Reciprocal theorem: From local equations to symmetry over the whole system

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Outline

Onsager’s reciprocal symmetry & Lorentz’s reciprocal theorem

Reciprocal symmetry: from the local to the global

1. A phase field model
2. Micropolar fluids
3. Heat conduction

Summary
Consider a closed system whose entropy is maximized at equilibrium. Close to equilibrium, the entropy can be expressed as

\[ S = S_e + \Delta S(\alpha) = S_e - \frac{1}{2} \beta_{ij} \alpha_i \alpha_j \]

where \( \beta_{ij} \) form a symmetric and positive definite matrix.

Spontaneous irreversible processes toward equilibrium driven by the thermodynamic forces

\[ X_i \equiv \frac{\partial \Delta S}{\partial \alpha_i} = -\beta_{ij} \alpha_j \]

In linear response regime, the kinetic equations are linear:

\[ \dot{\alpha}_i = L_{ij} X_j \]

Onsager’s Reciprocal Relations: \( L_{ij} = L_{ji} \)
For his discovery of the reciprocal relations, Lars Onsager was awarded the 1968 Nobel Prize in Chemistry.

Nobel Prize in Chemistry (1968)
Linear Kinetic Equations

\[ \dot{\alpha}_i = L_{ij} X_j, \]
\[ X_i = R_{ij} \dot{\alpha}_j, \quad L_{ij} R_{jk} = \delta_{ik}. \]

\[ \dot{S} (\alpha, \dot{\alpha}) - \Phi_S (\dot{\alpha}, \dot{\alpha}) \]

for closed systems
to be maximized with respect to the rates \( \{\dot{\alpha}_i\} \)
\[ \dot{S} = X_i \dot{\alpha}_i \]
is the rate of change of the entropy,
\[ \Phi_S (\dot{\alpha}, \dot{\alpha}) = R_{ij} \dot{\alpha}_i \dot{\alpha}_j / 2 \]
is half the rate of entropy production.

Variational Principle

\[ \mathcal{O} = \dot{S} + \dot{S}^* - \Phi_S (\dot{\alpha}, \dot{\alpha}), \]

for open systems
\[ \dot{S}^* \]
is the rate of entropy given by the system to the environment
Isothermal systems

\[
\dot{S}^* = -\dot{Q}/T = -\dot{U}/T,
\]

\[
\dot{F} \equiv \dot{U} - T\dot{S} = -T\left(\dot{S} + \dot{S}^*\right)
\]

the dissipation function \( \Phi_F (\dot{\alpha}, \dot{\alpha}) \equiv T\Phi_S (\dot{\alpha}, \dot{\alpha}) \)

\[
\mathcal{R} = \dot{F} (\alpha, \dot{\alpha}) + \Phi_F (\dot{\alpha}, \dot{\alpha}),
\]

to be minimized with respect to the rates \( \{\dot{\alpha}_i\} \)

Lord Rayleigh's
Principle of the Least Dissipation of Energy

HYDRODYNAMICS
\[ R = \dot{F}(\alpha, \dot{\alpha}) + \Phi_F(\dot{\alpha}, \ddot{\alpha}), \]

\[ R = \frac{\partial F}{\partial \alpha_i} \dot{\alpha}_i + \frac{1}{2} \zeta_{ij} \dot{\alpha}_i \dot{\alpha}_j, \]

Minimization of \( R \) with respect to the rates gives the kinetic equations

\[- \frac{\partial F}{\partial \alpha_i} = \zeta_{ij} \dot{\alpha}_j, \]

which can be interpreted as a balance between the **reversible** force and the **dissipative** force. \( \zeta_{ij} \) forming a symmetric and positive definite matrix

**The principle of the least energy dissipation (PLED) leads to** \( \dot{F} = -2\Phi_F \).
HYDRODYNAMICS

Stokes equation from the PLED

\[ \Phi_v = \int_{\Omega} d\vec{r} \frac{\eta}{4} \left( \partial_i v_j + \partial_j v_i \right)^2 \]  \text{in the bulk region}

\[ \nabla \cdot \vec{v} = 0 \] and no slip at liquid-solid interface

\[ R = \Phi_v \] with fluid velocity as the rate

\[ -\nabla p + \nabla \cdot \left[ \eta \left( \nabla \vec{v} + \nabla \vec{v}^T \right) \right] = 0 \]  \text{Stokes equation}

\[ \vec{\sigma} \equiv -p \hat{I} + \vec{\sigma}^{\text{vis}} \; ; \quad \vec{\sigma}^{\text{vis}} \equiv \eta \left[ \nabla \vec{v} + \left( \nabla \vec{v} \right)^T \right] ; \quad \hat{\tau} \equiv \hat{I} - \hat{n} \hat{n} \]
Hydrodynamic Reciprocal Relations (HRR)

