Maximum principle preserving ETD schemes for the nonlocal Allen-Cahn equation

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Joint work with

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Workshop on Modeling and Simulation of Interface-related Problems, National University of Singapore, Singapore, 30 April – 3 May, 2018
Outline

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   - Exponential time differencing (ETD) schemes
   - Discrete maximum principle
   - Error estimates and asymptotic compatibility
   - Discrete energy stability

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Allen-Cahn equation

(Local) Allen-Cahn equation:

\[ u_t - \varepsilon^2 \Delta u + u^3 - u = 0. \]  \hspace{1cm} (LAC)

As an \( L^2 \) gradient flow w.r.t. the free energy functional

\[ E_{local}(u) = \int \left( \frac{1}{4} (u(x)^2 - 1)^2 + \frac{\varepsilon^2}{2} |\nabla u(x)|^2 \right) \, dx, \]  \hspace{1cm} (1)

- energy stability:

\[ E_{local}(u(t_2)) \leq E_{local}(u(t_1)), \quad \forall \ t_2 \geq t_1 \geq 0. \]  \hspace{1cm} (2)

As a second order reaction-diffusion equation,

- maximum principle:

\[ \|u(\cdot, 0)\|_{L^\infty} \leq 1 \quad \Rightarrow \quad \|u(\cdot, t)\|_{L^\infty} \leq 1, \quad \forall \ t > 0. \]  \hspace{1cm} (3)
Allen-Cahn equation (continued)

Energy stable schemes:

- **Stabilized semi-implicit (SSI) scheme** [Shen-Yang, 2010]:
  find $u^{n+1}$ such that
  \[
  \frac{u^{n+1} - u^n}{\tau} - \varepsilon^2 \Delta_h u^{n+1} + (u^n)^3 - u^n + S(u^{n+1} - u^n) = 0. \tag{4}
  \]

- **Exponential time differencing (ETD) scheme** [Ju et al., 2015]:
  find $u^{n+1} = w(\tau)$ with $w(t)$ subject to
  \[
  \begin{cases}
  \frac{dw}{dt} + (S - \varepsilon^2 \Delta_h) w + (u^n)^3 - u^n - Su^n = 0, \quad t \in (0, \tau], \\
  w(0) = u^n.
  \end{cases} \tag{5}
  \]

Both schemes are easy to implement and conditionally energy stable.
Allen-Cahn equation (continued)

\[ F(u) = \frac{1}{4}(u^2 - 1)^2, \quad f(u) := F'(u) = u^3 - u. \]

What is the condition for energy stability?

\[ S \geq \frac{1}{2} \|f'(u)\|_{L^\infty}. \quad (6) \]

However, \[ f'(u) = 3u^2 - 1, \] unbounded in \( L^\infty \)!
Allen-Cahn equation (continued)

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However,

\[ f'(u) = 3u^2 - 1, \quad \text{unbounded in } L^\infty! \]

If we have that \( u \) is bounded in \( L^\infty \), then so does \( f'(u) \).

Discrete maximum principle (DMP) insures the \( L^\infty \) boundness of \( u \).
Allen-Cahn equation (continued)

Maximum principle preserving schemes:

- **first order semi-implicit scheme** [Tang-Yang, 2016]:
  \[
  \frac{u^{n+1} - u^n}{\tau} - \varepsilon^2 \Delta_h u^{n+1} + (u^n)^3 - u^n + S(u^{n+1} - u^n) = 0 \quad (7)
  \]

  condition for DMP: \( \frac{1}{\tau} + S \geq 2 \).

- **Crank-Nicolson scheme** [Hou-Tang-Yang, 2017]:
  \[
  \frac{u^{n+1} - u^n}{\tau} - \varepsilon^2 \Delta_h \frac{u^{n+1} + u^n}{2} + \frac{(u^{n+1})^3 + (u^n)^3}{2} - \frac{u^{n+1} + u^n}{2} = 0 \quad (8)
  \]

  condition for DMP: \( \tau \leq \frac{1}{2} \min\{\varepsilon^2, h^2\} \).

