Courant algebroid connections
& Poisson-Lie T-duality
of string effective actions

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Overview

Effective action:

\[ S[g, B, \phi] = \int e^{-2\phi} \left\{ R(g) - \frac{1}{2} \left< H', H' \right>_g + 4 \left< d\phi, d\phi \right>_g \right\} \]

\[ M \]

\( g, B, \phi \) - metric, B-field, dilaton

\( H' = dB + H, \ H \) - closed 3-form

Bosonic part of type II supergravity

(\( RR \)-fields omitted)

Motivation: Understand the generalized Riemannian geometry of EAs and their (quasi-) PLTD

\( M = \pi/G, \ \pi \) - principal 3-bundle

\( G \) - connected Lie, \( d \) - quadratic

\( G \subseteq D \) - connected, closed

Lagrangian w.r.t. \( \langle \cdot, \cdot \rangle_D \)

In this talk: \( p = D \), i.e., no spectators for simplicity
Definition CA data:
- Vector bundle $q : E \to M$
- Morphism $\varphi : E \to TM$ - anchor
- Fibre-wise metric $\langle \cdot, \cdot \rangle_E$ on $E$
- $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]_E$ on $\Gamma(E)$

Axioms:
- $[\cdot, \cdot]_E$ is a differential operator
  \[
  [\cdot + \cdot']_E = ([\cdot, \cdot']_E + \varphi(\cdot) \cdot \cdot',
  \]
- $[\cdot, \cdot]_E$ is a derivation of the bracket
- The bracket $[\cdot, \cdot]_E$ and pairing $\langle \cdot, \cdot \rangle_E$ are compatible
  \[
  \langle [\cdot, \cdot]_E \cdot \cdot \rangle_E = \frac{1}{2} \varphi(\cdot') \langle \cdot, \cdot \rangle_E
  \]

Example $E = TM \oplus T^*M$
- Projection to $TM$ $\langle \cdot, \cdot \rangle_E$ - canonical
  \[
  \left[ (x, \xi) \right]_1 (y, \beta) = \left[ (x, \xi) \right], L_x \beta - i_y \xi dx - H(x, \xi)
  \]
  $H \in \Omega^3_c(M)$, $H$-twisted Dorfman bracket

Exact CA
Definition (generalized metric)

- A maximal positive subbundle
  \[ V_+ \subseteq E \text{ w.r.t. } \langle \cdot, \cdot \rangle_E \equiv g_E. \]

- \[ E = V_+ \oplus V_- \quad \text{and} \quad V_- := V_+^\perp \]

- \[ \tau \in \text{End}(E) \quad \tau(V_\pm) = \pm 1 \quad \tau^2 = 1 \]

- \[ G(y, y') := \langle y, \tau(y') \rangle_E \quad \text{is fibre-wise metric on } E \]

\[ (E_1, \langle \cdot, \cdot \rangle_E) \text{ orthogonal v.b. has a } \text{generalized metric} \]

\[ O(E_1, g_E) \text{ acts transitively on space of g.m.s} \]

- \[ TM \oplus T^* M \quad V_+ \text{-graph of a bundle map } \]
  \[ TM \to T^* M. \]

- \[ \Gamma(V_+) = \left\{ (X_1, (g + B)(x)) : x \in \mathcal{X}(M) \right\} \]

- \( g \) - Riemannian metric \( B \in \mathcal{C}^2(M) \)
**CA connections**

**Definition** CA conn \( \nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E) \)

s.t. \( \nabla \psi = \nabla (\psi_1 \cdot) : \Gamma(E) \to \Gamma(E) \) satisfies

- \( \nabla (f \psi') = f \nabla \psi (\psi') + \rho(\psi) f \cdot \psi' \)
- \( \nabla f (\psi') = f \nabla \psi (\psi') \)

and is compatible with \( \langle \cdot, \cdot \rangle \in \Gamma_1 \) i.e.,
\[ \nabla g_E = 0. \]

**Example** \( \nabla' \) - ordinary Levi-Civita connection on \( E \)

\[ \nabla' \psi := \nabla_{\rho(\psi)} \text{ is a CA connection.} \]

**Definition (Guattieri)** Torsion 3-form

\[ T_\nabla (\psi, \psi', \psi'') = (\nabla \psi \psi' - \nabla \psi' \psi) - [\psi, \psi'] \psi'' \]

\[ + \langle \nabla \psi'' \psi', \psi'' \rangle \theta_e \]

- \( \mathcal{C}^\infty(M) \) - linear, skew-symmetric
- \( \nabla \) is torsion-free if \( T_\nabla = 0 \)
- Works for CA connections
Definition

$V_+ \subseteq \mathbf{E}$ - generalized metric.

- $\nabla$ is a Levi-Civita w.r.t. to $V_+$, $\nabla \in \text{LC}(\mathbf{E}, V_+)$
- $\nabla_\psi (V_+) \subseteq V_+$
- is torsion-free.

$\text{LC}(\mathbf{E}, V_+) \neq \emptyset$; there are many LC connections.