Lorentz Reciprocal Theorem for Stokes Flows

Consider two real flows \((\vec{v}^{(1)}, \vec{\sigma}^{(1)})\) and \((\vec{v}^{(2)}, \vec{\sigma}^{(2)})\)

\[
\int_{\partial\Omega} dS \; \hat{n} \cdot \vec{\sigma}^{(1)} \cdot \vec{v}^{(2)} = \int_{\partial\Omega} dS \; \hat{n} \cdot \vec{\sigma}^{(2)} \cdot \vec{v}^{(1)}
\]

\(\partial\Omega\) can be moving.

1. The system is derived from Onsager’s variational principle.
2. The system is linear.
\[ \dot{\mathbf{V}}_i = \mathbf{V}_i + \mathbf{w}_i \times \mathbf{r}_i \]

\[ \partial \Omega = \sum_{i=1}^{N} \partial \Omega^i \]

\[ F_{k}^{(1)} \ddot{x}_{k}^{(2)} = F_{k}^{(2)} \ddot{x}_{k}^{(1)} \]

Generalized velocities of the solid objects

\[ F_k = \zeta_{kl} \dot{x}_l \]

Conjugate forces in linear response regime

HRR: \[ \zeta_{kl} = \zeta_{lk} \]

\[ \int_{\partial \Omega} dS \mathbf{n} \cdot \mathbf{\sigma}^{(1)} \cdot \mathbf{V}^{(2)} \]

\[ = \sum_{i=1}^{N} \left( \int_{\partial \Omega^i} dS \mathbf{n} \cdot \mathbf{\sigma}^{(1)} \right) \cdot \mathbf{V}^{i(2)} + \sum_{i=1}^{N} \left[ \int_{\partial \Omega^i} dS \delta \mathbf{r}_i \times (\mathbf{\hat{n}} \cdot \mathbf{\sigma}^{(1)}) \right] \cdot \mathbf{\omega}^{i(2)} \]

Force by the particle on the fluid

Torque by the particle on the fluid
Bacterial Motility and HRR

Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)

\[ F = A \nu + B \Omega \]

\[ N = C \nu + D \Omega \]
I. A Phase Field Model

Consider a system enclosed by a solid boundary where $\partial_n \phi = 0$.

The free energy functional $F[\phi(r)]$ takes the Cahn-Hilliard form

Rate of change of the free energy

$$\dot{F} = \int \mu \frac{\partial \phi}{\partial t} \, dr = \int \mu (-\nabla \cdot J) \, dr = \int \nabla \mu \cdot J \, dr = \int \nabla \mu \cdot (\phi v + j) \, dr$$

in which

$$\mu = \frac{\delta F}{\delta \phi} = -K\nabla^2 \phi + \frac{d}{d\phi} f(\phi), \quad \frac{\partial \phi}{\partial t} = -\nabla \cdot J, \quad J = \phi v + j.$$  

Dissipation functional

$$\Phi = \int \eta \left[ \nabla v + (\nabla v)^T \right]^2 \, dr + \int \frac{j^2}{2M} \, dr$$

with $\eta = \eta(\phi)$ & $M = M(\phi)$.
Application of Onsager’s variational principle using the action

\[ R = \dot{F} + \Phi \]

**Momentum equation**

\[-\nabla p - \phi \nabla \mu + \nabla \cdot \sigma_{\text{visc}} = 0,\]

with \( \sigma_{\text{visc}} = \eta \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right] \) and \( p \) due to \( \nabla \cdot \mathbf{v} = 0 \)

**Diffusive current density**

\[ \mathbf{j} = -M \nabla \mu \]

