Exotic equivariant cohomology of loop spaces: momentum ⇔ winding under T-duality in an H-flux

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joint work with: Fei Han, NUS, Singapore.

[HM15] Fei Han and V. M.,
Exotic twisted equivariant cohomology of loop spaces, twisted Bismut-Chern character and T-duality.

[HM18] Fei Han and V. M.,
T-duality in an H-flux: Exchange of momentum and winding
joint work with: Peter Bouwknegt (ANU) and Jarah Evslin (Chinese Academy of Science, Langzhou).

[BEM]
Peter Bouwknegt, Jarah Evslin and V. M.,
*T-duality: Topology Change from H-flux.*
*Communications in Mathematical Physics,*

[BEM2]
Peter Bouwknegt, Jarah Evslin and V. M.,
*On the Topology and Flux of T-Dual Manifolds*
*Physical Review Letters,*
Review: T-duality in an H-flux
String theory in a background flux

Data for a partial definition for Type II string theory is:
Let $Z$ be spacetime:

1. A **background H-flux** $H \in \Omega^3(Z)$, $dH = 0$ with integral periods;
2. A **Ramond-Ramond (RR) field** $G \in \Omega^{even/odd}(Z)$, satisfying the equations of motion, $(d - H \wedge)G = 0$;
   $\Rightarrow$ twisted cohomology/twisted K-theory
3. A **complex-valued dilaton + axion**.

* In the first part, we will be concerned with T-duality between type IIA and type IIB string theories, for circle bundle compactifications.
T-duality - The case of circle bundles

In [BEM], compactify spacetime $Z$ as a principal $\mathbb{T}$-bundle over $M$, with first Chern class $c_1(Z) \in H^2(M, \mathbb{Z})$, and $H$-flux $H \in H^3(Z, \mathbb{Z})$.

\[
\begin{array}{ccc}
\mathbb{T} & \longrightarrow & Z \\
\downarrow \pi & & \downarrow \pi \\
M & & M
\end{array}
\] (1)

The **T-dual** is another principal $\mathbb{T}$-bundle over $M$, denoted by $\hat{Z}$,

\[
\begin{array}{ccc}
\hat{\mathbb{T}} & \longrightarrow & \hat{Z} \\
\downarrow \hat{\pi} & & \downarrow \hat{\pi} \\
M & & M
\end{array}
\] (2)

which has first Chern class $c_1(\hat{Z}) = \pi_* H$.

The **Gysin sequence** for $Z$ enables us to define a T-dual $H$-flux $\hat{H}$.
N.B. $\hat{H}$ is not fixed by this data, since any integer degree 3 cohomology class on $M$ that is pulled back to $\hat{Z}$ integrates to zero. However, $[\hat{H}]$ is determined uniquely upon imposing the condition $[H] = [\hat{H}]$ on the correspondence space $Z \times_M \hat{Z}$, otherwise known as the doubled space, $Z \times_M \hat{Z} = \{(x, \hat{x}) \in Z \times \hat{Z} : \pi(x) = \hat{\pi}(\hat{x})\}$.
T-duality in a background flux - the slogan

Thus a slogan for T-duality for circle bundles is the exchange,

\[
\text{background H-flux} \iff \text{Chern class}
\]

The surprising/striking new phenomenon that we discovered is that there is a change in topology when either

* the background $H$-flux topologically nontrivial,

* or the Chern class is topologically nontrivial.
T-duality in a background flux - Examples

**Example (Lens space)**

$L(p, 1) = S^3/\mathbb{Z}_p$, where $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ 

& $\mathbb{Z}_p$ acts on $S^3$ by

$$\exp(2\pi i k/p). (z_1, z_2) = (z_1, \exp(2\pi i k/p)z_2), \quad k = 0, 1, \ldots, p-1.$$ 

$L(p, 1)$ is the total space of the circle bundle over $S^2$ with Chern class equal to $p$ times the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$. 

Then $L(p, 1)$ is **never** homeomorphic to $L(q, 1)$ if $p \neq q$. Nevertheless

$$(L(p, 1), H = q) \quad \text{and} \quad (L(q, 1), H = p).$$

are T-dual pairs! Thus T-duality is the interchange

$$p \leftrightarrow q.$$
Since \( L(1, 1) = S^3 \) & \( L(0, 1) = S^2 \times S^1 \), we get the T-dual pairs:

\[
(S^2 \times S^1, H = 1) \quad \text{and} \quad (S^3, H = 0)
\]

A picture (suppressing one dimension) illustrating this is the *doughnut universe* \((H = 1)\) & the *spherical universe* \((H = 0)\)
T-duality in a background flux - isomorphism of charges

T-duality gives rise to a map inducing degree-shifting isomorphisms between the

* $H$-twisted cohomology of $Z$ and $\hat{H}$-twisted cohomology of $\hat{Z}$;
* $H$-twisted K-theory of $Z$ and $\hat{H}$-twisted K-theory of $\hat{Z}$;

where charges of RR-fields in background 3-flux fields live.

These are twisted generalizations of the smooth analog of the Fourier-Mukai transform = a geometric Fourier transform.

