Low degree cohomologies of congruence groups

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1. Cohomologies of arithmetic manifolds

- \( G \): real reductive group;
- \( K \subset G \): a maximal compact subgroup;
- \( \Gamma \subset G \): a lattice.

**Problem:**

\[
H^i(\Gamma; \mathbb{C}) = H^i(\Gamma \backslash G/K; \mathbb{C}) = ?
\]

Here

\[
\mathbb{C}(U) := \{ \Gamma \text{-invariant locally constant function } f : \tilde{U} \to \mathbb{C} \},
\]

and \( \tilde{U} \) is the pre-image of \( U \) under the map \( G/K \to \Gamma \backslash G/K \).
• $F$: an irreducible finite dimensional representation of $G$;

More generally,

$$\check{H}^i(\Gamma; F) = H^i(\Gamma \backslash G/K; F) =?$$

where

$$F(U) := \{ \Gamma\text{-equivariant locally constant function } f : \check{U} \to F \},$$

and $\check{U}$ is the pre-image of $U$ under the map $G/K \to \Gamma \backslash G/K$.

• Interesting in topology.

• Important in arithmetic.
There is a linear map

\[ F^\Gamma \otimes \mathit{H}^i_{\text{ct}}(G; \mathbb{C}) = \mathit{H}^i_{\text{ct}}(G; F^\Gamma) \to \mathit{H}^i_{\text{ct}}(\Gamma; F^\Gamma) \to \mathit{H}^i(\Gamma; F). \]

Remark: We have

\[ \mathit{H}^i_{\text{ct}}(G; \mathbb{C}) = \mathit{Hom}_K(\wedge^i(g_{\mathbb{C}}/k_{\mathbb{C}}), \mathbb{C}). \]
Problem
Is this an isomorphism?

Some earlier results:
$\Gamma \backslash G$ is compact: Matsushima, Raghunathan, Venkataramana, Millson-Raghunathan, · · ·

$\Gamma \backslash G$ is noncompact : Borel, Franke, Yang, · · ·
• \( G \): connected reductive linear algebraic group over \( \mathbb{Q} \);
• \( \Gamma \subset G(\mathbb{Q}) \): a congruence subgroup;

\[
G = \bigcap_{\chi: G \rightarrow GL(1)/\mathbb{Q}} \ker(|\chi| : G(\mathbb{R}) \rightarrow \mathbb{R}^\times_+).
\]

**Global Theorem** (Li-Sun, 2018, preprint): The above linear map is an isomorphism if \( i < r_G \).

Here \( r_G \) is the smallest integer, if it exists, such that the continuous group cohomology

\[
H_{ct}^{r_G}(G; F \otimes \pi) \neq \{0\}
\]

for some finite dimensional irreducible representations \( F \) of \( G \), and some infinite-dimensional irreducible unitary representations \( \pi \) of \( G \). Otherwise \( r_G = \infty \).
Remark:

- In many cases, $r_G$ is the optimal bound.
- The quantity $r_G$ is calculated in all cases (Enright, Kumaresan, Vogan-Zuckerman, Li-Schwermer).
Example 1

- \( G = SU(m, n) \).

Matsushima: Suppose \( \Gamma \backslash G \) is compact, and \( i < \frac{\min\{m,n\}}{2} \).
Then
\[
H^i(\Gamma; \mathbb{C}) \cong H^i_{ct}(G; \mathbb{C}).
\]

Our theorem: Suppose \( \Gamma \) is a congruence subgroup and \( i < \min\{m, n\} \). Then
\[
H^i(\Gamma; F) \cong F^\Gamma \otimes H^i_{ct}(G; \mathbb{C}).
\]
Example 2

- $G = \text{SL}_n(\mathbb{R})$.

  Borel: Suppose $\Gamma$ is a congruence subgroup, $n \geq 4$ and $i \leq \frac{n+2}{4}$. Then

  $$H^i(\Gamma; \mathbb{C}) \cong H^i_{\text{ct}}(G; \mathbb{C}).$$

  Our theorem: Suppose $\Gamma$ is a congruence subgroup and $i \leq n - 2$. Then

  $$H^i(\Gamma; F) \cong F^\Gamma \otimes H^i_{\text{ct}}(G; \mathbb{C}).$$
2. Continuous cohomologies

- $G$: a locally compact Hausdorff topological group.