**Boundary conditions:**

\[ \partial_n \phi = 0 \ \& \ \mathbf{v}_\tau = 0 \quad \text{(dynamic BCs)} \]

\[ j_n = 0 \ \& \ \nu_n = 0 \quad \text{(kinematic BCs)} \]
Consider a system with solid wall (where \( \frac{\partial}{\partial n} \phi = 0 \) is applied) and inlets/outlets. \( \text{An open system} \)

\[
\dot{F} = \int \mu \frac{\partial \phi}{\partial t} \, dr = \int \mu (-\nabla \cdot J) \, dr = \int \mu \left[ -\nabla \cdot (\phi v + j) \right] \, dr
\]

becomes

\[
\dot{F} = \int -\mu J_n \, dS + \int n \cdot \sigma \cdot v \, dS + \int \nabla \mu \cdot j \, dr + \int -\sigma : \nabla v \, dr \]

as it should be.

1. Free energy pumped into the system
2. Work done onto the system with \( \sigma = -p I + \sigma_{\text{visc}} \)
3. Diffusive dissipation \( \int \nabla \mu \cdot j \, dr = \int -\frac{j^2}{M} \, dr \)
4. Viscous dissipation \( \int -\sigma : \nabla v \, dr = \int -\sigma_{\text{visc}} : \nabla v \, dr = \int -\frac{\eta}{2} \left[ \nabla v + (\nabla v)^T \right]^2 \, dr \)
Rate of change of the free energy can be expressed as

\[
\dot{F} = \int -\mu J_n \, dS + \int n \cdot \sigma \cdot v \, dS + \int -\frac{j^2}{M} \, dr + \int -\frac{\eta}{2} \left[ \nabla v + (\nabla v)^T \right]^2 \, dr
\]

Below we consider stationary states with

\[
\dot{F} = 0, \quad \frac{\partial \phi}{\partial t} = -\nabla \cdot J = 0, \quad \nabla \cdot v = 0.
\]

Note that without inlet/outlet, the two \( \int \cdots dS \) would vanish, and the stationary states become the global equilibrium state.
At equilibrium state, we have

\[ \dot{\mathbf{j}}_{eq} = 0, \quad \mu_{eq} = \text{const.}, \quad \mathbf{v}_{eq} = 0, \quad p_{eq} = \text{const.}, \quad \mathbf{\sigma}_{eq} = -p_{eq} \mathbf{I} \]

\[ \int J_n \, dS = 0 \quad \& \quad \int \nu_n \, dS = 0 \]

\[ \dot{F} = 0 : \quad \text{Boundary contributions fully dissipated} \]

\[ \int -\mu J_n \, dS + \int \mathbf{n} \cdot \mathbf{\sigma} \cdot \mathbf{v} \, dS = \int \frac{j^2}{M} \, d\mathbf{r} + \int \frac{\eta}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]^2 \, d\mathbf{r} \]

becomes

\[ \int - (\mu - \mu_{eq}) J_n \, dS + \int \mathbf{n} \cdot (\mathbf{\sigma} + p_{eq} \mathbf{I}) \cdot \mathbf{v} \, dS = \int \frac{j^2}{M} \, d\mathbf{r} + \int \frac{\eta}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]^2 \, d\mathbf{r} \]

\[ \tilde{\mu} = \mu - \mu_{eq} \quad \& \quad \tilde{\mathbf{\sigma}} = \mathbf{\sigma} + p_{eq} \mathbf{I} \]

Net forces due to deviation from global equilibrium state

\[ \int -\tilde{\mu} J_n \, dS + \int \mathbf{n} \cdot \tilde{\mathbf{\sigma}} \cdot \mathbf{v} \, dS = \int \frac{j^2}{M} \, d\mathbf{r} + \int \frac{\eta}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]^2 \, d\mathbf{r} \]
If the system is very close to the global equilibrium state, then it can be linearized around the equilibrium state.

\[
\int -\tilde{\mu} J_n \, dS + \int \mathbf{n} \cdot \mathbf{\bar{\sigma}} \cdot \mathbf{v} \, dS = \int \frac{\mathbf{j}^2}{M} \, d\mathbf{r} + \int \frac{\eta}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]^2 \, d\mathbf{r}
\]

It can be expressed as

\[
\sum_{\alpha} F_{\alpha} I_{\alpha} = \sum_{m,n} j_m \mathcal{S}_{mn} j_n
\]

\[
\begin{align*}
F_{\alpha} & \quad \& \quad I_{\alpha} \quad \text{forces and fluxes at the boundary} \\
\sum_{m} j_m & \quad \text{fluxes in the bulk region}
\end{align*}
\]