T-duality map is assumed to be an isometry, relating

radius $R$ circle fibres of $Z \Leftrightarrow$ radius $1/R$ circle fibres of $\hat{Z}$,

a salient feature of T-duality.
Choosing connection 1-forms $A$ and $\hat{A}$, on the $\mathbb{T}$-bundles $Z$ and $\hat{Z}$, respectively, the rules for transforming the RR fields can be encoded in the \[\text{[BEM]}\] is a Twisted Fourier-Mukai transform,

$$T_\ast G = \int_{\mathbb{T}} e^{A \wedge \hat{A}} \, G,$$

where $G \in \Omega^\bullet(Z)^\mathbb{T}$ is the total RR fieldstrength,

$$G \in \Omega^{\text{even}}(Z)^\mathbb{T} \quad \text{for Type IIA;}$$
$$G \in \Omega^{\text{odd}}(Z)^\mathbb{T} \quad \text{for Type IIB},$$

and where the right hand side of (3) is an invariant differential form on $Z \times_M \hat{Z}$, and the integration is along the $\mathbb{T}$-fiber of $Z$. 
T-duality in a background flux

Let $F = dA$ and $\hat{F} = d\hat{A}$ be the curvatures of the connections, and we can assume wlog that $H$ is $\mathbb{T}$-invariant. Then on $Z$

$$H = A \wedge \hat{F} - \Omega,$$

(4)

for some $\Omega \in \Omega^3(M)$, while the T-dual $\hat{H}$ on $\hat{Z}$ is given by

$$\hat{H} = F \wedge \hat{A} - \Omega.$$

(5)

We note that

$$d(A \wedge \hat{A}) = -H + \hat{H},$$

(6)

$T_*$ indeed maps $d_H$-closed forms $G$ to $d_{\hat{H}}$-closed forms $T_* G$.

Recall that the twisted cohomology is defined as

$$H^\bullet(Z, H) = H^\bullet(\Omega^\bullet(Z), d_H = d - H\wedge).$$

So T-duality $T_*$ induces a map on twisted cohomologies, 

$$T_* : H^\bullet(Z, H) \to H^{\bullet-1}(\hat{Z}, \hat{H}).$$
We define the Riemannian metrics on $Z$ and $\hat{Z}$ by

$$g_Z = \pi^* g_M + R^2 A \odot A, \quad g_{\hat{Z}} = \hat{\pi}^* g_M + R^{-2} \hat{A} \odot \hat{A}.$$ 


Under the above choices of Riemannian metrics and flux forms,

$$T : \Omega^k(Z)^\mathbb{T} \to \Omega^{k+1}(\hat{Z})^{\hat{\mathbb{T}}},$$

for $k = 0, 1$, are isometries, inducing isometries on the spaces of twisted harmonic forms and twisted cohomology groups.

The circle fibre radius $R$ of $Z$ goes to circle fibre radius $1/R$ of $\hat{Z}$ and there is an induced degree-shifting isomorphism

$$T_* : H^\bullet(Z, H) \cong H^\bullet + 1(\hat{Z}, \hat{H}).$$
Exchange of winding and momentum missing?
One of the deficiencies of the T-duality isomorphism in equation (7) is that it is only defined on the smaller configuration space of invariant differential forms on spacetime $Z$, and therefore is not easy to formulate the exchange of momentum and winding in this framework, as pointed out to us by Maxim Zabzine.

We rectify this in the rest of the talk, by extending T-duality isomorphism in equation (7), from invariant differential forms, to all differential forms on one side, but on the T-dual side, we show that we get exotic differential forms. N.B. a naive extension of the FM transform gives ZERO.

Strikingly, the motivation for exotic differential forms arises from some previous constructions on loopspace in [HM15]!
Motivation for some constructions on loop space
Atiyah, working out an idea of Witten, revealed the remarkable fact that the **index of the Dirac operator** acting on the spin complex of a compact spin manifold can be formally interpreted as an **integral of an equivariantly closed differential form over loop space** (wrt the rotation circle action on loops).

It is only formal because integration over infinite dimensional loop space is not well defined.

Then a formal application of the **localisation formula of Duistermaat-Heckman** leads to the index theorem of Atiyah-Singer for the Dirac operator.
Bismut extended this approach to a Dirac operator twisted by a vector bundle with connection. In doing so, for a vector bundle with connection, he constructed an equivariantly closed form on the loop space, lifting to loop space the Chern character form of the vector bundle with connection.

Bismut’s construction of the loop space refinement of the Chern character form can be viewed roughly as the trace of the solution to an equation analogous to the holonomy of a connection, and is expressed as the trace of a path ordered exponential.
It was shown by Jones-Petrack that a completed version of equivariant cohomology of loopspace \( LZ \) with respect to the rotation circle action, localises to the ordinary cohomology of \( Z \), that is,

\[
h_T^\bullet(LZ) \xrightarrow{\text{res}} H^\bullet(Z)[z, z^{-1}]\]

Bismut refined the Chern character in such a way that the following diagram commutes,

\[
\begin{array}{ccc}
K^\bullet(Z) & \xrightarrow{\text{BCh}} & h_T^\bullet(LZ) \\
\downarrow \text{Ch} & & \downarrow \text{res} \\
H^\bullet(Z)[u, u^{-1}] & & \\
\end{array}
\]

where \( \text{res} \) is the localisation map,
Motivation for some constructions on loop space

[HM15] is concerned with the analog of this result is for twisted cohomology, $H^\bullet(Z, H)$ where $H$ is a closed degree 3 form on $Z$ with integral periods, i.e. $[H] \in H^3(Z; \mathbb{Z})$.

Here $H^\bullet(Z, H) = H^\bullet(\Omega^{\text{odd/even}}(Z), d + H \wedge)$ is a $\mathbb{Z}_2$-graded cohomology theory, coinciding with $H^\bullet(Z)$ when $H = 0$.

It was first studied by Rohm-Witten (1986), and arose in String Theory as the charge group classifying D-brane charges in an H-flux, at least rationally.

