From Little Strings to M5-branes via a Quasi-Topological Sigma Model on Loop Group

Meng-Chwan Tan

National University of Singapore

String and M-Theory: The New Geometry of the 21st Century
Outline of Talk

• Introduction and Motivation

• Summary of Results

• Main Body of the Talk

• Conclusion and Future Directions
In this talk, we will discuss a **quasi-topological twist** of a 2d $\mathcal{N} = (2, 2)$ nonlinear sigma model (NLSM) on $\mathbb{C}P^1$ with target space the based loop group $\Omega SU(k)$.

The motivations for doing so are to:

- Describe the ground and half-excited states of the 6d $A_{k-1} \mathcal{N} = (2, 0)$ little string theory.

- Obtain a physical **derivation and generalization** of a mathematical relation by Braverman-Finkelberg which defines a geometric Langlands correspondence for surfaces.

- Elucidate the $1/2$ and $1/4$ BPS sectors of the M5-brane worldvolume theory.
Introduction and Motivation

This talk is based on


Built on earlier insights in


1. In the quasi-topological sigma model with target $\Omega SU(k)$, there is a scalar supercharge $\bar{Q}_+$ which generated supersymmetry survives on a $\mathbb{C}P^1$ worldsheet, whereby in the $\bar{Q}_+$-cohomology, we have the following currents that generate the following toroidal algebra $su(k)_{\text{tor}}$:

\[
[J_{m_1}^{an_1}, J_{m_2}^{bn_2}] = if_c^{ab} J_{m_1+m_2}^c{\{n_1+n_2\}} + c_1 n_1 \delta^{ab} \delta{\{n_1+n_2\}} \delta{\{m_1+m_2\}} + c_2 m_1 \delta^{ab} \delta{\{n_1+n_2\}} \delta{\{m_1+m_2\}}
\]

2. In the topological subsector of the sigma model, we have instead the following affine algebra $su(k)_{\text{aff}}$:

\[
[J_0^{an_1}, J_0^{bn_2}] = if_c^{ab} J_0^c{\{n_1+n_2\}} + c_1 n_1 \delta^{ab} \delta{\{n_1+n_2\}}
\]
3. Via a theorem by Atiyah in [1], the left-excited states (in the DLCQ) of the 6d $A_{k-1} \mathcal{N} = (2, 0)$ little string theory (LST) on $\mathbb{R}^{5,1}$ can be related to the $\overline{Q}_+$-cohomology of the quasi-topological sigma model. In turn, we find that

\[
\text{left-excited spectrum of 6d } A_{k-1} (2, 0) \text{ LST} = \text{modules of } \mathfrak{su}(k)_{\text{tor}}
\]

4. Likewise, the ground states (in the DLCQ) of the 6d $A_{k-1} \mathcal{N} = (2, 0)$ LST on $\mathbb{R}^{5,1}$ can be related to the topological subsector of the sigma model. In turn, we find that

\[
\text{ground spectrum of 6d } A_{k-1} (2, 0) \text{ LST} = \text{modules of } \mathfrak{su}(k)_{\text{aff}}
\]
Summary of Results

5. This means (via the ground states) that we have

\[ \text{IH}^* \left( \mathcal{M}_\text{SU}(k)^N(\mathbb{R}^4) \right) = \widehat{\mathfrak{su}}(k)^N \]

i.e., the intersection cohomology of the moduli space \( \mathcal{M}_\text{SU}(k)^N(\mathbb{R}^4) \) of \( \text{SU}(k) \)-instantons forms a finite submodule over \( \mathfrak{su}(k)_{\text{aff}} \). This is just the Braverman-Finkelberg relation in [2].