- **ETD scheme** (in space-continuous version) [Du-Zhu, 2005].
Nonlocal diffusion operator

Nonlocal diffusion operator \( (x \in \mathbb{R}^d) \):

\[
\mathcal{L}_\delta u(x) = \frac{1}{2} \int_{B_\delta(0)} \rho_\delta(|s|) \left( u(x + s) + u(x - s) - 2u(x) \right) \, ds. \tag{9}
\]

Kernel \( \rho_\delta : [0, \delta] \rightarrow \mathbb{R} \) is nonnegative and

\[
\frac{1}{2} \int_{B_\delta(0)} |s|^2 \rho_\delta(|s|) \, ds = d. \tag{10}
\]

Consistency of \( \mathcal{L}_\delta \) with \( \mathcal{L}_0 := \Delta \) via [Du et al., 2012]

\[
\max_x |\mathcal{L}_\delta u(x) - \mathcal{L}_0 u(x)| \leq C \delta^2 \|u\|_{C^4}. \tag{11}
\]

In particular, in 1-D case,

\[
\mathcal{L}_\delta u(x) = \frac{1}{2} \int_{-\delta}^{\delta} |s|^2 \rho_\delta(|s|) \cdot \frac{u(x + s) + u(x - s) - 2u(x)}{|s|^2} \, ds. \tag{12}
\]
Nonlocal Allen-Cahn equation

Nonlocal Allen-Cahn (NAC) equation:

\[ u_t - \varepsilon^2 \mathcal{L}_\delta u + u^3 - u = 0. \]  \hspace{1cm} \text{(NAC)}

As an \( L^2 \) gradient flow w.r.t. the free energy functional

\[ E(u) = \int \left( \frac{1}{4} (u(x)^2 - 1)^2 - \frac{\varepsilon^2}{2} u(x) \mathcal{L}_\delta u(x) \right) \, dx, \]  \hspace{1cm} \text{(13)}

- energy stability:

\[ E(u(t_2)) \leq E(u(t_1)), \quad \forall \, t_2 \geq t_1 \geq 0. \]  \hspace{1cm} \text{(14)}

Similar to the case of local Allen-Cahn equation, we can prove

- maximum principle:

\[ \|u(\cdot, 0)\|_{L^\infty} \leq 1 \quad \Rightarrow \quad \|u(\cdot, t)\|_{L^\infty} \leq 1, \quad \forall \, t > 0. \]  \hspace{1cm} \text{(15)}
Consider the initial-boundary-value problem of the NAC equation

\[ u_t - \varepsilon^2 \mathcal{L}_\delta u + u^3 - u = 0, \quad x \in \Omega, \; t \in (0, T], \]
\[ u(\cdot, t) \text{ is } \Omega\text{-periodic}, \quad t \in [0, T], \]
\[ u(x, 0) = u_0(x), \quad x \in \bar{\Omega}, \]

where \( \Omega = (0, X)^d \) is a hypercube domain in \( \mathbb{R}^d \).

Purpose:
- establish the 1st and 2nd order ETD schemes for (NAC).

Main theoretical results:
- \textit{discrete maximum principle};
- maximum-norm error estimates;
- discrete energy stability.
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Quadrature-based finite difference discretization

Setting

- \( h = X/N \): uniform mesh size (\( N \) is a given positive integer);
- \( x_i = hi \): nodes in the mesh (\( i \in \mathbb{Z}^d \) is a multi-index).

