Mixed Hodge structures with modulus

Joint work with Florian Ivorra

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Our goal. To generalize Deligne’s category MHS of mixed Hodge structures to that of MHS with modulus MHSM;

“level ≤ 1 parts” (indicated by 1) admit geometric description:

\[
\text{(semi-abel.)} \subset (\text{Deligne } 1\text{-motives}) \cong \text{MHS}_1 \\
\cap \\
\text{(comm. alg. gp.)} \subset (\text{Laumon } 1\text{-motives}) \cong \text{MHSM}_1
\]

2/3 of this talk is devoted to “level ≤ 1 parts”, previously known by Barbieri-Viale, Kato, Russell in different languages.

We then explain the whole category MHSM, and its application to generalize Kato-Russell’s construction of Albanese varieties with modulus to 1-motives.
Commutative algebraic groups

A commutative algebraic group $G$ over $\mathbb{C}$ is an extension

$$0 \to G_{mul} \times G_{add} \to G \to G_{ab} \to 0,$$

$G_{ab}$: abel. var., $G_{mul} \cong \mathbb{G}^s_m$, $G_{add} \cong \mathbb{G}^t_a$ $(s, t \in \mathbb{Z}_{\geq 0})$.

$G$ is called semi-abelian if $G_{add} = 0$.

$AV = \{\text{ab. var.}\} \subset SA = \{\text{semi-ab.}\} \subset AG = \{\text{alg. gp.}\}$.

$X$: cpt. Riemann surface, $Y$: effective divisor on $X$ ($Y = a_1P_1 + \cdots + a_rP_r$; $P_i \in X$ distinct, $a_i \in \mathbb{Z}_{>0}$).

Generalized Jacobian $\text{Jac}(X, Y) \in AG$ classifies pairs $(L, \sigma)$ of a degree zero line b'dl. $L$ on $X$ and $\sigma: L|_Y \cong \mathcal{O}_Y$.

- $Y = \emptyset \Rightarrow \text{Jac}(X, \emptyset) = \text{Jac}(X) \in AV$;
- $Y$: reduced (i.e. $a_1 = \cdots = a_r = 1$) $\Rightarrow \text{Jac}(X, Y) \in SA$. 
Duality

\[ A \in AV \Rightarrow \exists A^\vee \in AV : \text{dual of } A; \quad \text{Jac}(X)^\vee \cong \text{Jac}(X). \]

To extend \( \vee \) to \( AG \), we need Laumon 1-motives:

**Def.** A Laumon 1-motive \( M \) is a two-term cpx. of the form

\[(*) \quad M = [\mathbb{Z}^s \times \hat{G}_a^t \rightarrow G] \quad (s, t \in \mathbb{Z}_{\geq 0}, \ G \in AG).\]

Regard \( G \in AG \) as a Laumon 1-motive by \([0 \rightarrow G]\).

\( M \) in (\( \ast \)) is called a Deligne 1-motive if \( t = 0 \) and \( G \in SA \).

\( M_1^D = \{\text{Deligne 1-motives}\} \subset M_1^L = \{\text{Laumon 1-motives}\} \)

**Generalized Jacobian can be generalized to 1-motives:**

Attach \( \text{Jac}(X, Y, Z) \in M_1^L \) to a triple \((X, Y, Z)\) of \( X : \text{cpt. Riemann surface and } Y, Z : \text{eff. divisors s.t. } |Y| \cap |Z| = \emptyset; \)

\[ \text{Jac}(X, Y, \emptyset) = \text{Jac}(X, Y), \quad \text{Jac}(X, Y, Z)^\vee \cong \text{Jac}(X, Z, Y). \]

This is best explained from the viewpoint of Hodge theory.
AV and HS\(_1\)

Recall. \(\exists\) equiv. of cat. AV \(\cong\) HS\(_1\) : Hodge str. of level \(\leq 1\).

Def. A Hodge structure of level \(\leq 1\) is a pair \(H = (H_{\mathbb{Z}}, F^0)\) of

a) \(H_{\mathbb{Z}}\) : free \(\mathbb{Z}\)-module of finite rank,

b) \(F^0 \subset H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C} : \mathbb{C}\)-subspace;

subject to conditions (\(H_{\mathbb{C}} = F^0 \oplus \overline{F^0}\) and “polarizable”).

- \(\text{Jac}(X) \leftrightarrow H_1(X, \mathbb{Z})\) for \(X\) : cpt. Riemann surf.
- \(\vee : \text{AV} \to \text{AV}\) corresponds to an easy linear algebra operation \(\text{Hom}(\_ , \mathbb{Z}(1))\) on HS\(_1\).
- \(J(X)^{\vee} \cong J(X)\) explained by Poincaré duality.