6. This also means (via the left-excited states) that we have

\[ H_{\check{\text{Cech}}}^* \left( \check{\Omega}^\text{ch}_N \mathcal{M}_\text{SU}(k)^N(\mathbb{R}^4) \right) = \widehat{\mathfrak{su}}(k)^N_{c_1,c_2} \]

i.e., the Čech-cohomology of the sheaf \( \check{\Omega}^\text{ch} \mathcal{M}_\text{SU}(k)^N(\mathbb{R}^4) \) of Chiral de Rham Complex on \( \mathcal{M}_\text{SU}(k)^N(\mathbb{R}^4) \) forms a submodule over \( \mathfrak{su}(k)_{\text{tor}} \). This is a novel, physically-derived generalization of the Braverman-Finkelberg relation.
7. Using the relevant SUSY algebras, one can show the correspondence between the ground states of the little string and the 1/2 BPS sector of the M5-brane worldvolume theory, from which we can compute the 1/2 BPS sector partition function to be

\[ Z_{1/2} = \sum_{\hat{\lambda}'} \chi_{\hat{su}(k)_{c_1}}^{\hat{\lambda}'} (p) \]

It is a cousin of a modular form which transforms as a representation of \( SL(2, \mathbb{Z}) \).

There is an intrinsic \( SL(2, \mathbb{Z}) \) symmetry in the M5-brane worldvolume theory on \( \mathbb{R}^{5,1} \)!

Emerges as gauge-theoretic S-duality of 4d \( \mathcal{N} = 4 \) SYM after compactifying on \( T^2 \).
8. Likewise, one can show the correspondence between the left-excited states of the little string and the 1/4 BPS sector of the M5-brane worldvolume theory, from which we can compute the 1/4 BPS sector partition function to be

\[ Z_{1/4} = q^{\frac{1}{24}} \sum_{\hat{\lambda}} \chi_{\hat{\lambda}}^{su(k)_c} (p) \frac{1}{\eta(\tau)} \]

It is a cousin of an automorphic form which transforms as a representation of \( SO(2, 2; \mathbb{Z}) \).

There is an intrinsic \( SO(2, 2; \mathbb{Z}) \) symmetry of the M5-brane worldvolume theory on \( \mathbb{R}^{5,1} \)!

Emerges as string-theoretic T-duality of little strings after compactifying on \( T^2 \).
LET’S EXPLAIN HOW WE GOT THESE RESULTS
Based Loop Group $\Omega G$

A **loop group** $LG$ is the group consisting of maps from the unit circle $S^1$ to a (Lie) group $G$:

$$f : S^1 \rightarrow G.$$  \hspace{1cm} (1)

Parametrize $S^1$ by $e^{i\theta}$. If we impose the based point condition

$$f(\theta = 0) \rightarrow I,$$  \hspace{1cm} (2)

we get the **based** loop group $\Omega G$. One can show that

$$\Omega G = LG/G,$$  \hspace{1cm} (3)

i.e., it is a $G$-equivariant subset of $LG$ endowed with an $LG$-action.

$\Omega G$ also admits a closed nondegenerate symplectic two-form $\omega$. The complex and symplectic structures of $\Omega G$ are compatible, and conspire to make it an infinite-dimensional Kähler manifold.
Based Loop Group $\Omega G$

Let $\xi$ and $\eta$ be elements of $\Omega g$, the based loop algebra. Then, expanding them in the $Lg$ basis gives

$$
\xi(\theta) = \xi_n e^{in\theta} = \xi_{an} T^a e^{in\theta},
$$
\[\eta(\theta) = \eta_n e^{in\theta} = \eta_{an} T^a e^{in\theta},\]

where $n \in \mathbb{Z}$ and $a = 1, \ldots, \dim g$. The based point condition (2), which can be written as $e^{i\xi(\theta=0)} = 1$, then translates to $\sum_n \xi_{an} T^a = 0$.