At any node \( x_i = hi \), we have

\[
\mathcal{L}_\delta u(x_i) = \frac{1}{2} \int_{B_\delta(0)} \frac{u(x_i + s) + u(x_i - s) - 2u(x_i)}{|s|^2} |s|_1 \cdot |s|^2 |s|_1 \rho_\delta(|s|) \, ds,
\]

where

- \(| \cdot |_1\): the vector 1-norm in \( \mathbb{R}^d \);
- \(| \cdot |\): the standard Euclidean norm.
At any node $x_i = hi$:

$$L_\delta u(x_i) = \frac{1}{2} \int_{B_\delta(0)} \frac{u(x_i + s) + u(x_i - s) - 2u(x_i)}{|s|^2} |s|_1 \cdot \frac{|s|^2}{|s|_1} \rho_\delta(|s|) \, ds.$$  \hfill (17)

Define the discrete version of $L_\delta$ by [Du-Tao-Tian-Yang, 2018]

$$L_{\delta, h} u(x_i) = \frac{1}{2} \int_{B_\delta(0)} I_h \left( \frac{u(x_i + s) + u(x_i - s) - 2u(x_i)}{|s|^2} |s|_1 \right) \frac{|s|^2}{|s|_1} \rho_\delta(|s|) \, ds.$$  \hfill (18)

For a function $v(s)$, the interpolation $I_h v(s)$ is piecewise linear w.r.t. each component of $s$ and

$$I_h v(s) = \sum_{s_j} v(s_j) \psi_j(s),$$

where $\psi_j$ is the piecewise $d$-multi-linear standard basis function.
Finite difference discretization of $\mathcal{L}_\delta$ reads

$$
\mathcal{L}_{\delta,h}u(x_i) = \sum_{0 \neq s_j \in B_\delta(0)} \frac{u(x_i + s_j) + u(x_i - s_j) - 2u(x_i)}{|s_j|^2} |s_j|_1 \beta_\delta(s_j),
$$

(19)

where

$$
\beta_\delta(s_j) = \frac{1}{2} \int_{B_\delta(0)} \psi_j(s) \frac{|s|^2}{|s|_1} \rho_\delta(|s|) \, ds.
$$

(20)

We have that $\mathcal{L}_{\delta,h}$ is self-adjoint and negative semi-definite.

**Lemma (Uniform consistency of $\mathcal{L}_{\delta,h}$ [Du-Tao-Tian-Yang, 2018])**

$$
\max_{x_i \in \Omega} |\mathcal{L}_{\delta,h}u(x_i) - \mathcal{L}_\delta u(x_i)| \leq C h^2 \|u\|_{C^4},
$$

(21)

where $C > 0$ is a constant independent of $\delta$ and $h$. 

Quadrature-based finite difference discretization (continued)
We order the nodes lexicographically,

- denote by $D_h \in \mathbb{R}^{dN \times dN}$ the matrix associated with $L_{\delta,h}$.

The space-discrete scheme: find $U : [0, T] \rightarrow \mathbb{R}^{dN}$ such that

$$\begin{cases}
\frac{dU}{dt} = \varepsilon^2 D_h U + U - U^3, & t \in (0, T], \\
U(0) = U_0.
\end{cases} \quad (22)$$

We know $D_h$ is

- symmetric and negative semi-definite;
- weakly diagonally dominant with all negative diagonal entries.
Introduce a stabilizing parameter $S > 0$ and define

$$L_h := -\varepsilon^2 D_h + SI, \quad f(U) := SU + U - U^3. \tag{23}$$

Then, we reach

$$\frac{dU}{dt} + L_h U = f(U), \tag{24}$$

whose solution satisfies

$$U(t + \tau) = e^{-L_h \tau} U(t) + \int_0^\tau e^{-L_h (\tau - s)} f(U(t + s)) \, ds. \tag{25}$$

We know $L_h$ is

- symmetric and positive definite;
- strictly diagonally dominant with all positive diagonal entries.
ETD methods for the temporal integration

Setting

- \( \tau = T/N_t \): uniform time step (\( N_t \) is a given positive integer);
- \( t_n = n\tau \): nodes in the time interval \([0, T]\).

