Inverse Scattering Problems: To Overcome the Ill-posedness

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Outline

- Introduction
- Inverse Scattering : Analysis and Computation
- Inverse Source Problems
- On-going Research
Scattering and Inverse Scattering

- Incidence
- Scattering
- Inverse scattering
- Scattered fields
Applications

Stealth

Geophysical inspection

Cloaking

Medical imaging

Super-resolution
Application: Radon Transform

Radon transform (1917)

\[ Rf(s, \theta) = g(s, \theta) = \int_{\langle x, \theta \rangle = s} f(x) dl = \int_L f \]

Inverse Radon transform

\[ f(x) = \frac{1}{4\pi^2} \text{p.v.} \int_{S^1} d\theta \int \frac{d}{ds} g(s, \theta) ds \]

Application: Radon Transform

1979 Nobel Prize in Medicine
Computed Tomography (CT)

A. Cormack  G. Hounsfield

Integral geometry

\[ T = \int_{\gamma} \frac{1}{c(x)} \, ds = \text{Travel time} \]
Application: Seismic Inversion

\[
\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta\right) p(x, y, z, t) = 0
\]

Data: \( p(x, y, z, t) \big|_{z=0} \)

To determine \( c(x, y, z) \)?

Boundary rigidity problem:
Michel ’81’, Gromov, ’83’, …

Lens rigidity problem:
Largely open!
Application: Near-Field Optics

Diffraction limit: $\theta = \lambda/2$

2014 Noble Prize in Chemistry

Eric Betzig (92’, 93’) evanescent $\theta_1 = \lambda/10$, (95’) PSF

$$\Delta_{\text{min}} = \frac{\Delta}{\sqrt{N}} = \frac{1}{\sqrt{N}} \frac{\lambda}{2n \sin \alpha}$$

Stefan W. Hell (94’, 95’) STED

$$\Delta_{\text{min}} \approx \frac{\lambda}{2n \sin \alpha(\sqrt{1 + I_0/I_{\text{sat}}})}$$

William E. Moerner experiment

$$\begin{cases} \Delta u + k^2 u = 0 \\ \frac{\partial u}{\partial r} - ikr = o\left(\frac{1}{r}\right) \quad \text{as} \quad r \to \infty \\ u + u^i + u^r = 0 \end{cases}$$

Measurements: $u(x_1, x_2, d), \quad d - \max f < \lambda$,

to determine: $f(x_1, x_2)$
Challenges for Inverse Scattering

- Ill-posedness
- Nonlinearity
- Computation
- Uncertainty
Calderón Problem

To determine $\gamma$ from $\Lambda$
(voltage to current map)

$$\nabla \cdot (\gamma(x)\nabla u) = 0$$

$$\Lambda: u \bigg|_\Gamma \rightarrow \gamma \frac{\partial u}{\partial n} \bigg|_\Gamma$$

Calderón Problem: Progress

- Kohn & Vogelius 1984, *CPAM*
- Imanuvilov, Uhlmann & Yamamoto 2010, *JAMS*
Calderón Problem: Challenges

- Unstable! Ill-posedness...
  - Alessandrini 1987,
  - Mandache 2001
Same ill-posedness for Helmholtz/Maxwell eqns at a fixed frequency! No lucky break...

- Remedy:
  - Hybrid, Multiple frequency data
Inverse Scattering Problems

- Models: Electromagnetic, Optics, Acoustics, Elasticity
- Measurements: boundary
- Multiple Frequency
Stability for the IP of Wave Equation

\[
\left( \frac{1}{c^2(x)} \frac{\partial^2}{\partial t^2} - \Delta \right) u = 0
\]

\[
\Lambda : u \bigg|_\Gamma \rightarrow \frac{\partial u}{\partial \nu} \bigg|_\Gamma , \text{ to find } c(x)
\]

**Uniqueness:**
Boundary control theory, Belishev & Kurylev 92'

**Stability (partial results):** Uhlmann, Lassas, Vasy et.al, 98’ --now GO, No Caustics!
For wave equation in time domain, WKB expansion for GO is:

$$u(x,t) = A(x,t)e^{i\omega \phi(x,t)}, \ x \in \Omega, t \in [0, T],$$

where the frequency $$\omega \gg 1$$, and the phase function $$\phi(x,t)$$ is real. The solution is defined on a ray which follows the characteristic of the eikonal equation.