If the system is very close to the global equilibrium state, then it can be linearized around the equilibrium state.

\[
F_{\alpha} = A_{\alpha m} j_m , \quad I_{\alpha} = B_{\alpha m} j_m
\]

Note that matrices \( A \) and \( B \) are independent of the dynamic state. They are properties of the equilibrium state.
\[ F^T I = j^T \zeta j, \quad F = A j, \quad I = B j \quad \rightarrow \quad A^T B = \zeta \]

Note that \( \zeta \) is not only symmetric but also independent of the dynamic state.

Consider two dynamic states labelled by superscripts (1) and (2)

\[
\begin{bmatrix} F^{(1)} \end{bmatrix}^T I^{(2)} = \begin{bmatrix} j^{(1)} \end{bmatrix}^T \zeta j^{(2)} = \begin{bmatrix} j^{(2)} \end{bmatrix}^T \zeta j^{(1)} = \begin{bmatrix} F^{(2)} \end{bmatrix}^T I^{(1)}
\]

Linear response: \( F_\alpha = L_{\alpha\beta} I_\beta \)

Reciprocal symmetry for forces and fluxes at the boundary

\[ L_{\alpha\beta} = L_{\beta\alpha} \]
Reciprocal symmetry exists for local constitutive equations and constitutive equations over the whole system.

Local constitutive equations are valid over a time interval that is macroscopically short yet microscopically long. Local constitutive equations are valid in the proximity of local equilibrium and applied at a time instant in a macroscopic model.

Constitutive equations over the whole system are valid for stationary states in the proximity of global equilibrium. Why the stationary states? Constitutive equations over the whole system are valid over a time interval that is macroscopically long.
\[ \sum_{\alpha} F_{\alpha} I_{\alpha} = \sum_{m,n} j_{m} \zeta_{mn} j_{n} \]

- \( F_{\alpha} \) \& \( I_{\alpha} \) forces and fluxes at the boundary
- \( j_{m} \) fluxes in the bulk region

\( I_{\alpha} \) does not vanish at the boundary if

- if the boundary is **open**
  — the inlets/outlets
  (for the phase field model in **Part I**),
- or
- if the solid boundary is **moving**
  — the surface of a moving solid particle
  (for the micropolar fluids in **Part II**).
II. Micropolar Fluids

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  
Mass

\[ \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = \nabla \cdot \mathbf{\sigma} \]  
Momentum

\[ \frac{\partial}{\partial t} (\rho I \mathbf{\omega}) + \nabla \cdot (\rho I \mathbf{v} \mathbf{\omega}) = \rho I \left( \frac{\partial \mathbf{\omega}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{\omega} \right) = \nabla \cdot \mathbf{C} + \mathbf{\Omega} \]  
Angular momentum

\( I \omega \): internal angular momentum per unit mass

\[ \mathbf{\sigma} = -p \mathbf{I} + \kappa \left( \nabla \cdot \mathbf{v} \right) \mathbf{I} + 2\eta \mathbf{E}_v + \frac{1}{2} \mathbf{\varepsilon} \cdot \mathbf{\Omega} \]  
Newtonian-like micropolar fluids

\[ \mathbf{C} = \nu_1 \left( \nabla \cdot \mathbf{\omega} \right) \mathbf{I} + 2\nu_2 \mathbf{E}_\omega \]

\[ \mathbf{E}_v = \frac{1}{2} \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right] - \frac{1}{3} (\nabla \cdot \mathbf{v}) \mathbf{I} \quad \& \quad \mathbf{E}_\omega = \frac{1}{2} \left[ \nabla \mathbf{\omega} + (\nabla \mathbf{\omega})^T \right] - \frac{1}{3} (\nabla \cdot \mathbf{\omega}) \mathbf{I} \]

\[ \mathbf{\Omega} = \zeta \left( \frac{1}{2} \nabla \times \mathbf{v} - \mathbf{\omega} \right) \]  
symmetric, traceless
Free energy functional is given by
$$F[\rho(r)] = \int f[\rho(r)] \, dr$$
from which the pressure is determined as
$$p[\rho(r)] = \mu(\rho) \rho - f(\rho) \text{ with } \mu(\rho) = df(\rho) / d\rho.$$  