At the time level \( t = t_n \), we have

\[
U(t_{n+1}) = e^{-L_h\tau} U(t_n) + \int_0^\tau e^{-L_h(\tau-s)} f(U(t_n+s)) \, ds. \tag{26}
\]

By

- approximating \( f(U(t_n+s)) \) by \( f(U(t_n)) \) in \( s \in [0, \tau] \),
- calculating the integral exactly,

we have the first order ETD scheme of (NAC):

\[
U^{n+1} = e^{-L_h\tau} U^n + \int_0^\tau e^{-L_h(\tau-s)} f(U^n) \, ds
\]

\[
= e^{-L_h\tau} U^n + L_h^{-1}(I - e^{-L_h\tau}) f(U^n). \tag{ETD1}
\]
ETD methods for the temporal integration (continued)

At the time level $t = t_n$:

$$U(t_{n+1}) = e^{-L_h \tau} U(t_n) + \int_0^\tau e^{-L_h (\tau - s)} f(U(t_n + s)) \, ds. \quad (27)$$

By approximating $f(U(t_n + s))$ by a linear interpolation based on $f(U(t_n))$ and $f(U(t_{n+1}))$, we have the second order ETD Runge-Kutta scheme of (NAC):

$$\begin{align*}
\tilde{U}^{n+1} &= e^{-L_h \tau} U^n + \int_0^\tau e^{-L_h (\tau - s)} f(U^n) \, ds, \\
U^{n+1} &= e^{-L_h \tau} U^n + \int_0^\tau e^{-L_h (\tau - s)} \left[ \left(1 - \frac{S}{\tau}\right) f(U^n) + \frac{S}{\tau} f(\tilde{U}^{n+1}) \right] \, ds.
\end{align*}$$

(ETDRK2)
Discrete maximum principle (DMP)

For both (ETD1) and (ETDRK2), we prove the DMP by induction:

- $\|U^0\|_\infty \leq \|u_0\|_{L^\infty} \leq 1$;
- assume $\|U^k\|_\infty \leq 1$, prove $\|U^{k+1}\|_\infty \leq 1$.

For the ETD1 scheme, we have

$$\|U^{k+1}\|_\infty \leq \|e^{-L_h\tau}\|_\infty \|U^k\|_\infty + \int_0^\tau \|e^{-L_h(\tau-s)}\|_\infty ds \cdot \|f(U^k)\|_\infty.$$  

We can prove

- $\|e^{-L_h\tau}\|_\infty \leq e^{-S\tau}$ for any $S > 0$ and $\tau > 0$;
- $\|f(U^k)\|_\infty \leq S$ when $S \geq 2$.

Then,

$$\|U^{k+1}\|_\infty \leq e^{-S\tau} \cdot 1 + \frac{1 - e^{-S\tau}}{S} \cdot S = 1.$$
Discrete maximum principle (continued)

- $\| e^{-L_h \tau} \|_\infty \leq e^{-S \tau}$ for any $S > 0$ and $\tau > 0$.

**Proof.** We know $L_h$ is strictly diagonally dominant with all positive diagonal entries, that is, $L_h = (\ell_{ij})$ has $\ell_{ii} > 0$, $\forall i$ and

$$|\ell_{ii}| \geq \sum_j |\ell_{ij}| + S, \quad \forall i.$$

For any $\theta(0) = \theta_0$, the solutions to $\frac{d\theta}{dt} = -L_h \theta$ satisfy [Lazer, 1971]

$$\| \theta(t_2) \|_\infty \leq e^{-S(t_2-t_1)} \| \theta(t_1) \|_\infty, \quad \forall t_2 \geq t_1 \geq 0.$$

In particular, noting that $\theta(t) = e^{-L_h t} \theta_0$, we have

$$\| e^{-L_h \tau} \theta_0 \|_\infty = \| \theta(\tau) \|_\infty \leq e^{-S \tau} \| \theta_0 \|_\infty, \quad \tau > 0.$$
Discrete maximum principle (continued)

\[ \|f(U^k)\|_\infty \leq S \text{ when } S \geq 2. \]

\[ f(U) = SU + U - U^3 \]

**Proof.** Obviously,

\[ f(-1) = -S, \quad f(1) = S. \]

For any \( \xi \in [-1, 1] \), we have

\[ f'(_\xi) = S + 1 - 3\xi^2 \geq S - 2 \geq 0. \]

Therefore,

\[ \max_{\xi \in [-1,1]} |f(\xi)| = S. \]
Discrete maximum principle (continued)