The metric of $\Omega G$ is

$$
g_{am,bn} = |n| \delta_{n+m,0} \text{Tr}(T_a T_b). \tag{5}
$$

If we denote $T^a e^{im\theta} \equiv T^{am}$, we have

$$
[T^{am}, T^{bn}] = if^{abc} T^{c(m+n)}. \tag{6}
$$
The \( \mathcal{N} = (2, 2) \) Sigma Model on \( \mathbb{C}P^1 \) with Target \( \Omega SU(k) \)

The action of the \( \mathcal{N} = (2, 2) \) supersymmetric sigma model on \( \mathbb{C}P^1 \) with \( \Omega SU(k) \) target space is

\[
S = \int d^2z \left( g_{am,bn} \left( \frac{1}{2} \partial_z \phi^a \partial_z \phi^b + \frac{1}{2} \partial_z \phi^a \partial_z \phi^b + \bar{\psi}^b D_z \psi^a + \psi^a D_z \bar{\psi}^b \right) \right. \\
\left. - R_{am,c\bar{p},bn,d\bar{q}} \psi^a \psi^{bn} \psi^{c\bar{p}} \psi^{d\bar{q}} \right),
\]

(7)

where

\[
\phi^a(-n) = \bar{\phi}^a, \\
\psi^a(-n) = \bar{\psi}^a.
\]

(8)

and

\[
D_z \psi^{am} = \partial_z \psi^{am} + \Gamma^{am}_{bn,cp} \partial_z \phi^{bn} \psi^{cp}, \\
D_z \bar{\psi}^{\bar{a}m} = \partial_z \bar{\psi}^{\bar{a}m} + \Gamma^{\bar{a}m}_{\bar{b}n,c\bar{p}} \partial_z \bar{\phi}^{\bar{b}n} \bar{\psi}^{\bar{c}p}.
\]

(9)
Quasi-Topological A-Model on $\mathbb{CP}^1$ with Target $\Omega SU(k)$

We may twist the $\mathcal{N} = (2, 2)$ sigma model, i.e., shift the spin of the fields by their $U(1)_R$-charges. Let us consider the A-twist. The fermionic fields then become the following scalars/one-forms

$$\begin{align}
\psi^{am}_+ & \to \rho^{am}_z, \\
\overline{\psi}^{am}_+ & \to \overline{\chi}^{am}, \\
\psi^{am}_- & \to \chi^{am}, \\
\overline{\psi}^{am}_- & \to \overline{\rho}^{am}_z,
\end{align}$$

(10)

and we can write

$$S = \int d^2 z \left( g_{am,bn}(\partial_z \phi^{am}_z \partial_z \phi^{bn} + \overline{\rho}^{bn}_Z D_Z \chi^{am} + \rho^{am}_Z D_Z \overline{\chi}^{bn} ) \\
- R_{cp,bn,dq,am} \overline{\rho}^{cp}_Z \chi^{bn}_Z \overline{\chi}^{dq}_Z \rho^{am}_Z \\
+ \int \Phi^* \omega \right)$$

(11)

$$= S_{pert.} + \int \Phi^* \omega,$$

where the map $\Phi : \mathbb{CP}^1 \to \Omega SU(k)$ is of integer degree $N$. 
Like the fermion fields, there are two (nilpotent) scalar supercharges $\overline{Q}_+$ and $Q_-$, which SUSYs are therefore preserved on a worldsheet of any genus. In particular, $\overline{Q}_+$ generates the transformations

$$
\begin{align*}
\delta \phi^{am} &= 0, \\
\delta \overline{\phi}^{am} &= \bar{\epsilon}_- \overline{\chi}^{am}, \\
\delta \rho_z^{am} &= -\bar{\epsilon}_- \partial_z \phi^{am}, \\
\delta \overline{\rho}_z^{am} &= -\bar{\epsilon}_- \Gamma_{bn,cp}^{am} \overline{\chi}^{bn} \overline{\rho}^{cp}_z, \\
\delta \chi^{am} &= 0, \\
\delta \overline{\chi}^{am} &= 0,
\end{align*}
$$

(12)

where $\bar{\epsilon}_-$ is a scalar grassmanian parameter.
Quasi-Topological A-Model on $\mathbb{C}P^1$ with Target $\Omega SU(k)$

- The action (11) can be cast into the form

$$S = \int d^2z \{ \overline{Q}_+, W'(t) \} + \cdots + tN$$ (13)

where $W'(t)$ is a metric-dependent combination of fields with metric scale $t$, and the ellipsis indicates additional terms which are metric-independent but depend on the complex structure of the target space.

