CONSTRUCTION OF $p$-ADIC ADAMS–BARBASCH–VOGAN PACKETS

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Abstract. This note is a proceeding of the IMS program “On the Langlands Program: Endoscopy and Beyond” held in National University of Singapore from 17 Dec. 2018 to 18 Jan. 2019. The purpose is to explain $p$-adic Adams–Barbasch–Vogan packets constructed in [CFMMX].

Contents

1. Overview 1
2. Vogan varieties and several groups associated to infinitesimal characters 5
3. $A$-parameters and strongly regular elements in the conormal bundle 12
4. Vanishing cycles functor of perverse sheaves on Vogan varieties 14
5. $A$-packets v.s. ABV packets 19
Appendix A. Supplements 20
Appendix B. Representations, $\mathcal{D}$-modules, and perverse sheaves 24
References 27

1. Overview

In this section, we recall Arthur’s results and we explain a motivation for this lecture note.

1.1. $L$-parameters and $A$-parameters. Let $F$ be a non-archimedean local field of characteristic zero. We denote the Weil group of $F$ by $W_F$. We consider a connected reductive split algebraic group $G$ over $F$ as in the following cases:

$(B_n)$: $G = \text{SO}_{2n+1}$ is the split special orthogonal group of size $2n + 1$.
$(C_n)$: $G = \text{Sp}_{2n}$ is the split symplectic group of size $2n$.

We denote by $\Pi(G(F))$ the set of equivalence classes of irreducible smooth representations of $G(F)$, and by $\Pi_{\text{temp}}(G(F))$ its subset consisting of equivalence classes of tempered representations.

Let $\widehat{G}$ be the complex dual group of $G$, i.e., $\widehat{G} = \text{Sp}_{2n}(\mathbb{C})$ in the case $(B_n)$; and $\widehat{G} = \text{SO}_{2n+1}(\mathbb{C})$ in the case $(C_n)$.

Definition 1.1. Let $G$ be in the case $(B_n)$ or $(C_n)$.

1. An infinitesimal character for $G$ is a homomorphism $\lambda: W_F \rightarrow \widehat{G}$ such that
   (a) $\lambda$ is smooth, i.e., $\lambda$ has an open kernel;
   (b) the image of $\lambda$ consists of semisimple elements in $\widehat{G}$.

2. An $L$-parameter for $G$ is a homomorphism $\phi: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow \widehat{G}$ such that
   (a) $\phi|_{W_F \times \{1\}}: W_F \rightarrow \widehat{G}$ is an infinitesimal character for $G$;
(b) \( \phi|_{\{1\} \times \text{SL}_2(\mathbb{C})}: \text{SL}_2(\mathbb{C}) \to \hat{G} \) is algebraic.

(3) An \( A \)-parameter for \( G \) is a homomorphism \( \psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \hat{G} \) such that

(a) \( \psi|_{W_F \times \{1\} \times \{1\}}: W_F \to \hat{G} \) is an infinitesimal character for \( G \);

(b) \( \psi(W_F) \) is bounded;

(c) \( \psi|_{\{1\} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})}: \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \hat{G} \) is algebraic.

Two infinitesimal characters (resp. \( L \)-parameters, \( A \)-parameters) for \( G \) are equivalent if they are conjugate under \( \hat{G} \). The set of equivalence classes of infinitesimal characters (resp. \( L \)-parameters, \( A \)-parameters) for \( G \) is denoted by \( \Lambda(G/F) \) (resp. \( \Phi(G/F) \), \( \Psi(G/F) \)).

We call an \( L \)-parameter \( \phi \) (resp. an \( A \)-parameter \( \psi \)) tempered if \( \phi(W_F) \subseteq \hat{G} \) is bounded (resp. \( \psi|_{\{1\} \times \{1\} \times \text{SL}_2(\mathbb{C})} = 1 \)). Namely, the notion of tempered \( L \)-parameters are the same as the one of tempered \( A \)-parameters. We denote by \( \Phi_{\text{temp}}(G/F) \) (resp. \( \Psi_{\text{temp}}(G/F) \)) the subset of \( \Phi(G/F) \) (resp. \( \Psi(G/F) \)) consisting of tempered \( L \)-parameters (resp. tempered \( A \)-parameters) for \( G \). Namely,

\[
\Phi(G/F) \supset \Phi_{\text{temp}}(G/F) = \Psi_{\text{temp}}(G/F) \subset \Psi(G/F).
\]

For \( \lambda \in \Lambda(G/F) \), \( \phi \in \Phi(G/F) \), and \( \psi \in \Psi(G/F) \), we define their component groups by

\[
A_\lambda = Z_{\hat{G}}(\text{Im}(\lambda))/Z_{\hat{G}}(\text{Im}(\lambda))^0, \\
A_\phi = Z_{\hat{G}}(\text{Im}(\phi))/Z_{\hat{G}}(\text{Im}(\phi))^0, \\
A_\psi = Z_{\hat{G}}(\text{Im}(\psi))/Z_{\hat{G}}(\text{Im}(\psi))^0,
\]

respectively. They are elementary 2-groups. When \( G = \text{SO}_{2n+1} \), i.e., in the case \( (B_n) \), the image of \( -1 \in \hat{G} \) in \( A_\lambda \) (resp. \( A_\phi \), \( A_\psi \)) is also denoted by \( -1 \).