It has many applications in mathematics such as twisted eta invariants, twisted analytic torsion, etc.
Motivation for some constructions on loop space

In [HM15], we defined an exotic equivariant cohomology. A key innovation is the construction of a canonical $S^1$-flat superconnection on the holonomy line bundle of a gerbe with connection, satisfying the localisation formula

$$h^\bullet_\mathbb{T}(LZ, \nabla^{\mathcal{L}^B} : \tilde{H}) \overset{\text{res}}{\cong} H^\bullet(Z, H)[u, u^{-1}]$$

where $\text{res}$ is the localisation map.

We also defined a loop space refinement of the twisted Chern character of Bouwknegt-Carey-VM-Murray-Stevenson in such a way that the following diagram commutes,

$$\begin{align*}
K^\bullet(Z, H) & \xrightarrow{BCh_H} h^\bullet_\mathbb{T}(LZ, \nabla^{\mathcal{L}^B} : \tilde{H}) \\
& \xrightarrow{\text{res}} H^\bullet(Z, H)[u, u^{-1}] \\
& \xleftarrow{\text{Ch}_H}
\end{align*}$$
Gerbes
Gerbes

Consider a pair \((Z, H)\), where \(Z\) is a spacetime and \(H\) is a background flux, i.e. a closed 3-form on \(Z\) with \(\mathbb{Z}\) periods.

We want to study open covers \(\{U_\alpha\}\) of \(Z\) such that the space of loops \(\{LU_\alpha\}\) is an open cover of \(LZ = C^\infty(S^1, Z)\). The usual Cech open cover of \(Z\) consisting of a convex open cover of \(Z\) does not satisfy this property.

Suppose that \(\{U_\alpha\}\) is a maximal open cover of \(Z\) with the property that \(H^i(U_{\alpha_I}) = 0\) for \(i = 2, 3\) where \(U_{\alpha_I} = \bigcap_{i \in I} U_{\alpha_i}\), \(|I| < \infty\). Such an open cover is a Brylinski open cover of \(Z\).

It is easy to see that \(\{LU_\alpha\}\) is an open cover of \(LZ\).

Let \(H\) a closed 3-form on \(Z\) with integral periods. Then \(H|_{U_\alpha} = dB_\alpha\) since \(H^3(U_\alpha) = 0\) where \(B_\alpha \in \Omega^2(U_\alpha)\). Also \(B_\beta - B_\alpha = dA_{\alpha\beta}\) since \(H^2(U_\alpha \cap U_\beta) = 0\). Then \((H, B, A)\) defines a connective structure (or connection) for a gerbe \(\mathcal{G}_B\) on \(Z\).
More precisely, a gerbe $\mathcal{G}$ on $Z$ is a collection of line bundles $\{L_{\alpha\beta}\}$ on double overlaps, $L_{\alpha\beta} \to U_{\alpha\beta} = U_\alpha \cap U_\beta$ such that on triple overlaps $U_{\alpha\beta\gamma}$ there is a trivialization

$$\phi_{\alpha\beta\gamma} : L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha} \xrightarrow{\sim} \mathbb{C}$$

Then $\{\phi_{\alpha\beta\gamma}\}$ is a $U(1)$-valued Cech 2-cocycle representing the Dixmier-Douady invariant of the gerbe in $H^3(Z, \mathbb{Z})$. Upto equivalence, gerbes on $Z$ are classified by $H^3(Z, \mathbb{Z})$.

A trivial gerbe $\{L_{\alpha\beta}\}$ is of the form $L_{\alpha\beta} = L_\alpha \otimes L^*_\beta$, where $\{L_\alpha \to U_\alpha\}$ is a collection of line bundles.
Gerbes

GERBE

\begin{align*}
  & \text{U}_\alpha \\
  & \text{U}_\beta \\
  & \text{U}_\gamma
\end{align*}

\begin{align*}
  & \text{L}_{\alpha\gamma} \\
  & \text{L}_{\alpha\beta} \\
  & \text{L}_{\gamma\beta}
\end{align*}
Let $\{g_{\alpha\beta} : U_{\alpha\beta} \to SO(n)\}$ denote the set of transition functions for the oriented orthonormal frame bundle of $Z$.

$U(1) \to Spin^C(n) \to SO(n)$

is the defining nontrivial central extension. Let $L \to SO(n)$ be the associated line bundle, $L = Spin^C(n) \times_{U(1)} \mathbb{C}$. Then the gerbe $\{L_{\alpha\beta} = g_{\alpha\beta}^*(L)\}$ is called the $Spin^C$-gerbe of $Z$. The Dixmier-Douady class of this gerbe is equal to $W_3(Z)$, the 3rd integral Stiefel-Whitney class of $Z$.

So every oriented manifold has a $Spin^C$-gerbe.

This construction also works for the oriented orthonormal frame bundle of any oriented vector bundle $E$ over $Z$. 

Example: $Spin^C$-gerbes
Let \( \{ g_{\alpha\beta} : U_{\alpha\beta} \to PU \} \) denote the set of transition functions for a principal \( PU \)-bundle \( P \) over \( \mathbb{Z} \),

\[
U(1) \to U \to PU
\]

is the defining nontrivial central extension.

Let \( L \to PU \) be the associated line bundle, \( L = U \times_{U(1)} \mathbb{C} \). Then the gerbe \( \{ L_{\alpha\beta} = g_{\alpha\beta}^*(L) \} \) is called the \( PU \)-gerbe of \( P \) over \( \mathbb{Z} \).