**Definition** A representation of $G$ is a quasi-complete Hausdorff locally convex topological vector space $V$ over $\mathbb{C}$, together with a continuous linear action

$$G \times V \rightarrow V.$$

All representations of $G$ form a category: $\text{Rep}_G$.

- Want: $H^i_{ct}(G; V)$.
- Defined by: Hochschild-Mostow.
Definition A homomorphism $\phi : V_1 \to V_2$ in $\text{Rep}_G$ is strong if, as topological vector spaces,

\[ \ker \phi \subset V_1 \text{ is a direct summand} \]

and

\[ \text{Im} \phi \subset V_2 \text{ is a direct summand.} \]
Definition A representation $V$ of $G$ is relatively injective if for all injective strong homomorphisms $V_1 \hookrightarrow V_2$ in $\text{Rep}_G$, every homomorphism in $\text{Hom}_G(V_1, V)$ extends to a homomorphism in $\text{Hom}_G(V_2, V)$. 
**Definition**  A strong injective resolution of a representation $V$ of $G$ is an exact sequence

$$0 \to V \to J_0 \to J_1 \to J_2 \to \cdots$$

in $\text{Rep}_G$ such that all $J_i$’s are relatively injective, all the arrows are strong homomorphisms.

**Definition**

$$H^i_{\text{ct}}(G; V) := H^i(J_\bullet).$$

This is a locally convex topological vector space.
3. Smooth cohomologies

- $G$: a Lie group.

**Definition** A representation $V$ of $G$ is said to be smooth if for every $X \in \mathfrak{g}$, the map

$$V \to V, \quad v \mapsto X.v := \frac{d}{dt}|_{t=0} \exp(tX).v$$

is well-defined and continuous.

All smooth representations of $G$ form a category: $\text{Rep}_G^\infty$.

All smooth representations of $G$ are representations of $\mathfrak{g}_\mathbb{C}$.
Definition A homomorphism $\phi : V_1 \to V_2$ in $\text{Rep}_G^\infty$ is strong if, as topological vector spaces,

$$\ker \phi \subset V_1 \text{ is a direct summand}$$

and

$$\text{Im} \phi \subset V_2 \text{ is a direct summand.}$$
**Definition** A representation $V$ in $\text{Rep}^\infty_G$ is relatively injective if for all injective strong homomorphisms

$$V_1 \hookrightarrow V_2$$

in $\text{Rep}^\infty_G$, every homomorphism in $\text{Hom}_G(V_1, V)$ extends to a homomorphism in $\text{Hom}_G(V_2, V)$. 

Binyong Sun (joint with Jian-Shu Li) Low degree cohomologies of congruence groups
**Definition** A strong injective resolution of a representation $V$ in $\text{Rep}^\infty_G$ is an exact sequence

$$0 \to V \to J_0 \to J_1 \to J_2 \to \cdots$$

in $\text{Rep}^\infty_G$ such that all $J_i$’s are relatively injective, all the arrows are strong homomorphisms.

**Definition**

$$H^i_{\text{sm}}(G; V) := H^i(J_\bullet).$$

This is a locally convex topological vector space.
Facts

- (Hochschild-Mostow) For every $V$ in $\text{Rep}^\infty_G$,
  \[ H^i_{\text{ct}}(G; V) = H^i_{\text{sm}}(G; V). \]

- (Hochschild-Mostow) Suppose $G$ has only finitely many connected components. Then for every $V$ in $\text{Rep}^\infty_G$,
  \[ H^i_{\text{sm}}(G; V) = H^i(g_\mathbb{C}, K; V). \]

- (Blanc) For every representation $V$ of $G$,
  \[ H^i_{\text{ct}}(G; V) = H^i_{\text{sm}}(G; V^\infty) \]
  at least when $V$ is Fréchet.
Remarks.