Application of Onsager’s variational principle using the action

$$R = \dot{F} + \Phi$$

$$\Phi = \frac{1}{2} \kappa (\nabla \cdot \mathbf{v})^2 + \eta \mathbf{E}_v : \mathbf{E}_v + \frac{1}{2} \nu_1 (\nabla \cdot \omega)^2 + \nu_2 \mathbf{E}_\omega : \mathbf{E}_\omega + \frac{1}{2} \zeta \left( \frac{1}{2} \nabla \times \mathbf{v} - \omega \right)^2$$

velocity spin cross coupling

$$\nabla \cdot \sigma = 0, \quad \sigma = -p(\rho) \mathbf{I} + \sigma_{\text{visc}}$$

$$\nabla \cdot \mathbf{C} + \mathbf{\Omega} = 0$$

with all the necessary constitutive equations given above.
It can be shown that the rate of change of the free energy is given by

\[
\dot{F} = \int_{\partial\Omega} \mathbf{n} \cdot \sigma \cdot \mathbf{v} \, dS + \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{C} \cdot \omega \, dS
\]

\[
- \left[ \kappa (\nabla \cdot \mathbf{v})^2 + 2\eta \mathbf{E}_v : \mathbf{E}_v + \nu_1 (\nabla \cdot \omega)^2 + 2\nu_2 \mathbf{E}_\omega : \mathbf{E}_\omega + \zeta \left( \frac{1}{2} \nabla \times \mathbf{v} - \omega \right)^2 \right]
\]

Consider stationary states in the proximity of global equilibrium

\[
\mathbf{v}_{eq} = 0, \quad \omega_{eq} = 0, \quad \sigma_{eq} = -p_{eq} \mathbf{I}, \quad p_{eq} = \text{const.}, \quad \rho_{eq} = \text{const.}
\]

In a stationary state, we have \( \dot{F} = 0 \) \& \( \nabla \cdot (\rho \mathbf{v}) = 0 \)

Near equilibrium, we have \( \nabla \cdot \mathbf{v} = 0 \)

\[
\int_{\partial\Omega} \mathbf{n} \cdot (\sigma - \sigma_{eq}) \cdot \mathbf{v} \, dS + \int_{\partial\Omega} \mathbf{n} \cdot \mathbf{C} \cdot \omega \, dS
\]

\[
= \kappa (\nabla \cdot \mathbf{v})^2 + 2\eta \mathbf{E}_v : \mathbf{E}_v + \nu_1 (\nabla \cdot \omega)^2 + 2\nu_2 \mathbf{E}_\omega : \mathbf{E}_\omega + \zeta \left( \frac{1}{2} \nabla \times \mathbf{v} - \omega \right)^2
\]
It can be expressed as
\[ \sum_{\alpha} F_{\alpha} I_{\alpha} = \sum_{m,n} j_m \zeta_{mn} j_n \]

\( F_{\alpha} \) & \( I_{\alpha} \) forces and fluxes at the boundary
\( j_m \) fluxes in the bulk region

If the system is very close to the global equilibrium state, then it can be linearized around the equilibrium state.

\[ F_{\alpha} = A_{\alpha m} j_m, \quad I_{\alpha} = B_{\alpha m} j_m \]

Note that matrices \( A \) and \( B \) are independent of the dynamic state. They are properties of the equilibrium state.

\[ F^T I = j^T \zeta j, \quad F = Aj, \quad I = Bj \quad \rightarrow \quad A^T B = \zeta = \zeta^T \]

Consider two dynamic states labelled by superscripts (1) and (2)

\[ \begin{bmatrix} F^{(1)} \end{bmatrix}^T \begin{bmatrix} I^{(2)} \end{bmatrix} = \begin{bmatrix} j^{(1)} \end{bmatrix}^T \zeta \begin{bmatrix} j^{(2)} \end{bmatrix} = \begin{bmatrix} j^{(2)} \end{bmatrix}^T \zeta \begin{bmatrix} j^{(1)} \end{bmatrix} = \begin{bmatrix} F^{(2)} \end{bmatrix}^T \begin{bmatrix} I^{(1)} \end{bmatrix} \]