**However, the GO solution blows up at the caustics!**

Since the GO solution is local, at the caustics, the rays are intersecting and $$A(x,t) \to \infty$$.
Gaussian Beams

Gaussian Beam solution is also based on WKB form. Instead of being a local solution, each GB is a global solution.

In particular, near the ray, the phase function admits the following expansion

$$\varphi(x, t) = \mathbf{p}(t) \cdot (x - x(t)) + \frac{1}{2} (x - x(t))^T \cdot M(t) \cdot (x - x(t)) + O(\|x - x(t)\|^3)$$

where the Hessian matrix $M(t)$ has a positive definite imaginary part.

**Since GB solution conserve the energy during the propagation and is linearly independent of the other GB solutions, it will not blow up at the caustics.**
New Stability Result


**New Method:**
Gaussian beam and microlocal analysis

**Key Ingredients:**
Linearized Hamiltonian with respect to the velocity,
Stability analysis of the X-ray transform,
Gaussian beams/microlocal analysis
New Stability Result

- First stability result for the inverse problem with caustics; general well-posedness result

- Lens rigidity, first stability result for the non-simple metric case, ARMA, 2017
Objective: Stable reconstruction methods:

- Ill-posedness
- Nonlinarity
- Systematic initial guesses
- Uncertainty principle, limitation of resolution

Inverse Scattering Algorithms
Related direct imaging approaches: Linear sampling, factorization methods, transmission eigenvalues, Colton, Kirsch, Monk, Cakoni, et al

Many issues not addressed!
Inverse Medium Problem

Maxwell’s Eq:

\[ \nabla \times (\nabla \times E^t) - k^2 (1 + q(x)) E^t = 0 \]

\( k \) : wavenumber

\( q(x) > -1 \). supported in \( \Omega \subset \mathbb{R}^3 \)

\[ E^t = E^i + E \]

\( E^i \) : incidence

\[ \nabla \times (\nabla \times E^i) - k^2 E^i = 0 \]

\( E \) : scattered field

\[ \nabla \times (\nabla \times E) - k^2 (1 + q(x)) E = k^2 q(x) E^i \]

\[ \lim_{r \to \infty} r \left[ \nabla \times E \times \frac{x}{r} - ikE \right] = 0, \quad r = |x| \]
Inverse Medium Problem

Abstract Setting:

\[ M(q,k) = \text{Data}(k) \]

Previous work:


- Optimization, initial guess/stability
Low Frequency Approximation

**Born**

\[ \nabla \times (\nabla \times E) - k^2 E = k^2 q(E^i + E) \]

\[ \nu \times (\nabla \times E) + ik \nu \times (\nu \times E) = 0 \]

**Important identity:**

\[ \int_{\Omega} q(x) p_1 \cdot p_2 e^{ikx \cdot (n_1 + n_2)} \, dx = \frac{i}{k} \int_{\Gamma} (\nu \times E) \cdot ((n_2 + \nu) \times p_2) e^{ikx \cdot n_2} \, ds - \int_{\Omega} q(x) p_2 \cdot E e^{ikx \cdot n_2} \, dx. \]

**Linearization:**

\[ \int_{\Omega} q(x) e^{ikx \cdot (n_1 + n_2)} \, dx = \frac{i}{(p_1 \cdot p_2)k} \int_{\Gamma} (\nu \times E) \cdot ((n_2 + \nu) \times p_2) e^{ikx \cdot n_2} \, ds \]

\[ \hat{q}(\xi) = \int_{\Omega} q(x) e^{ikx \cdot (n_1 + n_2)} \, dx \quad \xi = k(n_1 + n_2) \quad |\xi| \leq 2k \]
Algorithm beyond Born