Definition (Hohm, Zwiebach)

Curvature tensor

$$R_{\nabla} (\phi', \phi, \psi, \psi') =$$

$$\frac{1}{2} \left< \left[ \nabla_\psi, \nabla_\psi \right] - \nabla_\psi \nabla_{\psi'} \right> \phi, \phi' \rangle_{\mathbf{E}}$$

$$+ \frac{1}{2} \left( \psi \leftrightarrow \phi \psi' \leftrightarrow \phi' \right)$$

$$+ \frac{1}{2} \left< \nabla_\psi \psi, \psi' \right> \mathbf{E} \left< \nabla_\phi \phi, \phi' \right> \mathbf{E}$$

- geometrical meaning?
- $\nabla_\phi$ has all the usual symmetries including alg. Bianchi
Definition

Generalized Ricci tensor

\[ \text{Ric}_\nabla (y, y') = R_\nabla (g^{\nabla} (y^m), y, y_m, y') \]

- Symmetric in \( \text{C}^\infty (M) \)-bilinear.

\( \nabla \) is Ricci-compatible with \( V_+ = \text{Ric}_\nabla (V_+ V_-) = 0 \)

Definition

Scalar curvatures \( R_\nabla, R_\nabla^+ \)

\[ \begin{align*}
R_\nabla &= \text{Ric}_\nabla (y^m, g^{\nabla} (y^m)) \\
R_\nabla^+ &= \text{Ric} (y^m, g^{\nabla} (y^m))
\end{align*} \]

Lemma

Define \( \text{div}_\nabla (y) = \langle y^m, (y), y^m \rangle \).

If \( \nabla \) \( \nabla ' \) \( \in \text{LC}(E_1 V_+) \) \( \text{div}_\nabla ' = \text{div}_\nabla \)

then \( R_\nabla ' = R_\nabla \) \( R_\nabla^+ = R_\nabla^+ \)

and \( \text{Ric}_\nabla ' = \text{Ric}_\nabla^+ \)

Theorem

\( E = TM \oplus \mathcal{T}^* M \) with \( E_1 J_4 \).

Let \( V_+ \) corresponds to generalized metric \( (g, B) \) and \( \nabla \in \text{LC}(E_1 V_+) \)

satisfies \( \text{div}_\nabla y = \text{div}_\nabla \text{LC } g(y) - g(y) \phi \)

for some \( \phi \in \text{C}^\infty (M), \nabla \in \text{LC}(E_1 V_+ \phi) \).
Then \((g, b, \phi)\) satisfies EOM given by \(\mathcal{S}[g, b, \phi] \)
if \(R^+_\mathcal{D} = 0\) and \(\mathcal{D}\) is Ricci compatible with \(V_+\).

- Recall, \(R^+_\mathcal{D}\) and \(\text{Ric}^+_\mathcal{D}\) do not depend on the choice of \(\mathcal{D} \in \text{ELC}(E(V_+))\).
- Everything behaves nicely under CA isomorphism ("covariant") description.
KK reduction

\begin{itemize}
\item $P \to \mathbb{R} \quad \text{G-compact}$
\item $\pi \downarrow \quad \text{c = (-1, .)} \quad \text{-Killing form}$
\item $M$ \quad $g_P \quad \text{-adjoint bundle}$
\item $A \quad \text{-connection on } P, A \in \mathfrak{g}_1(p, g)$
\item $\text{with curvature } F \in \mathfrak{g}_2(M, g_P)$
\item $H_0 \in \mathfrak{g}_2(M)$
\item $E' = TM \oplus g_P \oplus T^*M \quad \text{(pre-)covariant algebroid}$
\item \quad \text{- pairing is the canonical one}
\item \quad \text{on } TM \oplus T^*M \text{ and induced by}
\item \quad \text{(-1, .) on } g_P.
\item \quad \text{- anchor - projection to } TM
\item \quad \text{- bracket - combination of}
\item \quad \text{H}_0\text{-twisted bracket and the}
\item \quad \text{Atiyah Lie-algebroid bracket}
\item \quad \text{on } TM \oplus g_P
\item \quad \text{It is a covariant algebroid if}
\item \quad \text{dH}_0 + \frac{1}{2} (F \wedge F)_g = 0$
\item \quad \text{generalized metric } V'_+ \in E'$
\item \quad \text{~ (g_0, B_0, 2\pi) } \quad V \in \mathfrak{g}_2(M, g_P)$
Effective action

\[ S_0 [g_0, B_0, \phi_0, A_1, \omega] = \]

\[ = \int e^{-2\phi_0[R(g_0) + \frac{1}{2} \langle F_1' F' \rangle_{g_0} - \frac{1}{2} \langle H_0' + \langle d\phi_0, d\phi_0 \rangle_{g_0} - 2\Lambda_0 \rangle}

F' = F'(A_1, \omega) \quad H'(B_0, F_1, \nu_1, H_0)

Einstein-Yang-Mills gravity

- \[ P = P_Y M \times M P_{spin} \]
  \[ S_0(32) \text{ or } E_8 \times E_8 \quad \text{Spin}(9,1) \]
  Heterotic supergravity