The DD-invariants of these gerbes exhaust \( H^3(\mathbb{Z}, \mathbb{Z}) \).
Gerbes, connections and their holonomy line bundle

holonomy line bundle
A connection on the gerbe $\mathcal{G}_B$ is $\{(L_{\alpha\beta}, \nabla^L_{\alpha\beta})\}$, a collection of line bundles $L_{\alpha\beta} \to U_{\alpha\beta}$ such that there is an isomorphism $L_{\alpha\beta} \otimes L_{\beta\gamma} \cong L_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$ and collection of connections $\{\nabla^L_{\alpha\beta}\}$ such that $\nabla^L_{\alpha\beta} = d + A_{\alpha\beta}$ (note that as $H^2(U_\alpha \cap U_\beta) = 0$, the bundle $L_{\alpha\beta}$ is trivial). Then we have

$$(\nabla^L_{\alpha\beta})^2 = F^L_{\alpha\beta} = B_\beta - B_\alpha. \quad (8)$$

The holonomy of this gerbe is a line bundle $\mathcal{L}^B \to LZ$ over the loop space $LZ$. $\mathcal{L}^B$ has $\mathbb{T}$-invariant Brylinski local sections $\{\sigma_\alpha\}$ with respect to $\{LU_\alpha\}$ such that the transition functions are $\{e^{-\sqrt{-1}\tau(A_{\alpha\beta})}\}$, i.e.

$$\sigma_\alpha = e^{-\sqrt{-1}\tau(A_{\alpha\beta})} \sigma_\beta,$$

$\tau : \Omega^\bullet(U_{\alpha_1}) \longrightarrow \Omega^{\bullet-1}(LU_{\alpha_1})$ is the transgression map defined as $\tau(\xi_1) = \int_\mathbb{T} \text{ev}^*(\xi_1), \quad \xi_1 \in \Omega^\bullet(U_{\alpha_1}).$ Here $\text{ev}$ is the evaluation map $\text{ev} : \mathbb{T} \times LU_{\alpha_1} \to U_{\alpha_1} : (t, \gamma) \to \gamma(t)$. 
The holonomy line bundle $\mathcal{L}^B$ on loopspace $LZ$ comes with a natural connection, whose definition with respect to the basis \{\sigma_\alpha\} is $\nabla^{\mathcal{L}^B} = d - \sqrt{-1} \tau(B_\alpha)$. The curvature of the connection $\nabla^{\mathcal{L}^B}$ is $F_B = (\nabla^{\mathcal{L}^B})^2 = -\sqrt{-1} \tau(H)$ is the transgression of the minus $i \times 3$-curvature $H$ of the gerbe $\mathcal{G}_B$.

Observe that $\mathcal{L}^B$ is never flat if $H \neq 0$.

Consider $\Omega^\bullet(LZ, \mathcal{L}^B)$ = the space of differential forms on loop space $LZ$ with values in the holonomy line bundle $\mathcal{L}^B \to LZ$ of the gerbe $\mathcal{G}_B$ on $Z$. 
Induced tensors on loop space

Let $\omega \in \Omega^i(Z)$. Define $\hat{\omega}_s \in \Omega^i(LZ)$ for $s \in [0, 1]$ by

$$\hat{\omega}_s(X_1, \ldots, X_i)(\gamma) = \omega(X_1|_{\gamma(s)}, \ldots, X_i|_{\gamma(s)})$$

for $\gamma \in LZ$ and $X_1, \ldots, X_i$ are vector fields on $LZ$ defined near $\gamma$. Then one checks that $d\hat{\omega}_s = \hat{d}\omega_s$.

The $i$-form

$$\bar{\omega} = \int_0^1 \hat{\omega}_s ds \in \Omega^i(LZ)$$

is the extension of $\omega$ on $Z$, to $LZ$. Then $\bar{\omega} = \text{ is } T$-invariant, that is, $L_K (\bar{\omega}) = 0$ and $d\bar{\omega} = \bar{d}\omega$.

Moreover $\tau(\omega) = i_K \bar{\omega}$ and that $\bar{\omega}$ restricts to $\omega$ on the submanifold of constant loops.
Exotic twisted equivariant cohomology of loop space
Let $H$ be as before and $\tilde{H} \in \Omega^3(LZ)$ be the associated closed 3-form on $LZ$. Define $D_{\tilde{H}} = \nabla^{L^B} - i_K + \tilde{H}$. Then we compute,

**Lemma**

$$(D_{\tilde{H}})^2 = 0 \text{ on } \Omega^\bullet(LZ, L^B)^T.$$

**Proof.**

Let $\{U_\alpha\}$ be a Brylinski open cover of $Z$. Then $\tilde{H}\mid_{LU_\alpha} = d\tilde{B}_\alpha$ on $LU_\alpha$. On $LU_\alpha$, we have

$$(D_{\tilde{H}})^2 = (\nabla^{L^B} - i_K + \tilde{H})^2$$

$$= (d - i_K\tilde{B}_\alpha - i_K + \tilde{H})^2$$

$$= ((d - i_K) + (d - i_K)\tilde{B}_\alpha)^2$$

$$= (\exp(-\tilde{B}_\alpha)(d - i_K)\exp(\tilde{B}_\alpha))^2$$

$$= -L_K - (L_K\tilde{B}_\alpha) = -L_K,$$
Exotic twisted equivariant cohomology of loop space

Proof.

where \( L_K \) denotes the Lie derivative of the vector field \( K \). As the Brylinski sections are invariant, we have \( L_K = L_K^B \) on \( LU_\alpha \). So \((D_{\tilde{H}})^2 = -L_{\tilde{K}}^B\), which vanishes on \( \Omega^\bullet(LZ, L^B)^T \) as claimed. 