The equivalence AV \(\cong\) HS\(_1\) is extended to \(\mathcal{M}_1^D\) by Deligne.
$\mathcal{M}^D_1$ and $\text{MHS}_1$

Thm. (Deligne). $\exists$ equiv. of cat. $\mathcal{M}^D_1 \cong \text{MHS}_1$.

Def. A mixed HS of level $\leq 1$ $H = (H_\mathbb{Z}, F^0, W_{-1}, W_{-2})$ is:

a) $H_\mathbb{Z}$ : free $\mathbb{Z}$-module of finite rank,
b) $F^0 \subset H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C} : \mathbb{C}$-subspace,
c) $W_{-2} \subset W_{-1} \subset H_\mathbb{Z}$ : $\mathbb{Z}$-submodules,

s.t. $H_\mathbb{Z}/W_{-1} \cong \mathbb{Z}^s$, $W_{-1}/W_{-2} \in \text{HS}_1$, $W_{-2} \cong \mathbb{Z}(1)^t$ ($s, t \in \mathbb{Z}_{\geq 0}$)

- $(X, Y, Z)$ as above, with $Y, Z$ reduced $\Rightarrow$
  $\text{Jac}(X, Y, Z) \leftrightarrow H_1(X \setminus Y, Z, \mathbb{Z}) :$ relative homology.
- $\vee : \mathcal{M}^D_1 \rightarrow \mathcal{M}^D_1 \leftrightarrow$ an easy lin. alg. operation on $\text{MHS}_1$.
- Poincaré duality explains $\text{Jac}(X, Y, Z)^\vee \cong \text{Jac}(X, Z, Y)$.

The equivalence $\mathcal{M}^D_1 \cong \text{MHS}_1$ is extended to $\mathcal{M}^L_1$. 
At least 3 known categories: $M^L_1 \cong FHS^1_1 \cong \mathcal{H}_1 \cong MHSM^1_1$.

- FHS$^1_1$: formal HS (Barbieri-Viale, 2007)
- $\mathcal{H}_1$: MHS with additive part (Kato and Russel, 2012)
- MHSM$^1_1$: MHS with modulus (Ivorra and Y...)

Def. A MHSM of level $\leq 1$ $\mathcal{H} = (H, U, V, F)$ consists of:

a) $H = (H_0, F, W_1, W_2) \in MHS^1_1$.

b) $U, V$: finite dim. $C$-vector sp.

c) $F \subset H \otimes U \otimes V$: $C$-subspace.

satisfying certain conditions (details come later).

If $[Z \times G\hat{a} \to G] \leftrightarrow (H, U, V, F)$ by $M^L_1 \cong MHSM^1_1$,

$U \cong \text{Lie}(G_{\text{add}})$, $V \cong \text{Lie}(\hat{G}_a)$.

$Z \times G\hat{a} \to G$ by $M^L_1 \cong MHSM^1_1$.
Jacobian 1-motives

For \((X, Y, Z)\) as above, \(Y, Z\) not. nec. reduced, \(\text{Jac}(X, Y, Z) \leftrightarrow (H, U, V, \mathcal{F})\) under \(\mathcal{M}_1^L \cong \text{MHSM}_1\) with

- \(H = H_1(X \setminus Y_{\text{red}}, Z_{\text{red}}, \mathbb{Z}) \in \text{MHS}_1\) (Deligne),
- \(U = H^0(X, \mathcal{O}_X(-Y_{\text{red}})/\mathcal{O}_X(-Y))\).
- \(V = H^0(X, \mathcal{O}_X(Z - Z_{\text{red}})/\mathcal{O}_X)\).
- \(\mathcal{F} := \text{Im}(H^0(X, \Omega^1_X(Z)) \to H^1(X, [\mathcal{O}_X(-Y) \to \Omega^1_X(Z)])) \cong H_C \oplus U \oplus V\). (This map turns out to be injective.)

N.B. \(Y, Z\): reduced \(\Rightarrow U = V = 0\), i.e. \(\text{Jac}(X, Y, Z) \in \mathcal{M}_1^D\).

\(\forall : \mathcal{M}_1^L \to \mathcal{M}_1^L \leftrightarrow\) an easy lin. alg. operation on \(\text{MHSM}_1\).

Poincaré and Serre dualities imply

\[ \text{Jac}(X, Y, Z)^\forall \cong \text{Jac}(X, Z, Y). \]
MHS of arbitrary level

**MHS**: Deligne’s cat. of mixed Hodge structures.

**Formal property.** \( \text{MHS} \) is abelian; it contains \( \text{MHS}_1 \).

**Geometry.** \( X \): smooth proper variety of dimension \( d \), \( Y, Z \subset X \): effective reduced divisors, \( |Y| \cap |Z| = \emptyset \).

\( \exists H^n(X, Y, Z) \in \text{MHS} \) s.t. \( H_Z = H^n(X \setminus Z, Y, \mathbb{Z}) \) (\( n \in \mathbb{Z} \)).