**Born + Recursive linearization**

Born $q_{k_0}$

Do $i = 1, 2, \ldots$ (wave number)

$q_{k_i}^0 = q_{k_{i-1}}$

Do $j = 1, \ldots, m$ (incidence)

$$
\delta q_j = \frac{1}{\beta_k} DM_j^*(q_{k_i}^{j-1}) R_j(q_{k_i}^{j-1})
$$

$q_{k_i}^j = q_{k_i}^{j-1} + \delta q_j$

End

End
Numerical Results

2D smooth medium

$q(x)$

$q_{k_0}$

$\eta = 10.2$

$\eta = 8.4$
Numerical Results

\[ \eta = 6.6 \]
\[ \eta = 4.8 \]
\[ \eta = 3.0 \]
\[ \eta = 0 \]
Features on the Algorithm I

- Multiple frequency data is crucial, spectral information
- Systematic selection of initial guesses
- Convergence, analysis is related to the stability analysis
- Other types of inverse problems in wave propagation
- Uncertainty based continuation method
Features on the Algorithm II

- Not distorted Born approximation (W. Chew et al)
- Not direct frequency hopping,…
- It is a continuation method!

At each frequency, optimization, regularization are involved, the problem needs not be solved precisely.
Uncertainty Based Continuation Method

- Inverse medium scattering:
  2-D Full aperture, Chen & Rokhlin 97’, Chen 97’, B., Chen, & Ma 00’; Limited aperture, B. & Liu 03’, B. & Li 07’; Fixed frequency, B. & Li 05’, 06’, 07’; Convergence, B. & Triki 08’; 3-D, B. & Li 04’, 05’, 07’, 08’, 09’
- Inverse source problems: B., Lin, &Triki 10’, 11’, B., Lu, Rundell, Xu, 15’
- Inverse obstacle scattering:

Inverse scattering problems with multi-frequency data
B., Li, Lin, Triki, Topical Review, Inverse Problems, 15’
Inverse Source Problems

Model problem: 2D Helmholtz equation

\[
\begin{align*}
\Delta u + k^2 u &= S \quad \text{in } R^2 \\
\frac{\partial u}{\partial r} - iku &= o(r^{-1/2}) \quad \text{as } r \to +\infty
\end{align*}
\]

(1) \( S \) compact supported

(2) Boundary measurements \( \left\{ u_k, \frac{\partial u_k}{\partial n} \right\} \) for \( k \in [k_{\min}, k_{\max}] \)

ISP:

From \( \left\{ u_k, \frac{\partial u_k}{\partial n} \right\} \) for \( k \in [k_{\min}, k_{\max}] \)

to determine \( S \)
Earlier Work

Bleistein & Cohen 77’
He & Romanov 98’,
Ammari, Gang Bao, Fleming 02’
Albanese & Monk 06’
Davaney et al 04’, 07’, …

Difficulties:
Nonuniqueness/ill-posedness at fixed frequency
Uniqueness: B., Lin, Triki, JDE, 10’

Let \( \{k_j\}_{j=1}^{\infty} \) be a bounded and strictly monotone sequence, then

\[
\begin{align*}
\left\{ u_k, \frac{\partial u_k}{\partial n} \right\}_{k=1}^{\infty}
\end{align*}
\]

uniquely determine \( S \).

Stability:

There is a critical number, when the highest frequency exceeds the number. The stability is Lipschitz; otherwise it is logarithmic.
Numerical Results for ISP

(a) The source function $S(x)$

(b) Reconstruction of $S(x)$

(c) Reconstruction of $S(x)$
$\quad k = 9, 17, 25$
$\quad k = 33, 41, 61$
Inverse Obstacle Problems

- **Multiscale**
  B., J. Lin, SIAP, 11’

- **Rough surface**
  B., P. Li, et al., 13’, 14’...
Ongoing and Future Research

- Inverse medium problems: Stability
- Inverse problems with uncertainty: Inverse random source problems, rough surface scattering
- Inverse scattering for elastic waves
- Inverse problems via AI
Stability for Multi-frequency IMP 1-D