For the ETDRK2 scheme, we have

\[
\| U^{k+1} \|_\infty \leq \| e^{-L_h \tau} \|_\infty \| U^k \|_\infty 
+ \int_0^\tau \| e^{-L_h (\tau-s)} \|_\infty \left( 1 - \frac{S}{\tau} \right) f(U^k) + \frac{S}{\tau} f(\tilde{U}^{k+1}) \|_\infty \, ds.
\]

Note that \( \tilde{U}^{k+1} \) is exactly the solution to ETD1 scheme, so

\[
\| \tilde{U}^{k+1} \|_\infty \leq 1 \quad \Rightarrow \quad \| f(\tilde{U}^{k+1}) \|_\infty \leq S.
\]

For \( s \in [0, \tau] \),

\[
\left\| \left( 1 - \frac{S}{\tau} \right) f(U^k) + \frac{S}{\tau} f(\tilde{U}^{k+1}) \right\|_\infty \leq \left( 1 - \frac{S}{\tau} \right) \| f(U^k) \|_\infty + \frac{S}{\tau} \| f(\tilde{U}^{k+1}) \|_\infty \leq S.
\]

Then,

\[
\| U^{k+1} \|_\infty \leq e^{-S \tau} \cdot 1 + \frac{1 - e^{-S \tau}}{S} \cdot S = 1.
\]
Error estimates

Error estimates of ETD1 scheme

For a fixed $\delta > 0$, if $\|u_0\|_{L^\infty} \leq 1$ and $S \geq 2$, then we have

$$\|U^n - u(t_n)\|_{\infty} \leq C e^{t_n} (h^2 + \tau), \quad t_n \leq T,$$

(28)

where $C > 0$ depends on the $C^1([0, T]; C^4_{\text{per}}(\Omega))$ norm of $u$.

Error estimates of ETDRK2 scheme

For a fixed $\delta > 0$, if $\|u_0\|_{L^\infty} \leq 1$ and $S \geq 2$, then we have

$$\|U^n - u(t_n)\|_{\infty} \leq C e^{t_n} (h^2 + \tau^2), \quad t_n \leq T,$$

(29)

where $C > 0$ depends on the $C^2([0, T]; C^4_{\text{per}}(\Omega))$ norm of $u$. 
Error estimates (continued)

**Sketch of the proof for the ETD1 scheme:**

\[
U^{n+1} = e^{-L_h \tau} U^n + \int_0^\tau e^{-L_h (\tau - s)} f(U^n) \, ds.
\] (ETD1)

For given \( U^n \), the solution \( U^{n+1} \) is actually given by \( U^{n+1} = W_1(\tau) \) with the function \( W_1 : [0, \tau] \to \mathbb{R}^{dN} \) determined by

\[
\begin{cases}
\frac{dW_1(s)}{ds} = -SW_1(s) + \varepsilon^2 D_h W_1(s) + f(U^n), & s \in (0, \tau], \\
W_1(0) = U^n.
\end{cases}
\] (30)

For given \( u(x, t_n) \), the solution \( u(x, t_{n+1}) \) is determined by \( u(x, t_{n+1}) = w(x, \tau) \) with the \( \Omega \)-periodic function \( w(x, s) \) satisfying