- Although the stress tensor $T_{zz}$ (i.e. $\delta S/\delta g_{zz}$) is $\overline{Q}_+$-closed, it is generically not $\overline{Q}_+$-exact; only $T_{zz}$ is $\overline{Q}_+$-exact. So, the correlation function of $\overline{Q}_+$-closed (but not exact) observables $\tilde{O}$ is not completely independent of arbitrary deformations the worldsheet metric $g$. This is the quasi-topological A-model.

- Path integral localizes to $\overline{Q}_+$-fixed points, and from (12), these are holomorphic maps from $\mathbb{C}P^1$ to $\Omega SU(k)$. 
The $\overline{Q}_+$-cohomology of the model has **ground and left-excited states**, and the relevant operator observables $\widetilde{O}$ of **holomorphic dimension zero and positive** are Čech cohomology classes of the sheaf $\hat{\Omega}^{ch}$ of chiral de Rham complex on $M(\mathbb{C}P^1 \xrightarrow{N} \Omega SU(k))$ [3].

A correlation function of observables $\widetilde{O}$ has the form

$$\langle \prod_{\gamma} \widetilde{O}_\gamma \rangle = \sum_{N} e^{-tN} \left( \int_{F_N} D\phi D\overline{\phi} D\rho_z D\overline{\rho}_z D\chi D\overline{\chi} \ e^{-\int d^2z(\{\overline{Q}_+,W'(t)\}+\ldots)} \prod_{\gamma} \widetilde{O}_\gamma \right).$$

(14)

Notice that

$$\frac{d}{dt} \left( \int_{F_N} D\phi \ldots D\overline{\chi} \ e^{-\int d^2z(\{\overline{Q}_+,W'(t)\}+\ldots)} \prod_{\gamma} \widetilde{O}_\gamma \right) = \langle \{\overline{Q}_+,\ldots\} \rangle_{pert.} = 0$$

(15)

so we can compute the path integral over $F_N$ in (14), henceforth denoted as $\langle \prod_{\gamma} \widetilde{O}_\gamma \rangle_{pert.}$, at any convenient value of $t$, whilst keeping the original value of $t$ in the constant factor $e^{-tN}$ (due to worldsheet instantons).
Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- **Isometries** of the target space inherited as *worldsheet symmetries* of the sigma model.

- Since $\Omega G \cong LG/G$, our sigma model ought to have an $LSU(k)$ symmetry on the worldsheet.

- Indeed, the corresponding **Noether currents**, the $J$’s, which charges generate a symmetry of the sigma model, can be shown to obey a current algebra associated with $LSU(k)$.

- As the $J$’s generate a symmetry, they act to leave the $\overline{Q}_+$-cohomology of operator observables invariant. Thus, they ought to be $\overline{Q}_+$-closed (but not exact), and are therefore also **in the $\overline{Q}_+$-cohomology**, as one can verify.
• We can conveniently compute the correlation functions of the $J$’s and $T_{zz}$ via a large $t$ limit, as explained in (14)-(15), and as OPEs, they are (in worldsheet instanton sector $\mathcal{N}$)

\[
J_{z}^{an_{1}}(z) J_{z}^{bn_{2}}(w) \sim \frac{i f_{c} J_{z}^{c\{n_{1}+n_{2}\}}(w)}{z - w},
\]

(16)

and

\[
T_{zz}(z) J_{z}^{ak}(w) \sim \frac{J_{z}^{ak}(w)}{(z - w)^2} + \frac{\partial J_{z}^{ak}(w)}{(z - w)}.
\]