Notice that
\[
D_{\tilde{H}} = \nabla L^B - i_K + \tilde{H}
\]
is a flat \( T \)-equivariant superconnection (in the sense of Quillen) on \( \Omega^\bullet(LZ, L^B)^T \).
Therefore \((\Omega^\bullet(LZ, L^B)^T, D_{\tilde{H}})\) is a \( \mathbb{Z}_2 \)-graded complex. We call the cohomology of this complex the exotic twisted \( T \)-equivariant cohomology of loop space, denoted by
\[
H^\bullet_T(LZ, \nabla L^B : \tilde{H}).
\]
Completed exotic twisted equivariant cohomology of loop space

Define the completed periodic exotic twisted $\mathbb{T}$-equivariant cohomology $h_T^*(LZ, \nabla^{L^B} : \bar{H})$ to be the cohomology of the complex

$$(\Omega^\bullet(LZ, \mathcal{L}^B)^\mathbb{T}[u, u^{-1}], \nabla^{\mathcal{L}^B} - ui_K + u^{-1}\bar{H}).$$

NB the holonomy line bundle $\mathcal{L}^B$ is trivial when restricted to $Z$, the constant loop space, we have

**Theorem (Localisation)**

The restriction to the constant loops

$$res : h_T^*(LZ, \nabla^{\mathcal{L}^B} : \bar{H}) \cong H^*(Z, H)[u, u^{-1}]$$

is an isomorphism.
This justifies the following 2 proposals:

**RR fields in type II String Theory in a background H-flux, are exotic differential forms in**
\[ \Omega^\bullet(LZ, L^B)^{S^1} \]  and are closed wrt the exotic differential \( D_{\overline{H}} \). (EOM)

It also includes massive RR-fields.

Also

**Over the rationals, D-brane charges on space-time Z in a background H-flux, take values in**
\[ h_T^*(LZ, \nabla^B : \overline{H}) \]
Let \( \{ U_\alpha \} \) be a Brylinski cover of \( Z \) and \( E = \{ E_\alpha \} \) be a collection of (infinite dimensional) Hilbert bundles \( E_\alpha \to U_\alpha \) whose structure group is reduced to \( U_{tr} \), which are unitary operators on the model Hilbert space \( \mathcal{H} \) of the form (identity + trace class operator). Here \( tr \) denotes the Lie algebra of trace class operators on \( \mathcal{H} \).

In addition, assume that on the overlaps \( U_{\alpha\beta} \) that there are isomorphisms \( \phi_{\alpha\beta} : L_{\alpha\beta} \otimes E_\beta \cong E_\alpha \), which are consistently defined on triple overlaps because of the gerbe property. Then \( \{ E_\alpha \} \) is said to be a **gerbe module** for the gerbe \( \{ L_{\alpha\beta} \} \).
A gerbe module connection $\nabla^E$ is a collection of connections $\{\nabla^E_\alpha\}$ is of the form $\nabla^E_\alpha = d + A^E_\alpha$ where $A^E_\alpha \in \Omega^1(U_\alpha) \otimes tr$ whose curvature $F^E_\alpha$ on the overlaps $U_{\alpha\beta}$ satisfies

$$\phi^{-1}_{\alpha\beta}(F^E_\alpha)\phi_{\alpha\beta} = F^L_{\alpha\beta}I + F^E_{\beta}.$$ 

Using $F^L_{\alpha\beta} = B_\beta - B_\alpha$, this becomes

$$\phi^{-1}_{\alpha\beta}(B_\alpha I + F^E_\alpha)\phi_{\alpha\beta} = B_\beta I + F^E_{\beta}.$$ 

It follows that $\exp(-B) \Tr(\exp(-F^E) - I)$ is a globally well defined differential form on $Z$ of even degree. Notice that $\Tr(I) = \infty$ which is why we need to consider the subtraction.
Gerbe modules Twisted K-theory

Let $E = \{ E_\alpha \}$ and $E' = \{ E'_\alpha \}$ be a gerbe modules for the gerbe $\{ L_{\alpha\beta} \}$. Then an element of twisted K-theory $K^0(Z, H)$ is represented by the pair $(E, E')$. Two such pairs $(E, E')$ and $(G, G')$ are equivalent if $E \oplus G' \oplus K \cong E' \oplus G \oplus K$ as gerbe modules for some gerbe module $K$ for the gerbe $\{ L_{\alpha\beta} \}$. We can assume without loss of generality that these gerbe modules $E, E'$ are modeled on the same Hilbert space $\mathcal{H}$.

Suppose that $\nabla^E, \nabla^{E'}$ are gerbe module connections on the gerbe modules $E, E'$ respectively. Then we can define the twisted Chern character of [BCMMS] as

$$Ch_H : K^0(Z, H) \to H^{even}(Z, H)$$

$$Ch_H(E, E') = \exp(-B) \text{Tr} \left( \exp(-F^E) - \exp(-F^{E'}) \right)$$
Path ordered exponential

Let $\mathcal{A}$ be a unital Banach algebra and $a : [0, 1] \rightarrow \mathcal{A}$ be a continuous function. Define the **path ordered exponential**, denoted $\mathcal{T}(t) = \mathcal{T}(\exp(\int_0^1 a(s)ds))$ as the unique solution to

$$
\frac{d}{dt} \mathcal{T}(t) = a(t)\mathcal{T}(t)
$$

$$\mathcal{T}(0) = 1
$$

Then it has a convergent power series expansion

$$
\mathcal{T}(t) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(t)} a(s_1) \cdots a(s_n) ds_1 \cdots ds_n
$$

where $\Delta_n(t)$ is the n-simplex of size $t$, ie

$$
\Delta_n(t) = \{0 \leq s_1 \leq \cdots \leq s_n \leq t\}.
$$
Twisted Bismut-Chern character