\[
\begin{cases}
\frac{\partial w}{\partial s} = -Sw + \varepsilon^2 L_\delta w + f(w), & x \in \Omega, \ s \in (0, \tau], \\
w(x, 0) = u(x, t_n), & x \in \overline{\Omega}.
\end{cases}
\] (31)
Let $e_1(s) = W_1(s) - w(s)$. Then,
\[
\begin{cases}
\frac{de_1}{ds} = -L_h e_1 + f(U^n) - f(u(t_n)) + R_{ht}^{(1)}(s), & s \in (0, \tau], \\
e_1(0) = U^n - u(t_n) =: e^n,
\end{cases}
\tag{32}
\]
with
\[
\|R_{ht}^{(1)}(s)\|_\infty \leq C(h^2 + \tau), \quad \forall s \in (0, \tau],
\]
where $C$ depends on $\varepsilon$, $S$, and $u$. Then,
\[
e_1(t) = e^{-L_{ht}t} e_1(0) + \int_0^t e^{-L_{ht}(t-s)} [f(U^n) - f(u(t_n)) + R_{ht}^{(1)}(s)] \, ds, \quad t \in [0, \tau].
\]
Setting $t = \tau$, we have
\[
\|e_1^{n+1}\|_\infty \leq e^{-S\tau} \|e_1^n\|_\infty + \frac{1 - e^{-S\tau}}{S} [(S + 1) \|e_1^n\|_\infty + C(h^2 + \tau)] \\
\leq (1 + \tau) \|e_1^n\|_\infty + C\tau(h^2 + \tau).
\]
An application of the Gronwall’s inequality leads to the result.
Asymptotic compatibility

\[ \max_{x_i \in \Omega} |\mathcal{L}_{\delta,h} u(x_i) - \mathcal{L}_\delta u(x_i)| \leq Ch^2 \|u\|_{C^4}, \quad C \text{ independent of } \delta; \]

\[ \max_{x \in \Omega} |\mathcal{L}_\delta u(x) - \mathcal{L}_0 u(x)| \leq C\delta^2 \|u\|_{C^4}. \]

Then,

\[ \max_{x_i \in \Omega} |\mathcal{L}_{\delta,h} u(x_i) - \mathcal{L}_0 u(x_i)| \leq C(\delta^2 + h^2) \|u\|_{C^4}. \quad (33) \]
Asymptotic compatibility (continued)

Let $\hat{e}(s) = W_1(s) - \varphi(s)$, where $\varphi(x, s)$ denotes the exact solution to the local Allen-Cahn equation. Then,

$$
\begin{align*}
\frac{d\hat{e}}{ds} &= -L_h\hat{e} + f(U^n) - f(\varphi(t_n)) + \hat{R}_{h\tau}^\delta(s), \quad s \in (0, \tau], \\
\hat{e}(0) &= U^n - \varphi(t_n) =: \hat{e}^n,
\end{align*}
$$

where

$$
\|\hat{R}_{h\tau}^\delta(s)\|_\infty \leq C(\delta^2 + h^2 + \tau), \quad \forall \ s \in (0, \tau],
$$

where $C > 0$ depends on $\varepsilon$, $S$, and $\varphi$, but independent of $\delta$, $h$ and $\tau$.

Asymptotic compatibility of ETD1 scheme

If $\|\varphi_0\|_{L^\infty} \leq 1$ and $S \geq 2$, then we have

$$
\|U^n - \varphi(t_n)\|_\infty \leq Ce^{t_n}(\delta^2 + h^2 + \tau), \quad t_n \leq T,
$$

where $C > 0$ depends on the $C^1([0, T]; C^4_{\text{per}}(\overline{\Omega}))$ norm of $\varphi$. 

Discrete energy stability

We define the discretized energy $E_h$:

$$E_h(U) = \frac{1}{4} \sum_{i=1}^{dN} F(U_i) - \frac{\varepsilon^2}{2} U^T D_h U, \quad F(s) = \frac{1}{4}(s^2 - 1)^2. \quad (36)$$

Discrete energy stability of the ETD1 scheme

Under the condition $S \geq 2$, for any $\tau > 0$, we have

$$E_h(U^{n+1}) \leq E_h(U^n).$$

The proof includes two steps.
Discrete energy stability (continued)

**Step 1.** We have

\[ F(U^{n+1}) - F(U^n) = F'(U^n)(U^{n+1} - U^n) + \frac{1}{2}F''(\xi)(U^{n+1} - U^n)^2, \]

where \( \|F''(\xi)\|_\infty = \|3\xi^2 - 1\|_\infty \leq 2 \) since \( \|\xi\|_\infty \leq 1 \) due to DMP. Then, we obtain

\[ E_h(U^{n+1}) - E_h(U^n) \leq (U^{n+1} - U^n)^T(L_hU^{n+1} - f(U^n)). \quad (37) \]