(17)
Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- Laurent expanding, these correspond to the **double loop algebra** $LLsu(k)$

$$[J_{m_1}, J_{m_2}] = if_{abc} J_{m_1+m_2}^{c\{n_1+n_2\}},$$

and

$$[L_n, J_m] = -m J_{n+m}^{ak}.$$  \hspace{1cm} (18)

- In the **holomorphic dimension zero sector**, the corresponding operator $L_0 = \oint dzz T_{zz}$ must act trivially, i.e., be $\overline{Q}_+$-exact, and from (19), we see that $m = 0$, whence $LLsu(k)$ reduces to the **loop algebra** $Lsu(k)$:

$$[J_{0m_1}, J_{0m_2}] = if_{abc} J_{0}^{c\{n_1+n_2\}}.$$  \hspace{1cm} (20)

This is also the **topological sector**, since $T_{zz}$ is also $\overline{Q}_+$-exact.
Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- Our aforementioned $J$’s were derived from a classical Lagrangian density, and there would be quantum corrections.

- This means that the aforementioned algebras ought to modified as well. Specifically, they will acquire central extensions.

- This leads us to a **toroidal lie algebra** $\mathfrak{su}(k)_{\text{tor}}$:

  $$ [J_{m_1}^{\{n_1+n_2\}}, J_{m_2}^{\{n_1+n_2\}}] = i f_{c}^{a b} J_{m_1+m_2}^{c \{n_1+n_2\}} + c_1 n_1 \delta^{a b} \delta^{\{n_1+n_2\}0} \delta^{\{m_1+m_2\}0} + c_2 m_1 \delta^{a b} \delta^{\{n_1+n_2\}0} \delta^{\{m_1+m_2\}0} $$

  (21)

and **affine Lie algebra** $\mathfrak{su}(k)_{\text{aff}}$:

  $$ [J_{0}^{\{n_1+n_2\}}, J_{0}^{\{n_1+n_2\}}] = i f_{c}^{a b} J_{0}^{c \{n_1+n_2\}} + c_1 n_1 \delta^{a b} \delta^{\{n_1+n_2\}0} $$

  (22)

in the $\overline{Q}_+$-cohomology (for some $c_{1,2}$).
Now, acting on a ground state $|0\rangle$ (which is $\overline{Q}_+$-closed) with the generators of $\mathfrak{su}(k)_{\text{tor}}$, we have the states

$$J^a\{-n_1\} J^b\{-n_2\} J^c\{-n_3\} \ldots |0\rangle,$$

where $m_j, n_i \geq 0$.

They span a module $\widehat{\mathfrak{su}}(k)^N_{c_1,c_2}$ over the toroidal Lie algebra $\mathfrak{su}(k)_{\text{tor}}$ of levels $c_1$ and $c_2$.

These states have nonzero holomorphic dimension (according to (19)), and can be shown to be elements of the $\overline{Q}_+$-cohomology.

Thus, via the state-operator correspondence, we have

$$H^*_\text{Čech} \left( \widehat{\Omega}^{\text{ch}} \right) = \widehat{\mathfrak{su}}(k)^N_{c_1,c_2}. \quad (24)$$
In the topological sector where $m_i = 0$, the states are

$$J_0^a \{ -n_1 \} J_0^b \{ -n_2 \} J_0^c \{ -n_3 \} \ldots |0\rangle,$$

(25)

where $n_i \geq 0$.

They span a module $\widehat{su}(k)^N_{c_1}$ over the affine Lie algebra $su(k)_{\text{aff}}$ of level $c_1$.

These states have zero holomorphic dimension (according to (19)), and persist as elements of the $Q_+$-cohomology.

Thus, via the state-operator correspondence, and the fact that the zero holomorphic dimension chiral de Rham complex is just the de Rham complex [3], we have

$$H^*_L(M(\mathbb{C}P^1 \overset{N}{\longrightarrow} \Omega SU(k))) = \widehat{su}(k)^N_{c_1}.$$  

(26)
Little string theories (LST) exist in 6d spacetimes, and reduce to interacting local QFTs when string length $l_s \to 0$.