Reciprocal symmetry for forces and fluxes at the boundary
III. Non-isothermal Situation: Thermal Diffusion

\[ \mathbf{j} = \tilde{\mathbf{\alpha}}(T) \cdot \nabla \frac{1}{T} \]

for heat current density with \( \tilde{\mathbf{\alpha}}(T) = \left[ \tilde{\mathbf{\alpha}}(T) \right]^T \)

\[
\int_{\partial \Omega} \frac{j_n^{(1)}}{T^{(2)}} \, dS = \int_{\Omega} \nabla \cdot \frac{j^{(1)}}{T^{(2)}} \, d\mathbf{r} = \int_{\Omega} \nabla \cdot \frac{j^{(1)}}{T^{(2)}} \, d\mathbf{r} + \int_{\Omega} \nabla \frac{1}{T^{(2)}} \cdot j^{(1)} \, d\mathbf{r} \\
\n\n\n\n\n\n
\[ \nabla \cdot \mathbf{j} = 0 \] \text{ for stationary states}

\[
\int_{\partial \Omega} \frac{j_n^{(1)}}{T^{(2)}} \, dS = \int_{\Omega} \nabla \frac{1}{T^{(2)}} \cdot j^{(1)} \, d\mathbf{r} = \int_{\Omega} \nabla \frac{1}{T^{(2)}} \cdot \tilde{\mathbf{\alpha}}(T^{(1)}) \cdot \nabla \frac{1}{T^{(1)}} \, d\mathbf{r} \\
\int_{\partial \Omega} \frac{j_n^{(2)}}{T^{(1)}} \, dS = \int_{\Omega} \nabla \frac{1}{T^{(1)}} \cdot j^{(2)} \, d\mathbf{r} = \int_{\Omega} \nabla \frac{1}{T^{(1)}} \cdot \tilde{\mathbf{\alpha}}(T^{(2)}) \cdot \nabla \frac{1}{T^{(2)}} \, d\mathbf{r} \\
\n\text{They are equal if} \left[ \tilde{\mathbf{\alpha}}(T^{(1)}) \right]^T = \tilde{\mathbf{\alpha}}(T^{(2)}). \]
\[
\left[ \tilde{\lambda}(T^{(1)}) \right]^T = \tilde{\lambda}(T^{(2)})
\]

\[
\tilde{\lambda}(T^{(1)}) = \tilde{\lambda}(T^{(2)}) \approx \tilde{\lambda}(T_{eq}) = \left[ \tilde{\lambda}(T_{eq}) \right]^T
\]

Stationary states in the proximity of global equilibrium AGAIN!

\[
\int_{\partial \Omega} \frac{j_n^{(1)}}{T^{(2)}} |_{T^{(1)}} dS = \int_{\Omega} \nabla \frac{1}{T^{(2)}} \cdot \tilde{\lambda}(T_{eq}) \cdot \nabla \frac{1}{T^{(1)}} dr = \int_{\partial \Omega} \frac{j_n^{(2)}}{T^{(1)}} |_{T^{(2)}} dS
\]

Reciprocal symmetry over the whole system

\[
\int_{\partial \Omega} j_n^{(1)} \left( \frac{1}{T^{(2)}} - \frac{1}{T_{eq}} \right) dS = \int_{\partial \Omega} j_n^{(2)} \left( \frac{1}{T^{(1)}} - \frac{1}{T_{eq}} \right) dS
\]

rate of entropy flowing out of the system = rate of entropy production in the system

\[
\int_{\partial \Omega} \frac{j_n}{T} dS = \int_{\Omega} \nabla \frac{1}{T} \cdot \tilde{\lambda}(T_{eq}) \cdot \nabla \frac{1}{T} dr \quad \text{for} \ (1) = (2)
\]
Systems with Inlets & Outlets

Reciprocal theorem for forces and fluxes at open boundaries, which are directly related to the operation of the “device”.

\[ F_\alpha = L_{\alpha \beta} I_\beta \]
\[ L_{\alpha \beta} = L_{\beta \alpha} \]

Electro-osmotic flows
(from Wikipedia)
Concluding remarks

1. The system is derived from Onsager’s variational principle.

2. The system is in the immediate proximity of global equilibrium.

3. Open or moving boundary for non-equilibrium stationary states.

Electro-osmotic flows
Micropolar fluids
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Non-isothermal situation
(Thermal diffusion)
Thank You
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