**Step 2.** Solve \( f(U^n) \) from (ETD1) to get

\[ f(U^n) = (I - e^{-L_h\tau})^{-1}L_h(U^{n+1} - U^n) + L_hU^n, \]

and then,

\[ L_hU^{n+1} - f(U^n) = B_1(U^{n+1} - U^n) \]

with \( B_1 = L_h - (I - e^{-L_h\tau})^{-1}L_h \) symmetric and negative definite. So,

\[ E_h(U^{n+1}) - E_h(U^n) \leq (U^{n+1} - U^n)^T B_1(U^{n+1} - U^n) \leq 0. \]
Discrete energy stability (continued)

Discrete energy stability of the ETDRK2 scheme

Under the condition $S \geq 2$,

- for any $h > 0$ and $\tau \leq 1$, we have

$$E_h(U^{n+1}) \leq E_h(U^n) + \tilde{C}h^{-\frac{1}{2}}(h^2 + \tau)^2,$$

where $\tilde{C}$ is independent of $h$ and $\tau$;

- if $h \leq 1$ and $\tau = \lambda \sqrt{h}$ for some constant $\lambda > 0$, we have

$$E_h(U^n) \leq E_h(U^0) + \hat{C},$$

where $\hat{C}$ is independent of $h$ and $\tau$, i.e., the discrete energy is uniformly bounded.
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3. Numerical experiments
We consider the 2-D case in all the experiments.

Fractional power kernel:

\[
\rho_\delta(r) = \frac{2(4 - \alpha)}{\pi^{\delta 4 - \alpha} r^\alpha}, \quad r > 0, \quad \alpha \in [0, 4),
\]

which satisfies

\[
\frac{1}{2} \int_{B_\delta(0)} |s|^2 \rho_\delta(|s|) \, ds = d = 2.
\]

- \( \alpha \in [0, 2) \): integrable, \( \rho_\delta(|s|) \in L^1(B_\delta(0)) \), \( \mathcal{L}_\delta \) is bounded;
- \( \alpha \in [2, 4) \): non-integrable.
Convergence tests

Setting

- $\Omega = (0, 2\pi) \times (0, 2\pi)$, $T = 0.5$, $\varepsilon = 0.1$;
- smooth initial data $u_0(x, y) = 0.5 \sin x \sin y$;
- kernel: $\alpha = 1$ (integrable) and $\alpha = 3$ (non-integrable).

We consider

1. temporal convergence rate, i.e., $\tau \to 0$;
2. spatial convergence rate, i.e., $h \to 0$;
3. convergence to the local limit, i.e., $\delta \to 0$. 
Convergence tests (continued)

1. Temporal convergence rate.
Setting
- $\delta = 0.2$ and $\delta = 2$, respectively;
- $N = 256$;
- $\tau = 0.05 \times 2^{-k}$ with $k = 0, 1, \ldots, 7$;
- benchmark: ETDRK2 scheme with $\tau = 10^{-6}$.

The computed errors are almost independent of choices of $\delta$ and $\alpha$. 
Convergence tests (continued)

2. Spatial convergence rate.
Setting
- $\delta = 2$ and $\tau = T$;
- $N = 2^k$ with $k = 4, 5, \ldots, 10$;
- benchmark: $N = 4096$.

The $O(h^2)$ convergence rate is observed as $h \to 0$. 
3. Convergence to the local limit.

Setting

- $N = 4096$ and $\tau = T$;
- local solution: ETDRK2 scheme for LAC equation.