The 6d $A_{k-1} \mathcal{N} = (2, 0)$ LST, in particular, reduces to the 6d $A_{k-1} \mathcal{N} = (2, 0)$ superconformal field theory - has no known classical action. Rich theory, so corresponding LST must be at least just as rich.

It is also the worldvolume theory of a stack of NS5-branes in type IIA string theory, whereby the fundamental strings which reside within the branes with coupling $g_s \to 0$ (whence bulk d.o.f., including gravity, decouple) and $l_s \not\to 0$, are the little strings.
The Ground and Left-Excited Spectrum of the 6d \( A_{k-1} \) \((2, 0)\) LST

- The discrete lightcone quantization (DLCQ) of the LST on \( \mathbb{R}^{5,1} \) describes it as a 2d \( \mathcal{N} = (4, 4) \) sigma model on \( S^1 \times \mathbb{R} \) with target \( \mathcal{M}^N_{SU(k)}(\mathbb{R}^4) \), the moduli space of \( SU(k) \) \( N \)-instantons on \( \mathbb{R}^4 \). Here, \( k = \) no. of branes, \( N = \) units of discrete momentum along the \( S^1 \) [4].

- The ground states of the LST are given by sigma model states annihilated by all the supercharges, i.e., they correspond to harmonic forms and thus \( L^2 \)-cohomology classes of \( \mathcal{M}^N_{SU(k)}(\mathbb{R}^4) \).

- The left-excited states of the LST are given by sigma model states annihilated by the four chiral supercharges, i.e., they correspond to \( \check{\text{Č}}ech \) cohomology classes of the sheaf \( \check{\Omega}^ch_{SU(k)}(\mathbb{R}^4) \).
According to Atiyah [1], we have the identification

$$\mathcal{M}_G^N(\mathbb{R}^4) \cong \mathcal{M}(\mathbb{C}P^1 \xrightarrow{N \text{ hol.}} \Omega G).$$ \hspace{1cm} (27)

In turn, this means we can identify

$$H^*_{L^2}(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) \cong H^*_{L^2}(\mathcal{M}(\mathbb{C}P^1 \xrightarrow{N \text{ hol.}} \Omega SU(k)))$$ \hspace{1cm} (28)

and

$$H^*_{\text{Čech}}(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) \cong H^*_{\text{Čech}}(\hat{\Omega}^{ch}_{\mathcal{M}(\mathbb{C}P^1 \xrightarrow{N \text{ hol.}} \Omega SU(k))})$$ \hspace{1cm} (29)
The Ground and Left-Excited Spectrum of the 6d $A_{k-1}$ $(2,0)$ LST

- Thus, from (28) and (26), we find that

$$\text{ground spectrum of 6d } A_{k-1} \ (2,0) \ \text{LST} = \text{modules of } su(k)_{\text{aff}}$$

(30)

- Similarly, from (29) and (24), we find that

$$\text{left-excited spectrum of 6d } A_{k-1} \ (2,0) \ \text{LST} = \text{modules of } su(k)_{\text{tor}}$$

(31)
Deriving the Braverman-Finkelberg Relation and its Generalization

- From (26) and (28), we also find (c.f. [5]) that

\[
\text{IH}^* (\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) = \hat{s}u(k)^N_{c_1},
\]

(32)

This is the Braverman-Finkelberg relation in [2].

- From (24) and (29), we also find that

\[
H^*_\text{Čech} (\Omega^N_{SU(k)}(\mathbb{R}^4)) = \hat{s}u(k)^N_{c_1,c_2}.
\]

(33)

This is a novel generalization of the Braverman-Finkelberg relation.
The M5-brane Worldvolume Theory

- M-theory does not have a perturbative coupling constant, since there is no dilaton-field in 11d supergravity.

- The dual relationship between M2-branes and M5-branes must therefore be different from more standard weak-strong coupling dualities between strings and fivebranes in $d = 10$.

- It is thus reasonable to view M-theory as self-dual in the sense that the dynamics of the M2-branes ought to describe the dynamics of the M5-branes and vice-versa.