1-D Case:

\[ \varphi''(x, k) + k^2 (1 + q(x)) \varphi(x, k) = 0, \quad x \in (0, 1) \]

IP: Given the reflection coefficients \( k \in [0, k_0] \)

to determine \( q(x) \)

Theorem(B., Triki, 17')

Let \( q, \tilde{q} \in C^m_0 (0, 1), \quad \|q\|, \|\tilde{q}\| \leq M, \quad q, \tilde{q} > q_0 > -1. \)

Then for any \( k_0 \geq k_{M,q_0}, \)

\[ \|q - \tilde{q}\|_{L^\infty} \leq C_{M,q_0} \left( \|d(k) - \tilde{d}(k)\|_{L^1(-k_0,k_0)} + \frac{1}{k_0^{m-1}} \right) \]

Key: Trace formula
Inverse Scattering for Elastic Waves

Model
\[ \nabla \cdot \sigma(u) + \rho \omega^2 u = 0 \]
\[ \sigma(u) = C : \varepsilon(u) \]
Kupadze radiation condition
Lame parameters: \( \lambda, \mu \)

**Shear + pressure**

- Forward problem
  Solution \( u^S = u - u^i \)
- Inverse Problem
  Reconstruct medium/obstacle/source from boundary measurements
Research for Scattering in Elasticity

➢ **Scattering:**

Finite element method, transparent BC  
B., G. Hu, J. Sun and T. Yin, JMPA, 18’.

Boundary integral equation method  
B., L. Xu and T. Yin, JCP, 17’

Time-domain  
B., Y. Gao, P. Li, ARMA, 18’

➢ **Inverse scattering:**

Uniqueness/stability: Beretta, Yamamoto, Uhlmann, De Hoop…  
Object: Recursive linearization algorithm for inverse elastic scattering problems with multi-frequency data.

Inverse source problems: stability analysis, B., P. Li et al.
Inverse Source Problem for Elasticity

Model problem: elasticity equation

\[
\begin{align*}
\Delta^* u + \omega^2 u &= f(x) \quad \text{in } B_R, \\
Tu &= T u \quad \text{on } \Gamma_R.
\end{align*}
\]

\[\Delta^* = \mu \Delta + (\lambda + \mu) \text{grad div}\] is the Lamé operator

ISP. Let \( f \) be a complex function with a compact support \( \Omega \subset B_R \). The ISP is to determine \( f \) from the data \( u(x, \omega), x \in \Gamma_R, \omega \in (0, K) \), where \( K > 1 \) is a constant.
\[ \| u(\cdot, \omega) \|_{\Gamma_R}^2 = \int_{\Gamma_R} (|\mathcal{T} u(x, \omega)|^2 + \omega^2 |u(x, \omega)|^2) \, ds_x \]

\[ \mathcal{F}_M = \{ f \in H^m(\Omega)^3 : \| f \|_{H^m(\Omega)^3} \leq M, \text{ supp } f = \Omega \subset B_R \} \]

**Theorem (B.-Li-Zhao)**

Let \( f \in \mathcal{F}_M \) and \( u \) be the solution of the scattering problem corresponding to \( f \). Then

\[ \| f \|_{L^2(\mathbb{R}^3)}^2 \lesssim \epsilon^2 + \frac{M^2}{K^{\frac{2}{3}} \ln c \left[ \frac{1}{4} \ln (R+1) \left( \frac{6m-15}{6m} \right)^3 \right]^{2m-5}}, \]

where

\[ \epsilon = \left( \int_0^K \omega^2 \| u(\cdot, \omega) \|_{\Gamma_R}^2 \, d\omega \right)^{\frac{1}{2}}. \]
Concluding Remarks

- Inverse scattering problems are an exciting and fast growing area of mathematics driven by interdisciplinary applications.
- Many basic math questions to be addressed; effective computational methods are in demand.
- Interactions of crossed-disciplinary efforts, Prof. Bolomey…
- New problems are emerging; AI, deep learning, big data,…
Thanks!