**Duality.** \( H^n(X, Y, Z)^\vee \cong H^{2d-n}(X, Z, Y)(d)/(\text{tor}) \).

**Albanese.** One has \( H^{2d-1}(X, Y, Z)(d)/(\text{tor}) \in \text{MHS}_1 \), and it corresponds to the Albanese 1-motive \( \text{Alb}(X, Y, Z) \in \mathcal{M}^D_1 \) via \( \text{MHS}_1 \cong \mathcal{M}^D_1 \). [\( d = 1 \Rightarrow \text{Alb}(X, Y, Z) = \text{Jac}(X, Y, Z) \).]

**NB.** Deligne constructed \( H^n(S) \in \text{MHS} \) for any variety \( S \). One has \( H^{2d-1}(S)(d)/(\text{torsion}) \in \text{MHS}_1 \) if \( d = \text{dim } S \), and it corresponds to the Albanese 1-motive \( \text{Alb}(S) \in \mathcal{M}^D_1 \).
MHSM of arbitrary level

Def. A MHS with modulus is $\mathcal{H} = (H, U^*, V^*, \{F_p\}_{p \in \mathbb{Z}})$,

a) $H \in \text{MHS}$,

b) $\cdots \to U^p \xrightarrow{u^p} U^{p-1} \xrightarrow{u^{p-1}} \cdots$ : chain of $\mathbb{C}$-linear maps,

c) $\cdots \to V^p \xrightarrow{v^p} V^{p-1} \xrightarrow{v^{p-1}} \cdots$ : chain of $\mathbb{C}$-linear maps,

d) $F^p \subset H_{\mathbb{C}} \oplus U^p \oplus V^p : \mathbb{C}$-subspaces ($\forall p \in \mathbb{Z}$),

subject to the following conditions:

- $\dim_{\mathbb{C}}[\oplus_p(U^p \oplus V^p)] < \infty$.
- $(\text{id}_{H_{\mathbb{C}}} \oplus u^p \oplus v^p)(F^p) \subset F^{p-1}$.
- For $x \in H_{\mathbb{C}}$: $x \in F^p H_{\mathbb{C}} \iff \exists u \in U^p$ s.t. $x + u \in F^p$.
- $F^p \hookrightarrow H_{\mathbb{C}} \oplus U^p \oplus V^p \to V^p : \text{surj.}$
- $U^p \hookrightarrow H_{\mathbb{C}} \oplus U^p \oplus V^p \to H_{\mathbb{C}} \oplus U^p \oplus V^p / F^p : \text{inj.}$
Main results

Formal property. **MHSM** is abelian; it contains **MHSM_1**.

Geometry. **X** : smooth proper variety of dimension **d**.
**Y, Z ⊂ X** : eff. divisors, not nec. reduced, \(|Y| \cap |Z| = \emptyset\),
\((Y + Z)_{\text{red}}\) : strict normal crossing. \(n \in \mathbb{Z}\).
\(\exists \mathcal{H}^n(X, Y, Z) \in \text{MHSM} \) s.t. \(H = \mathcal{H}^n(X, Y_{\text{red}}, Z_{\text{red}}) \in \text{MHS}\).

Duality. \(\mathcal{H}^n(X, Y, Z)^\vee \cong \mathcal{H}^{2d-n}(X, Z, Y)(d)/(\text{tor})\).

Albanese. \(\mathcal{H}^{2d-1}(X, Y, Z)(d)/(\text{tor}) \in \text{MHSM}_1\), corresponding
to **Albanese** 1-motive \(\text{Alb}(X, Y, Z) \in \mathcal{M}_1^L\) via \(\text{MHSM}_1 \cong \mathcal{M}_1^L\).

Jacobian. When \(d = 1\), we have \(\text{Alb}(X, Y, Z) = \text{Jac}(X, Y, Z)\),
and \(\text{Jac}(X, Y, Z)^\vee \cong \text{Jac}(X, Z, Y)\) by **Duality** (above).
Relation with previous works

Kato and Russell (2012) constructed $\text{Alb}(X, Y) \in \text{AG}$ for smooth proper $X$, eff. divisor $Y$ (generalizing $\text{Jac}(X, Y)$ for curves). It agrees with our $\text{Alb}(X, Y, \emptyset)$.

Bloch and Srinivas (2000) constructed enriched HS and $H^n(S) \in \text{EHS}$ for proper (but possibly singular) var. $S$. Barbieri-Viale (2007) and Mazzari (2011) generalized EHS to formal HS. This is (almost) same as MHSM with $V^* = 0$.

Deligne constructed $H^n(S) \in \text{MHS}$ for arbitrary (possibly singular/non-proper) $S$, which is basis of all constructions.

Problem. Can one define $H^n(S) \in \text{MHSM}$ for such $S$?