- The aforementioned point, and the fact that type IIA fundamental strings (which originate from M2-branes that wrap the 11th circle) and NS5-branes (which originate from M5-branes) are dual in $d=10$, suggest that the M5-brane worldvolume theory ought to be described by M-strings arising from M2-branes ending on it and wrapping the 11th circle.
• The setup of the $k$ NS5-branes with type IIA fundamental strings bound to it as little strings, has an M-theoretic interpretation.

• They can be regarded as $k$ M5-branes with M2-branes ending on them in one spatial direction (as M-strings) and wrapping the 11th circle of radius $R$ in the other spatial direction, observed at low energy scales $<< R^{-1}$.

• As such, the low energy DLCQ of this worldvolume theory of $k$ M5-branes can also be understood via the LST described as a $\mathcal{M}^{N}_{SU(k)}(\mathbb{R}^4)$ sigma model [6].

• The 6d $\mathcal{N} = (2, 0)$ supersymmetry algebra of the M5-brane worldvolume theory in DLCQ is

$$\{Q^{a\alpha}, Q^{b\beta}\} = \epsilon^{\alpha\beta}(\mathcal{H}^{ab} + \mathcal{P}^m_L \Gamma^{ab}_m),$$

$$\{Q^{a\dot{\alpha}}, Q^{b\dot{\beta}}\} = \epsilon^{\dot{\alpha}\dot{\beta}}(\mathcal{H}^{ab} + \mathcal{P}^m_R \Gamma^{ab}_m),$$

(34)

where $a = 1, \ldots, 4$ and $\alpha(\dot{\alpha}) = 1, 2$ are Lorentz and chiral (anti-chiral) $R$-symmetry indices, respectively.

• The 1/2 BPS sector of the worldvolume theory consists of states which obey the four chiral relations

$$\epsilon_{A\alpha} Q^{A\alpha} |BPS\rangle = 0, \quad A = 3, 4,$$

(35)

and anti-chiral relations

$$\epsilon_{A\dot{\alpha}} Q^{A\dot{\alpha}} |BPS\rangle = 0, \quad A = 3, 4.$$

(36)
Since the M-strings that live on the M5-brane worldvolume are 1/2 BPS objects, they only preserve eight of the sixteen supersymmetries of the worldvolume theory on their worldsheet.

The eight preserved supercharges on each M- and therefore little string worldsheet obey the supersymmetry algebra

\[
\{ \mathcal{D}^{\hat{a}\alpha}, \mathcal{D}^{\hat{b}\beta} \} = 2\epsilon^{\hat{a}\hat{b}}\epsilon^{\alpha\beta} \mathcal{L}_0, \\
\{ \overline{\mathcal{D}}^{\dot{a}\dot{\alpha}}, \overline{\mathcal{D}}^{\dot{b}\dot{\beta}} \} = 2\epsilon^{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}} \overline{\mathcal{L}}_0,
\]

where \( \hat{a}, \hat{b} = 1, 2 \) are Lorentz indices, and \( \mathcal{L}_0 \) and \( \overline{\mathcal{L}}_0 \) are the left and right-moving parts of the Hamiltonian on the worldsheet.
• Observe that the eight preserved worldsheet supercharges are evenly divided into four chiral \((\mathcal{Q}^a_\alpha)\) and four anti-chiral \((\overline{\mathcal{Q}}^\dot{a}\dot{\alpha})\) under the R-symmetry of the worldvolume theory. In fact, they correspond to the worldvolume supercharges in (35) and (36), respectively, since \(\dot{a}, \dot{b} = 1, 2\) is actually \(A, B = 3, 4\).

• Hence, the 1/2 BPS sector, spanned by states obeying the four chiral and four anti-chiral relations (35) and (36), is also spanned by states annihilated by the four left-moving and four right-moving supercharges on the worldsheet.