- $H^i_{ct}(g_\mathbb{C}, K; V)$ is calculated by the cochain complex

\[ \text{Hom}_K(\wedge^\bullet g_\mathbb{C}/\mathfrak{k}_\mathbb{C}, V). \]

- The smoothing $V^\infty$ of $V$ is defined by the Cartesian diagram

\[
\begin{array}{ccc}
V^\infty & \longrightarrow & C^\infty(G; V) \\
\downarrow & & \downarrow \subset \\
V & \xrightarrow{v \mapsto (g \mapsto g \cdot v)} & C(G; V).
\end{array}
\]
4. A key local result

- **$G$:** real reductive group;
- **$F$:** an irreducible finite dimensional representation of $G$;
- **$P \subset G$:** a proper parabolic subgroup of $G$;
- **$N$:** the unipotent radical of $P$;
- **$L := P/N$.**

**Local Theorem** (Li-Sun, 2018, preprint) For every irreducible unitarizable Casselman-Wallach representation $\sigma$ of $L$, and every dominant positive character $\nu : L \to \mathbb{R}_+^\times$,

$$H^i_{ct}(G; F \otimes \text{Ind}^G_P(\nu \otimes \sigma)) = \{0\} \quad \text{for all } i < r_G.$$
• Take a splitting of $P \to L$ and write $P = LN$.
• $\mathfrak{h} \subset \mathfrak{l}$ : a maximally split Cartan subalgebra of $\mathfrak{l}$.

**Definition** A positive character $\nu : L \to \mathbb{R}_+^\times$ is said to be dominant if

$$\langle \nu|_{\mathfrak{h}_C}, \alpha^\vee \rangle \geq 0 \quad \text{for all } \alpha \in \Delta(\mathfrak{h}_C, \mathfrak{n}_C).$$

Here the complexified differential of $\nu : L \to \mathbb{R}_+^\times$ is denoted by $\nu : \mathfrak{l}_C \to \mathbb{C}$. 
Proof: Case by case analysis of the root systems.
5. A result of Franke

- $G$: real reductive group;
- $K \subset G$: a maximal compact subgroup;
- $\Gamma \subset G$: a lattice;
- $F$: an irreducible finite dimensional representation of $G$.

Shapiro’s Lemma:

$$H^i(\Gamma; F) = H^i_{\text{sm}}(G; \text{Ind}_G^\Gamma F) = H^i_{\text{sm}}(G; C^\infty(\Gamma \backslash G) \otimes F)$$
Suppose

- \( G \): connected reductive linear algebraic group over \( \mathbb{Q} \);
- \( \Gamma \subset G(\mathbb{Q}) \): a congruence subgroup;
- \[ G = \bigcap_{\chi:G \to GL(1)/\mathbb{Q}} \ker(|\chi|: G(\mathbb{R}) \to \mathbb{R}^*_+). \]

\( \mathcal{A}(\Gamma \backslash G) \): the space of smooth automorphic forms. Then

\[ \mathcal{A}(\Gamma \backslash G) = \lim_{\text{I is a finite codimensional ideal of } \mathbb{Z}(g_{\mathbb{C}})} \mathcal{A}(\Gamma \backslash G)^I, \]

where

\[ \mathcal{A}(\Gamma \backslash G)^I := \{ f \in \mathcal{A}(\Gamma \backslash G) : f \text{ is annihilated by } I \} \]

is a Casselman-Wallach representation of \( G \).
Theorem: (Franke)

\[ H^i_{\text{sm}}(G; C^\infty(\Gamma \backslash G) \otimes F) = H^i_{\text{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F). \]

Remark.

- Extend \( F \) to a representation of \( G(\mathbb{R}) \). Then

\[ H^i_{\text{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F) = H^i_{\text{sm}}(G(\mathbb{R}); \mathcal{A}(\Gamma \backslash G(\mathbb{R})) \otimes F). \]
6. Franke’s filtration

- \( P_0 \) : a minimal split torus in \( G \);
- \( A_0 \) : a maximal split torus in \( P_0 \);
- \( P = L_P \ltimes N_P \) : a standard parabolic;
- \( A_P \) : the largest split central torus in \( L \);
- \( \alpha_P \) : the Lie algebra of \( A_P(\mathbb{R}) \);
- \( \alpha_P, \mathbb{C} \) : the complexification;
- \( \check{\alpha}_P, \mathbb{C} \) : the dual space;
- \( \mathcal{A}_P(G) \) : the space of smooth automorphic forms on \( (L_P(\mathbb{Q})N_P(\mathbb{A})) \backslash G(\mathbb{A}) \). This is a representation of \( G(\mathbb{A}) \). It is smooth as a representation of \( G(\mathbb{R}) \).
- \( \mathcal{A}(G) := \mathcal{A}_G(G) \).
A smooth representation

\[ A_P(\mathbb{R}) \curvearrowright A_P(G), \quad (a.\phi)(x) := a^{-\rho_P} \phi(ax), \]

where \( \rho_P \in \check{a}_P \) denotes the half sum of the weights (with the multiplicities) associated to \( N_P \).

