{blue}}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \text{Wind}=0, \quad H_{\text{red}}(k) = \begin{pmatrix} 0 & e^{ik} \\ e^{-ik} & 0 \end{pmatrix} \to \text{Wind}=1. \]

Actually \( H_{\text{blue}} \sim_{\text{unitary}} H_{\text{red}} \), so Wind\((U)\) has no bulk meaning.

A boundary “fixes the gauge”, and also cuts a red link, leaving behind one “dangling zero mode” of A-type.
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SSH model and index theorem

Truncation to $n \geq 0 \Leftrightarrow \text{Hardy space } \ell^2(\mathbb{N}) \cong \mathcal{H}^2 \subset L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$.

$$H = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \quad \text{half-space} \quad \widetilde{H} = \begin{pmatrix} 0 & T_U \\ T_U^\dagger & 0 \end{pmatrix}.$$

$T_U$ is Toeplitz operator on $l^2(\mathbb{N}) \otimes \mathbb{C}^N$ with invertible symbol $U \in C(\mathbb{T}, \text{GL}(N))$ representing a $K^{-1}(\mathbb{T}) \cong \mathbb{Z}$ class.

F. Noether index theorem (1921): $T_U$ is Fredholm iff $U$ invertible, and $\text{Ind}(T_U) = -\text{Wind}(U) = \int_{\mathbb{T}} \text{ch}(U)$.

Analytic Fredholm $\text{Ind}(T_U) = \#B - \#A$ zero modes of $\widetilde{H}$, which is topological because of index theorem!

$$H_{\text{blue}}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_{\text{red}}(k) = \begin{pmatrix} 0 & e^{ik} \\ e^{-ik} & 0 \end{pmatrix} \quad \Leftrightarrow \begin{pmatrix} 0 & T_{e^{ik}} \\ T_{e^{-ik}} & 0 \end{pmatrix}.$$

Wind=0 \quad \text{Ind}=0 \quad \text{Wind}=1 \quad \text{Ind}=−1
Crystallographic BBC and mod-2 index theorem

Run the story in reverse: Physically reasonable BBC heuristic suggests the index theorem justifying the heuristic.

The torsion-free wallpaper group $\text{pg}$ is the fundamental group of the Klein bottle. Baum–Connes magic gives easy computation

$$K_1(\mathbb{C}^*_r(\text{pg})) \cong K_1(B\text{pg}) \equiv K_1(\text{Klein}) \cong H_1(\text{Klein}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$  

$\Rightarrow \mathbb{Z}/2$ topological phase for $\text{pg}$-invariant, supersymmetric $H$.

Next few slides will pictorially justify why there is a $\mathbb{Z}/2$-index map

$$K_1(\mathbb{C}^*_r(\text{pg})) \rightarrow K^\text{gr}_0(\mathbb{C}^*_r(p11g)) \cong \mathbb{Z}/2.$$  

bulk phases \hspace{1cm} boundary zero modes
Super-symmetric Hamiltonians: connect Black-Brown

\[ H_{\text{blue}} \text{ has no zero modes when truncated: trivial phase.} \]
The $\mathbb{Z}/2$ “Klein bottle” phase

Zero modes along glide axis edge have graded p11g symmetry!
**$\mathbb{Z}/2$ super-index theorem**

Theorem [Gomi+T, Lett. Math. Phys. '18]: A pg supersymmetric $H$ has symbol class in $K_{\mathbb{Z}_2}^{1+\tau}(\mathbb{T}^2)$. Its truncation gives a twisted Toeplitz family over $\mathbb{T}$. The following index maps coincide:

1. topological Gysin map $K_{\mathbb{Z}_2}^{1+\tau}(\mathbb{T}^2) \xrightarrow{\pi!} K_{\mathbb{Z}_2}^{0+\tau+c}(\mathbb{T}) \cong \mathbb{Z}/2$.

2. analytic “twisted index bundle”, in $K_0^{gr}(C_r^*(p11g)) \cong \mathbb{Z}/2$.

In physics terms, $H_{\text{purple}}$ represents a topologically non-trivial $\mathbb{Z}/2$ phase with pg symmetry. It is detected by p11g-symmetric zero modes induced when sample is cut along a glide axis.

$\mathbb{Z}/2$ index theorems in complex $K$-theory are very rare! We found this by going to graded $K$-theory, guided by physics.
Crystallographic T-duality [Gomi+T, 1806:11385]

In string theory, one could transform a $T$ bundle $E \to X$ into a $\mathbb{T}$ bundle $\hat{E} \to X$. Even though $E \not\cong \hat{E}$, their twisted $K$-theories coincide, up to a degree shift [Bouwknegt–Mathai–Evslin ’04].

Some ad-hoc incorporation of $\mathbb{Z}/2$-actions, e.g. Witten–Atiyah–Hopkins $K_\pm$ groups, orientifolds, $KR$-theory,...

Crystallographic T-duality upgrades these to general finite groups acting on tori (or torus bundles).

In fact, crystallographic space group $G \leftrightarrow$ affine $F$ action on $T^d$!

Dually, $F$ acts on $\mathbb{T}^d$ with a twist $\tau$ from $G \not\leftarrow F$. 
Crystallographic groups

A crystallographic space group $\mathcal{G}$ is a discrete cocompact subgroup of isometries of Euclidean space $\mathbb{R}^d$.

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathbb{R}^d & \rightarrow & \text{Euc}(d) & \rightarrow & \text{O}(d) & \rightarrow & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \rightarrow & \mathbb{Z}^d & \rightarrow & \mathcal{G} & \rightarrow & F & \rightarrow & 1
\end{array}
\]

$\mathcal{G}$ is an extension of finite point group $F$ by lattice subgroup $\mathbb{Z}^d$.