Via the path ordered exponential method, lift the twisted Chern character of [BCMMS] to loop space $LZ$ by defining

$$BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LU_\alpha, \mathcal{L}^B)^\mathbb{T}[u, u^{-1}]$$

by

$$BCh_{H,\alpha}(\nabla^E, \nabla^{E'}) =$$

$$\left(1 + \sum_{n=1}^{\infty} (-u)^{-n} \int_{\Delta_n(1)} \widehat{B}_{\alpha s_1} \cdots \widehat{B}_{\alpha s_n} \right) \left( BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'}) \right) \sigma_\alpha$$

$$= \mathcal{T} \left( \exp \left( \frac{-1}{u} \int_0^1 \widehat{B}_{\alpha s} ds \right) \right) \left( BCh_\alpha(\nabla^E) - BCh_\alpha(\nabla^{E'}) \right) \sigma_\alpha$$

$BCh_\alpha(\nabla^E)$ is the path ordered exponential lift of the Chern character to loop space $LZ$ due to Bismut. Since the curvature of $\nabla^E$ is vector valued therefore parallel transport wrt $\nabla^E$ has to be inserted into the curvature factors before taking the trace.
Define the **twisted Bismut-Chern character form**

\[ BCh_H(\nabla^E, \nabla^{E'}) \in \Omega^\bullet(LZ, \mathcal{L}^B)^\mathbb{T}[u, u^{-1}] \]

to be the global form patched together from the local forms constructed above.

**Theorem**

(i) We have

\[ (\nabla^{\mathcal{L}^B} - u \iota_K + u^{-1} \bar{H})BCh_H(\nabla^E, \nabla^{E'}) = 0; \]

(ii) The exotic twisted \( \mathbb{T} \)-equivariant cohomology class \[ [BCh_H(\nabla^E, \nabla^{E'})] \text{ does not depend on the choice of connections } \nabla^E, \nabla^{E'} \).

(iii) One has a commutative diagram

\[
\begin{array}{ccc}
K^\bullet(Z, H) & \xrightarrow{BCh_H} & h^\bullet(\mathbb{L}Z, \nabla^{\mathcal{L}^B} : \bar{H}) \\
\downarrow{Ch_H} & & \downarrow{\text{res}} \\
H^\bullet(Z, H)[u, u^{-1}] & & \\
\end{array}
\]
Consider the diagram:

\[(Z, H) \quad \xleftarrow{p} \quad (\hat{Z}, \hat{H}) \quad \xrightarrow{\hat{p}} \quad X\]

where \(Z, \hat{Z}\) are principal circle bundles over a base \(X\) with fluxes \(H\) and \(\hat{H}\), respectively, satisfying

\[p_*(H) = c_1(\hat{Z}), \quad \hat{p}_*(\hat{H}) = c_1(Z)\]

and \(H - \hat{H}\) is exact on the correspondence space \(Z \times_X \hat{Z}\). The T-duality Theorem for circle bundles states that there is an isomorphism of twisted K-theories \(K^\bullet(Z, H) \cong K^{\bullet+1}(\hat{Z}, \hat{H})\) and an isomorphism of twisted cohomology theories, \(H^\bullet(Z, H) \cong H^{\bullet+1}(\hat{Z}, \hat{H})\).

As a consequence of our Localisation Theorem, properties of the twisted Bismut-Chern character, T-duality Theorem for circle
Theorem (T-duality: a loop space perspective)

In the notation above, there is an isomorphism

$$T : h_\mathcal{T}^\bullet(LZ, \nabla^L B : \bar{H}) \xrightarrow{\cong} h_\mathcal{T}^{\bullet+1}(\hat{LZ}, \nabla^{\hat{L} B} : \hat{H}),$$

such that the following diagram commutes,

$$\begin{array}{ccc}
K^\bullet(Z, H) & \xrightarrow{T} & K^{\bullet+1}(\hat{Z}, \hat{H}) \\
\downarrow BCh_H & & \downarrow BCh_{\hat{H}} \\
h_\mathcal{T}^\bullet(LZ, \nabla^L B : \bar{H}) & \xrightarrow{T} & h_\mathcal{T}^{\bullet+1}(\hat{LZ}, \nabla^{\hat{L} B} : \hat{H}) \\
\downarrow res & & \downarrow res \\
H^\bullet(Z, H)[u, u^{-1}] & \xrightarrow{T} & H^{\bullet+1}(\hat{Z}, \hat{H})[u, u^{-1}] \\
\end{array}$$

(14)
Solution to exchange of momentum and winding
The precise relation between [HM15] and this talk is that when we are in the setting of T-duality in the presence of an H-flux, that is, $Z$ is the total space of a principal circle bundle with flux form $H$ and $\xi$ the associated line bundle to $Z$, similarly $\hat{Z}$ be the T-dual circle bundle with T-dual H-flux $\hat{H}$ and associated line bundle $\hat{\xi}$.

Then there is a natural infinite sequence of embeddings $\iota_n : Z \to LZ$ defined by

$$\iota_n(x) = \{S^1 \ni t \mapsto \gamma_x(t) = t^n \cdot x\} \quad x \in Z, \, n \in \mathbb{Z}$$

We consider such sequence of embeddings motivated by the fact that there are $\mathbb{Z}$ many connected components in the (vertical) loop space.
We have $\iota_n^*(L) \cong \pi^*(\hat{\xi}) \otimes n$ since they have the same Chern class, since $\iota_1^*c_1(L) = \iota_1^*\tau(H)$, the transgression of $H$, and

$c_1(\pi^*\hat{\xi}) = \pi^*\pi_*(H) = \pi^*\int_{S^1} H$ are equal.