- Anomaly cancellation inflow Green-Schwarz mechanism

- Every heterotic CA \( E' \) is obtained by reduction from the brane
  \[ E = TP \oplus TP^* \quad \text{with} \quad H = \pi^*(H_0) + \frac{1}{2} CS_3(A) \]

- \( G \)-invariant generalized metric on \( E \) reduces to a generalized metric on \( E' \)
  \[ (g_1 B) \rightarrow (g_0, B_0, \omega) \]
Proposition

Let $\phi = \pi^* \phi_0$. Then there exist connections $\nabla \in \text{LC}(E, V_+, \phi)$ and $\nabla' \in \text{LC}(E, V'_+, \phi')$ s.t. $\nabla$ reduces to $\nabla'$. $\nabla$ is Ricci compatible with $V_+$ iff $\nabla'$ is Ricci compatible with $V'_+$ and

$$R^+_{\nabla} = R^+_{\nabla'} + \frac{1}{6} \dim(g)$$

Theorem (KK reduction)

For $(g, B, \phi)$ and $(g_0, B_0, \phi_0)$ related as above and for $\Lambda = \Lambda_0 + \frac{\dim(g)}{6}$ EOM for $S$ are equivalent to those of $S_0$. 
Poisson-Lie T-duality

KLIMČÍK, ŠEVERA 1994
ŠEVERA - in terms of CA and their reductions 2015 ...

- Simplest setting
  \((d, g)\) - Kac-Moody pair
  \((d, \langle \cdot, \cdot \rangle, E_1, \omega)\) - quadratic Lie Alg
  \(g \subset d\) Lagrangian
  Assume \((D, \Theta)\) integrate \((d, g)\)
  \(G \subset D\) - closed subgroup

- \(P = D \leftarrow D\)
- \(P = D \leftarrow G\)
- \(\Pi_0\)
- \(N = D/G\)

- CA \(E = TP \circ \Theta * P\)
- \(H = \frac{1}{2} C_{S_3}(\Theta)\)
- reduction of \(\Theta\) by \(D\)
- \(E'_d = (d, \langle \cdot, \cdot \rangle, E_1, \omega)\)
- reduction by \(G\)
- \(E'_g = N \times d \cong TN \circ \Theta * N\)
- Exact
- given by the left action of \(d\) on \(N\) (dressing action)
- bracketed and pairing - fibre-wise extension from \(d\)
**PLTD - Sigma models**

- Choose a g.m. \( E \subset E' \subset d \), i.e. a maximal positive subspace w.r. to \( \langle 1, d \rangle \)
  
  - \( V'_+ := N \times E'_+ \) is a g.m. in \( E'_+ = N \times d \)
  
  - \( E'_+ \cong TN + T^*N \) \( H \in H(N) \) - its Severa class

- Consider sigma model with W2W term targeted in \( N = D/\mathfrak{g} \) with backgrounds \( (g, B, H) \)

**Proposition (Severa 2017)**

Fix \( E \subset d \). Then all sigma models (for any \( G \)) are (modulo some technical assumptions) equivalent.

- In particular for a Manin triple \( (d, \mathfrak{g}, \mathfrak{g}^*) \) integrating to \( (D, g, g^*) \) this leads to PLT duality \( (g \leftrightarrow g^*) \)
PLTD - Effective actions

• in addition to the above
  fix a connection $\nabla^0 \in \text{LC}(d, E_+)$
  similar as above one finds:
  $\nabla \in \text{LC}(\text{T}N \otimes T^*N, V_+).$ By construction:
  $R^+_\nabla^0 = R^+_\nabla$ and $\nabla^0$ is Ricci compatible
  with $E_+$ iff $\nabla$ is Ricci compatible
  with $V_+$

• If $g$ is uni-modular, i.e.
  $\text{Tr}(\text{ad}_x) = 0 \quad \forall x \in g$ then
  we can choose a divergence-free $\nabla^0$
  and the corresponding
  $\nabla \in \text{LC}(\text{T}N \otimes T^*N, \phi)$
  $\phi$ - an explicit expression
  terms of $g_{1B}$ and $\Pi$ - the
  quasi-Poisson str.
  at $\text{Ad}$ of $\text{D}$ on $g.$
  on $N,$ and
  (unique up to an additive constant)

• For Mamin triple $(d, g, \mathbb{H}^*)$ formulas
  agree with von Kluge (2002, p. 17)
Theorem

\((g, B, \phi)\) satisfy EOM on \(N\) iff \(\nabla^0 \in \text{LC (} d, E_+\) is Ricci compatible with \(E_+\) and \(\nabla^0 \cdot E_+ = 0\).

These are algebraic equations for \(E_+\). From solutions of these we directly obtain solutions of EOM on \(N = D/\mathbb{G}\) for any choice of a Lagrangian \(G\).
OUTLOOK

- Explicit solutions
- Including spectaturs
- Topological T-decality
- Understand a relate to the recent approach of Severa & Valach

- Hopefully coming soon.