Classification of $\mathcal{G}$-symmetric Hamiltonians

$\Leftrightarrow$ $\tau_{\mathcal{G}}$-twisted $F$-equivariant $K$-theory of $\mathbb{T}^d$

$\Leftrightarrow$ $K$-theory of $C^*_r(\mathcal{G})$

[Freed–Moore ’13, T’16].
Crystallographic T-duality

[Gomi+T’ 18] There is a zoo of “crystallographic T-dualities”. Each $d$-dimensional space group $G$ defines an isomorphism

$$T_G : K_F^{d-\bullet+\sigma_G} (T_{\text{affine}}^d) \cong K_F^{-\bullet+\tau_G} (\mathbb{T}_{\text{dual}}^d)$$

Technical subtlety: $\sigma_G, \tau_G$ are graded, equivariant twists (physics gave a clue). As a set (not as a group), these are classified by

$$H_F^3(\cdot; \mathbb{Z}) \times H_F^1(\cdot; \mathbb{Z}_2).$$

Geometric formulation is a souped-up version of the Fourier–Mukai transform. Gives many previously unknown isomorphisms between twisted equivariant cohomology theories of tori.
Crystallographic T-duality and super Baum–Connes

Another formulation uses Baum–Connes for \( G \): Euclidean space \( R^d \) is a proper universal space \( E^G \). Quotient by \( \mathbb{Z}^d \) gives orbifold \( F \setminus T^d = B^G \).

So there is an assembly isomorphism

\[
K^F_\bullet(T^d) \cong K_\bullet(B^G) \xrightarrow{\mu_G} K_\bullet(C^*_r(G)).
\]

LHS is \( K^F_\bullet(T^d) \cong K^d_F(\bullet + \sigma^G) (T^d) \) by Poincaré duality, where the twist \( \sigma^G \) compensates for the failure of equivariant \( K \)-orientability.

RHS is \( K_\bullet(C^*_r(G)) \cong K^{-\bullet}_F(\mathbb{T}^d) + \tau^G \) by a Fourier transform, and twist \( \tau^G \) due to \( G \nleftarrow F \).
Crystallographic T-duality and super Baum–Connes

Overall, \( T^d_G : K_F^{d-\bullet+\sigma G}(T^d) \overset{\sim}{\to} K_F^{\bullet+\tau G}(\mathbb{T}^d) \).

Giving \( G \) a \( \mathbb{Z}_2 \)-grading \( \rightarrow \) extra \( H^1 \)-type twist \( c \) on both sides.

\( \Rightarrow \) a super Baum–Connes assembly map for the graded group \( G \) implements a morphism

\[
\mu_{G,c} : K_{\bullet+c}^G(T^d) \longrightarrow K_{\bullet}^{gr}(C_r^*(G)).
\]

“Ordinary Baum–Connes conjecture \( \Rightarrow \) Super version” is not known or obvious (according to experts I’ve asked). Instead, let me give some examples/applications.
Crystallographic T-duality and super Baum–Connes

We’ve seen how $K_0^{\text{gr}}(C^*(p11g)) \cong K^{0+\tau_{p11g}+c}_{\mathbb{Z}_2}(\mathbb{T})$ on the RHS appears for crystallographic BBC for pg $\to$ p11g.

Gomi+physicists Shiozaki–Sato computed this to be $\mathbb{Z}/2$ (hard!).

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

LHS is much easier: $Bp11g = R/p11g = \mathbb{Z}_2 \setminus (R/\mathbb{Z}) = \mathbb{Z}_2 \setminus (T_{\text{free}})$ is a circle $S^1$, but remember that the up/down gets flipped when looping once. Twice a point is homologically trivial, so that

$\mathbb{Z}/2 \cong H_{0+c}(S^1) \cong K_{0+c}(Bp11g) \equiv LHS \overset{s-BC}{\cong} RHS \equiv K_0^{\text{gr}}(C^*(p11g)) \cong \mathbb{Z}/2$
Crystallographic T-duality — orbifold exchange

$F$-actions on $T^d$ and $\mathbb{T}^d$ are generally inequivalent:

Application: Topological phases for $p3m1$ are dual to those for $p31m$. Similar exchange of FCC $\leftrightarrow$ BCC (well-known in physics).

For the $CT$ group (charge-conjugation, time-reversal), get $KO$-$KR$-theory exchange and dual tenfold way [M+T, JPhysA '15].

String theory interpretation?
Crystallographic T-duality — computational trick

Application: For $d \geq 3$ odd, $T_{\mathcal{G}} : K_{F}^{0+\sigma_{\mathcal{G}}} (T^{d}) \sim K_{F}^{-1+\tau_{\mathcal{G}}} (\mathbb{T}^{d})$.

AHSS computes both sides, but RHS has extension problems. Solved by simply inspecting $K^{0}$ on the LHS!
T-duality and BBC

In simplest models of BBC, the boundary is codim-1 hyperplane dividing Euclidean space into “bulk” and “vacuum”.

Machinery of **Toeplitz extensions** produces an index map $\partial$ from bulk to boundary “momentum space” $\sim$ integration-along-$k_\perp$

[Hannabuss+Mathai+T, CMP’16, LMP’18] $\partial$ is simply the T-dual of a geometric restriction-to-boundary map in position space.

Intuitively “obvious” because Fourier transform converts integration along a circle into restriction to 0-th Fourier coefficient.
T-duality and BBC in non-Euclidean geometries

In Nil-geometry, lattice $\sim \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in \mathbb{Z} \right\}$ models screw dislocations \cite{HM+16}, and T-dual has H-flux. Get “screw modes” as argued in \cite{Ran+19}

In hyperbolic plane, lattice $\sim$ surface/Fuchsian group, and there is fractional BBC \cite{MT+17}