The loop space $LZ$ has the natural circle action by rotating loops, and $Z$ has a circle action as the total space of circle bundle. To tell the difference of these two circle actions, we use $S^1$ for the circle action by rotating loops. We have that

$\iota_n^*: \Omega^k(LZ, \mathcal{L})^{S^1} \longrightarrow \Omega^k(Z, \pi^*(\hat{\xi} \otimes n))^T$ intertwines the equivariantly flat superconnections on both spaces.
Assembling these morphisms $i^* = \bigoplus i_n^*$ gives a morphism $i^* : \Omega^\kappa(LZ, L)^{S^1} \longrightarrow \mathcal{A}^\kappa(Z)^\mathbb{T}$, where

$$\mathcal{A}^\kappa(Z)^\mathbb{T} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n^\kappa(Z)^\mathbb{T} := \bigoplus_{n \in \mathbb{Z}} \Omega^\kappa(Z, \pi^*(\hat{\xi} \otimes n))^\mathbb{T},$$

that commutes with the equivariantly flat superconnections on the left hand side and the ordinary differential on the RHS, which is given by $-(\hat{\pi}^* \nabla \xi \otimes n - i_n \hat{\nu} + \hat{H})$.

Define the subspace of weight $-n$ differential forms on $Z$,

$$\Omega^*_{-n}(Z) = \{ \omega \in \Omega^*(Z) | L_v \omega = -n \omega \}.$$
It is easy to see that

\[ \Omega^k_0(Z) = \Omega^k(Z)^T, \quad A^{k+1}_0(\hat{Z})^\hat{T} = \Omega^{k+1}(\hat{Z})^\hat{T}. \]

Under the above choices of Riemannian metrics and flux forms, our results show that the Fourier-Mukai transform \( T \) can be extended to a sequence of isometries,

\[ \tau_n: \Omega^\bar{k}_n(Z) \to A^{\bar{k}+1}_n(\hat{Z})^\hat{T}, \quad (15) \]

for \( \bar{k} = k \mod 2 \), and is defined by the \textit{exotic Hori formula} to be given later. When \( n = 0 \), \( \tau_0 = T \) and we get the [BEM2003] isomorphism (2).
Our main Theorem [HM18] shows that the twisted de Rham differential \( d + H \) maps to the differential \(-\left(\hat{\pi}^* \nabla \xi \otimes n - \iota_n \hat{\nu} + \hat{H}\right)\).

One similarly has a sequence of isometries,

\[
\sigma_n: \mathcal{A}_n^\bar{k}(\hat{Z})^\mathbb{T} \to \Omega_{-n}^{\bar{k}+1}(\hat{Z}), \quad (16)
\]

for \( \bar{k} = k \mod 2 \), and is defined by the **exotic Hori formula**
given later and the differential \( \pi^* \nabla \xi \otimes n - \iota_n \nu + H \) maps to the twisted de Rham differential \(- (d + \hat{H})\). Note that \( \sigma_0 = T \).

Similarly, the sequences of isometries \( \hat{\tau}_n, \hat{\sigma}_n \) on \( \hat{Z} \) exist.
Although the extension of the Fourier-Mukai transform to all differential forms on $Z$ is slightly asymmetric, one has the crucial identities,

\[-\text{Id} = \hat{\sigma}_n \circ \tau_n : \Omega^k_{-n}(Z) \rightarrow \Omega^k_{-n}(Z), \quad (17)\]

\[-\text{Id} = \hat{\tau}_n \circ \sigma_n : \mathcal{A}^k_n(Z)^\mathbb{T} \rightarrow \mathcal{A}^k_n(Z)^\mathbb{T}. \quad (18)\]

This is interpreted as saying that T-duality when applied twice, returns one to minus of the identity. It was verified in the special case when $n = 0$ in [BEM2003].
This shows that for each of either $Z$ or $\hat{Z}$, there are two theories (at degree 0 the two theories coincide), and there are also graded isomorphisms between the two theories of both sides.

Theorem [HM18] tells us that when $n \neq 0$, the complex $(A_n^k(\hat{Z}), \hat{\pi}^* \nabla \xi^n - \iota_{n\hat{v}} + \hat{H})$ has vanishing cohomology (acyclic). Therefore, when $n \neq 0$, the complex $(\Omega_{-n}^k(Z), d + H)$ also has vanishing cohomology (acyclic).

We also construct a cochain homotopy to show this.
Now we are able to define **momentum** and **winding** in the much larger configuration space of invariant exotic differential forms, $A^{k}(Z)^\mathbb{T}$ as follows.

The multiple $w$ of the infinitesimal generator $\nu$ of the circle action on $Z$ is the **winding**, as it agrees with winding around the circle direction when the circle bundle is a product, cf. [Hori et al].

The tensor power $m$ of the line bundle $\xi$ is the **momentum**, as it agrees with momentum when the circle bundle is a product, cf. [Hori et al].
In Theorem [HM18], we show that one must have the momentum on spacetime $Z = \text{winding}$ on T-dual spacetime $\hat{Z}$, in order that the exotic differential is an equivariantly flat superconnection - a consistency check.

The T-dual side also exhibits this phenomena. Thus our slogan,

\textit{The momentum (on spacetime $Z$) gets exchanged with the winding number (on the T-dual spacetime $\hat{Z}$) and vice versa}
Twisted integration along the fibre
We introduce a **twisted version** of integration along the fiber.

Let $\pi : P \to M$ be a principal circle bundle over $M$, and $\Theta$ a connection one form on $P$. Let $L$ be the associated Hermitian line bundle over $M$. Let $\nabla^L$ be the connection on $L$ corresponding to $\Theta$.