<table>
<thead>
<tr>
<th>$\delta = 0.2$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>error</td>
<td>rate</td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.076e-5</td>
<td>*</td>
</tr>
<tr>
<td>$\delta/2$</td>
<td>2.703e-6</td>
<td>1.9927</td>
</tr>
<tr>
<td>$\delta/4$</td>
<td>6.250e-7</td>
<td>2.1124</td>
</tr>
<tr>
<td>$\delta/8$</td>
<td>1.580e-7</td>
<td>1.9835</td>
</tr>
</tbody>
</table>

The $O(\delta^2)$ convergence rate is observed as $\delta \to 0$. 
Stability tests

For the case $\rho_\delta(|s|) \in L^1(B_\delta(0))$, i.e., $\alpha \in [0, 2)$, denote

$$C_\delta = \int_{B_\delta(0)} \rho_\delta(|s|) \, ds = \frac{4(4 - \alpha)}{(2 - \alpha)\delta^2}.$$

**Theorem [Du-Yang, 2016]**

The steady state solution $u^*$ to (NAC) is continuous if $\varepsilon^2 C_\delta \geq 1$.

Setting

- $\Omega = (0, 2\pi) \times (0, 2\pi), \varepsilon = 0.1$;
- $N = 512, \tau = 0.01$;
- random initial data ranging from $-0.9$ to $0.9$ uniformly;
- integrable kernel: $\alpha = 1$ (now $\varepsilon^2 C_\delta \geq 1$ leads to $\delta \leq 2\sqrt{3}\varepsilon$);
- $\delta = 0, \delta = 3\varepsilon, \delta = 4\varepsilon$. 
Stability tests (continued)

From top to bottom: $\delta = 0, 3\varepsilon, 4\varepsilon$.
From left to right: $t = 6, 14, 50, 180$. 
Stability tests (continued)

From left to right: \( \delta = 0, 3\varepsilon, 4\varepsilon \).
Top: maximum norms; bottom: energies.
Discontinuity in the steady state solution

For the case $\rho_\delta(|s|) \in L^1(B_\delta(0))$, i.e., $\alpha \in [0, 2)$, denote

$$C_\delta = \int_{B_\delta(0)} \rho_\delta(|s|) \, ds = \frac{4(4 - \alpha)}{(2 - \alpha)\delta^2}.$$

**Theorem [Du-Yang, 2016]**

Under certain assumptions, if $\varepsilon^2 C_\delta < 1$, the locally increasing $u^*$ has a discontinuity at $x^*$ with the jump

$$[u^*](x^*) = 2 \sqrt{1 - \varepsilon^2 C_\delta}. \quad (40)$$

Setting

- $\Omega = (0, 2\pi) \times (0, 2\pi), \varepsilon = 0.1$;
- $N = 2048, \tau = 0.01$;
- smooth initial data;
- integrable kernel: $\alpha = 1$. 
Discontinuity in the steady state solution (continued)

\[ \text{theoretical jump} = 2\sqrt{1 - \frac{0.12}{\delta^2}}, \quad \delta > \delta_0 = \sqrt{0.12} \approx 0.3464. \]

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>0.2</th>
<th>0.8</th>
<th>1.6</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>theoretical jumps</td>
<td>0</td>
<td>1.802776</td>
<td>1.952562</td>
<td>1.988247</td>
</tr>
<tr>
<td>numerical jumps</td>
<td>0</td>
<td>1.804496</td>
<td>1.952713</td>
<td>1.988242</td>
</tr>
</tbody>
</table>
Discontinuity in the steady state solution (continued)

(a) $\delta = 0.2$: solutions at $t = 1, 40, 55$, cross-sections with $y = \pi$

(b) $\delta = 0.8$: solutions at $t = 1, 3, 20$, cross-sections with $y = \pi$

(c) $\delta = 3.2$: solutions at $t = 1, 3, 20$, cross-sections with $y = \pi$
For the NAC equation

\[ u_t - \varepsilon^2 \mathcal{L}_\delta u + u^3 - u = 0, \quad \text{(NAC)} \]

we present the first and second order ETD schemes by using

- quadrature-based difference method for spatial discretization,
- exponential time differencing methods for temporal integration,

and obtain

- discrete maximum principle,
- error estimates and asymptotic compatibility,
- discrete energy stability.
Something to consider further:
- high-order and other schemes preserving the maximum principle;
- not DMP, but $L^\infty$ stable schemes for high-order PDE;
- asymptotic compatibility for nonlocal C-H equation or others.

Thanks for your attention!