• From the supersymmetry algebra (37), we find that these states are necessarily ground states annihilated by the Hamiltonian \(\mathcal{H} = \mathcal{L}_0 + \overline{\mathcal{L}}_0\).

• Thus, the low energy 1/2 BPS sector of the M5-brane theory is captured by the ground states of the LST.
• Therefore, according to (30), the partition function of the 1/2 BPS sector ought to be given by summing representations of $su(k)_{\text{aff}}$. In particular, it is computed to be

$$Z_{1/2} = \sum_{\lambda'} \chi_{\hat{su}(k)_{c_1}}(p)$$

(38)

where $\chi$ is a character of the module in complex parameter $p$, and $\hat{\lambda'}$ is a dominant highest weight.

• This is a cousin of a modular form which transforms as a representation of $SL(2, \mathbb{Z})$.

• There is an intrinsic $SL(2, \mathbb{Z})$ symmetry in the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

• Emerges as gauge-theoretic S-duality of 4d $\mathcal{N} = 4$ SYM after compactifying on $T^2$. 
The 1/4 BPS sector of the worldvolume theory consists of states which obey just the four anti-chiral relations in (36)

$$\varepsilon_{A\dot{\alpha}} Q^{A\dot{\alpha}} |BPS\rangle = 0, \quad A = 3, 4.$$ (39)

Hence, the 1/4 BPS sector is also spanned by states annihilated by just the four right-moving supercharges on the worldsheet.

From the supersymmetry algebra (37), we find that these states have eigenvalues $\overline{L}_0 = 0$ but $L_0 \neq 0$.

Thus, the low energy 1/4 BPS sector of the M5-brane theory is captured by the left-excited states of the LST.
Therefore, according to (31), the partition function of the 1/4 BPS sector ought to be given by summing representations of $\mathfrak{su}(k)_{\text{tor}}$. In particular, it is computed to be

$$Z_{1/4} = q^{\frac{1}{24}} \sum_{\hat{\lambda}} \chi_{\widehat{\mathfrak{su}(k)c_1}}^{\hat{\lambda}}(p) \frac{1}{\eta(r)}$$

(40)

where $q$ is a complex parameter and $\eta$ is the Dedekind eta function.

This is a cousin of an automorphic form which transforms as a representation of $SO(2, 2; \mathbb{Z})$.

There is an intrinsic $SO(2, 2; \mathbb{Z})$ symmetry of the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

Emerges as string-theoretic T-duality of little strings after compactifying on $T^2$. 
Conclusion

• We have explained how a quasi-topological $\Omega SU(k)$ sigma model can be used to help us (i) understand the 6d $A_{k-1} (2,0)$ LST; (ii) derive and generalize the Braverman-Finkelberg relation; (iii) understand the M5-brane worldvolume theory.

• Notably, we find that the chiral spectrum of the little string is furnished by representations of a toroidal algebra, and the BPS spectra of the M5-brane worldvolume theory are closely related to modular and automorphic forms.

• Consistent with these aforementioned physical results is a geometric Langlands correspondence for surfaces – the Braverman-Finkelberg relation – and its generalization, which we also physically derived.

• We see a nice interconnection between string theory, M-theory, geometric representation theory and number theory.
Future Directions

• To ascertain the **full chiral plus anti-chiral spectrum** of the the 6d $A_{k-1} (2,0)$ LST. We expect it to be furnished by representations of a holomorphic plus antiholomorphic (positive-moded) toroidal algebra.

• Gauge the $\Omega SU(k)$ sigma model to obtain a derivation and **generalization of the AGT correspondence**, which we expect will relate the equivariant Čech-cohomology of the sheaf of chiral de Rham complex on $M^N_{SU(k)}(\mathbb{R}^4)$ to toroidal $W$-algebras.

• Go **beyond the BPS sector** of the M5-brane worldvolume theory as captured by the full spectrum of the LST. We expect the corresponding worldvolume partition function to consist of the 1/4 BPS partition function with an extra Dedekind eta function in $\tilde{r}$. 
THANKS FOR LISTENING!