Differential :

\[ a_{P,C} \curvearrowright A_P(G). \]

The generalized eigenspace decomposition :

\[ A_P(G) = \bigoplus_{\lambda \in \check{a}_P, C} A_P(G)_{\lambda}. \quad (1) \]
A result of Langlands:

\[ \tilde{\alpha}_{P_0} = \bigsqcup_{P \text{ is a standard parabolic}} \left( \tilde{\alpha}^+_P - \overline{\tilde{\alpha}^+_P} \right) \]

Since

\[ \overline{\alpha}^+_P = \bigsqcup_{P \text{ is a standard parabolic}} \tilde{\alpha}^+_P, \]

we get a map (the dominant part)

\[ (\cdot)^+: \tilde{\alpha}_{P_0} \rightarrow \overline{\tilde{\alpha}^+_P} \]
Franke’s filtration: for each $t \geq 0$, define

$$\mathcal{A}(G)[\leq t] := \left\{ \phi \in \mathcal{A}(G) : \phi_P \in \bigoplus_{\lambda \in \check{\mathfrak{a}}, \langle \Re(\lambda) \rangle_+ \rho^\vee_0} \mathcal{A}_P(G)_\lambda, \text{ for all } P \right\},$$

where $\phi_P \in \mathcal{A}_P(G)$ denotes the constant term of $\phi$ along $P$, and $\rho^\vee_0 \in \check{\mathfrak{a}}_0$ is the sum of the fundamental coweights.

Remark. The spaces of almost square integrable automorphic forms:

$$\mathcal{A}(G)[\leq 0] = \mathcal{A}^2(G) := \left\{ \phi \in \mathcal{A}^2(G) : \phi(\cdot g) \in L^{2+\epsilon}(G(\mathbb{Q}) \backslash G^1(\mathbb{A})) \text{ for all } g \right\}.$$
Define

\[ \mathcal{A}(G)[<t] := \left\{ \phi \in \mathcal{A}(G) : \phi_P \in \bigoplus_{\lambda \in \mathfrak{a}_P, \mathbb{C}} \mathcal{A}_P(G)_\lambda, \text{ for all } P \right\} , \]

and

\[ \mathcal{A}(G)[t] := \frac{\mathcal{A}(G)[\leq t]}{\mathcal{A}(G)[<t]} . \]
Theorem (Franke)

\[ \mathcal{A}(G)[t] = \bigoplus_{P} \bigoplus_{\lambda \in \mathfrak{a}_P, \text{Re}(\lambda) \in \mathfrak{a}_P^+} \langle \text{Re}(\lambda), \rho_0 \rangle = t \mathcal{A}_P^2(G) \lambda, \]

where

\[ \mathcal{A}_P^2(G) := \left\{ \phi \in \mathcal{A}_P(G) : \phi(\cdot g) \in L^{2+\epsilon}(L_P(Q) \backslash L_P^1(A)) \text{ for all } g \right\} = \text{Ind}_{P(A)}^{G(A)} \mathcal{A}_P^2(L_P). \]
Franke’s theorem + the local theorem implies that
\[ H^i_{\text{sm}}(G(\mathbb{R}); \mathcal{A}(G) \otimes F) = H^i_{\text{sm}}(G(\mathbb{R}); \tilde{\mathcal{A}}(G) \otimes F) \]
for \( i < r_G \).

The theory of Eisenstein series + the local theorem implies that
\[ H^i_{\text{sm}}(G(\mathbb{R}); \tilde{\mathcal{A}}(G) \otimes F) = H^i_{\text{sm}}(G(\mathbb{R}); \mathcal{A}(G) \otimes F) \]
for \( i < r_G \).

This implies that
\[ H^i_{\text{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F) = H^i_{\text{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F) \]
for \( i < r_G \).

Finally,
\[ H^i_{\text{sm}}(G; \mathcal{A}(\Gamma \backslash G) \otimes F) = F^{\Gamma} \otimes H^i_{\text{sm}}(G; \mathbb{C}). \]
Thank you!