Note that $C^\infty(P) = \bigoplus_{n \in \mathbb{Z}} C^\infty(M, L^\otimes n)$. Define a twisted integration along the fibre $\Omega^\bullet(P) \mapsto \Omega^{\bullet-1}(M, L^\otimes n)$ such that $n = 0$ is just usual integration along the fibre.

Choose a good cover $\{U_\alpha\}$ on $M$ such that $\pi^{-1}(U_\alpha) \cong U_\alpha \times S^1$. Let $\{f_\alpha\}$ be a local basis of $L$ corresponding to the constant map $U_\alpha \to \{1\} \subset S^1$. 

**Twisted integration along the fibre**
∀ \(n \in \mathbb{Z}\), define the **twisted integration along the fiber** as follows: for

\[
\Omega^*(P) \ni \omega \mapsto \int_{\mathcal{P}/\mathcal{M}, n} \omega \in \Omega^{*-1}(\mathcal{M}, L \otimes n),
\]

such that

\[
\left. \left( \int_{\mathcal{P}/\mathcal{M}, n} \omega \right) \right|_{\mathcal{U}_\alpha} = \left( \int_{\pi^{-1}(\mathcal{U}_\alpha)/\mathcal{U}_\alpha} \omega_\alpha e^{2\pi i n \theta_\alpha} \right) \otimes f_\alpha \otimes n,
\]

where \(\omega_\alpha = \omega|_{\pi^{-1}(\mathcal{U}_\alpha)}\), \(\theta_\alpha\) is the vertical coordinates of \(\pi^{-1}(\mathcal{U}_\alpha)\).

Note that as on \(\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta\), \(f_\alpha/f_\beta = e^{2\pi i (\theta_\beta - \theta_\alpha)}\) (a function on \(\mathcal{U}_{\alpha\beta}\)), the above construction patches to be a global section of the bundle \(\wedge^*(T^*\mathcal{M}) \otimes L \otimes n\). Moreover, it is not hard to see that this definition is independent of choice of the good cover \(\{\mathcal{U}_\alpha\}\) and local trivializations.
Twisted integration along the fibre

Theorem (intertwining property)

Let $Y$ be a vector field on $M$ and $\tilde{Y}$ a lift of $Y$ on $P$. Let $H$ be a differential form on $M$. Then $\forall n \in \mathbb{Z}$

$$(\nabla^{\otimes n} - \iota_Y + H) \int_{P/M,n} \omega = -\int_{P/M,n} (d + n\Theta - \iota_{\tilde{Y}} + H)\omega.$$
The exotic Hori formulae
The exotic Hori formulae

Recall that the weight \(-n\) subspace

\[
\Omega_{-n}^*(Z) = \{ \omega \in \Omega^*(Z) | L_v \omega = -n \omega \}.
\]

Note that \(\Omega_0^*(Z) = \Omega^*(Z)^T\), i.e. the \(T\)-invariant forms on \(Z\).

Let \(\omega_{-n} \in \Omega_{-n}^*(Z)\). Define the **exotic Hori formula** by

\[
\tau_n(\omega_{-n}) = \int_{\mathbb{T}^n} \omega_{-n} e^{-A \wedge \hat{A}} \in \Omega^*(\hat{Z}, \hat{\pi}^*(\xi) \otimes n),
\]

where \(\int_{\mathbb{T}^n}\) stands for \(\int (Z \times x \hat{Z}) / \hat{Z}, n\) for simplicity.

NB Let \(\{s_\alpha\}\) be local sections of the line bundle \(\xi\) and \(\theta_\alpha\) be the vertical coordinate function on \(\pi^{-1}(U_\alpha)\). Then, locally, \(\omega_{-n}\) must be of the form

\[
(\omega_{-n,\alpha,0} + \omega_{-n,\alpha,1}(d\theta_\alpha + A_\alpha)) e^{-2\pi in\theta_\alpha},
\]

where \(\omega_{-n,\alpha,0}\) and \(\omega_{-n,\alpha,1}\) are both forms on \(U_\alpha\).
So if \( m \neq n \),

\[
\left. \int_{\mathbb{T}^m} \left( \omega - n e^{-A \wedge \hat{A}} \right) \right|_{U_\alpha} = \left( \int_{\mathbb{T}} (\omega_{-n,0} + \omega_{-n,1}(d\theta_\alpha + A_\alpha)) \right) \times \\
\left( 1 - (d\theta_\alpha + A_\alpha) \wedge \hat{A} \right) e^{-2\pi i n\theta_\alpha} \cdot e^{2\pi i m\theta_\alpha} \otimes \hat{\pi}^* (s_\alpha)^\otimes n
\]

\[= 0\]

This explains why we only define \( \tau_m(\omega_{-n}) \) for \( m = n \).

One similarly has a sequence of isometries,

\[
\sigma_n: \mathcal{A}_n(Z)^{\mathbb{T}} \rightarrow \Omega_{-n}^{k+1}(\hat{Z}),
\]

(19)

for \( \bar{k} = k \mod 2 \), defined by an analogous exotic Hori formula.
Conclusions, outlook and open questions
Conclusions, outlook and open questions

- We extended T-duality in an H-flux in the literature, to an isometry of exotic differential forms. The payoff is that one can now incorporate the exchange of momentum and winding to this context.

- The next goal is to do the analogous result in (twisted) K-theory. This is trickier, but a first step is taken in the paper, F. Han and V.M., Exotic Twisted Equivariant K-Theory, 16 pages, [1712.06267]

- What about enhancing T-duality for torus bundles?

- Question: Can T-duality in an H-flux be done in the supersymmetric setting? This may be interesting to work out.