Let \( I_F \) be the inertia subgroup of \( W_F \), and \( \text{Frob}_F \in W_F \) be an arithmetic Frobenius element, i.e., a lift of \( [x \mapsto x^q] \in \text{Gal}({\overline{\mathbb{Q}}}_F/k_F) \), where \( k_F \) is the residue field of \( F \) and \( q \) is its cardinality. We normalize the norm map \( | \cdot |: W_F \to \mathbb{R}^\times \) so that \( |\text{Frob}_F| = q \). For \( w \in W_F \), we put

\[
d_w = \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix} \in \text{SL}_2(\mathbb{C}).
\]

For \( \psi \in \Psi(G/F) \) and \( \phi \in \Phi(G/F) \), we define \( \phi_\psi \in \Phi(G/F), \lambda_\psi \in \Lambda(G/F) \) and \( \lambda_\phi \in \Lambda(G/F) \) by

\[
\phi_\psi(w, a) = \psi(w, a, d_w), \quad \lambda_\psi(w) = \psi(w, d_w, d_w), \quad \lambda_\phi(w) = \phi(w, d_w).
\]

We call \( \phi_\psi \) the \( L \)-parameter for \( \psi \), and \( \lambda_\psi \) (resp. \( \lambda_\phi \)) the infinitesimal character for \( \psi \) (resp. \( \phi \)).

1.2. \( L \)-packets and \( A \)-packets. Set \( G = \text{SO}_{2n+1} \) in the case \( (B_n) \), and \( G = \text{Sp}_{2n} \) in the case \( (C_n) \). The local main theorem in Arthur’s book is stated as follows.

**Theorem 1.2** ([Ar13, Theorem 2.2.1]). (1) For each \( \psi \in \Psi(G/F) \), there is a finite multiset \( \Pi_\psi \) over \( \Pi(G(F)) \) with a map

\[
\Pi_\psi \to \text{Irr}(A_\psi), \quad \pi \mapsto \langle \cdot, \pi \rangle_\psi
\]

enjoying certain endoscopic character identities. When \( G = \text{SO}_{2n+1} \), i.e., in the case \( (B_n) \), the pairing satisfies \( \langle -1, \pi \rangle_\psi = 1 \). We call \( \Pi_\psi \) the \( A \)-packet of \( G(F) \) associated with \( \psi \).
(2) When \( \psi = \phi \in \Phi_{\text{temp}}(G/F) \), the A-packet \( \Pi_{\phi} \) is in fact a subset of \( \Pi_{\text{temp}}(G(F)) \). Moreover the map \( \pi \mapsto \langle \cdot, \pi \rangle_{\phi} \) is injective, and the image is

\[
\begin{cases}
\text{Irr}(A_{\psi}/\{\pm1\}) & \text{in the case } (B_n), \\
\text{Irr}(A_{\psi}) & \text{in the case } (C_n).
\end{cases}
\]

In addition, \( \Pi_{\phi} \cap \Pi_{\phi'} = \emptyset \) for \( \phi \neq \phi' \), and

\[
\Pi_{\text{temp}}(G(F)) = \bigcup_{\phi \in \Phi_{\text{temp}}(G/F)} \Pi_{\phi}.
\]

By the Langlands classification, one can extend Theorem 1.2 (2) to any L-parameter \( \phi \in \Phi(G/F) \). Namely, there is a finite subset \( \Pi_{\phi} \) of \( \Pi(G(F)) \) with an injection

\[
\Pi_{\phi} \rightarrow \text{Irr}(A_{\phi}), \quad \pi \mapsto \langle \cdot, \pi \rangle_{\phi}
\]
such that \( \Pi_{\phi} \cap \Pi_{\phi'} = \emptyset \) for \( \phi \neq \phi' \), and

\[
\Pi(G(F)) = \bigcup_{\phi \in \Phi(G/F)} \Pi_{\phi}.
\]

We call \( \Pi_{\phi} \) the L-packet of \( G(F) \) associated with \( \phi \). When \( \pi \in \Pi_{\phi} \), we say that \( \phi \) is the L-parameter of \( \pi \). The map \( \pi \mapsto \langle \phi, \eta \rangle \), where \( \phi \in \Phi(G(F)) \) is the L-parameter of \( \pi \) and \( \eta = \langle \cdot, \pi \rangle_{\phi} \in \text{Irr}(A_{\phi}) \), is called the local Langlands correspondence.

The A-packets \( \Pi_{\phi} \) were constructed explicitly by Mœglin. See also Xu’s paper [X17] and its references. As a consequence, Mœglin showed the following deep result.

**Theorem 1.3** (Mœglin [Mœ11], Xu [X17, Theorem 8.12]). The A-packet \( \Pi_{\phi} \) is multiplicity-free, i.e., \( \Pi_{\phi} \) is a subset of \( \Pi(G(F)) \).

### 1.3. The main conjecture.

The L-packets give a classification of \( \Pi(G(F)) \) but the endoscopic character identities fail for non-tempered L-packets in general. On the other hand, the A-packets are “local factors” of global A-packets which classify the discrete spectrum of automorphic representations, so that the local meaning of the A-packets is unclear. Furthermore, Mœglin’s construction of L-packets might not be so hard (up to constructing supercuspidal representations), but the one of A-packets should be quite difficult (see also [X17]).

One may desire to understand A-packets more directly. Since a classification of \( \Pi(G(F)) \) is given by L-packets, it may be desirable that A-packets are described in terms of L-packets.

After Arthur formulated conjectures about A-packets in 1980’s, in [ABV92], Adams, Barbasch and Vogan constructed A-packets \( \Pi_{\psi}^{\text{ABV}} \) for real reductive groups purely locally. These packets are called the Adams–Barbasch–Vogan packets (or shortly, the ABV packets). Vogan [V93] had also given a similar definition of A-packets \( \Pi_{\psi}^{\text{ABV}} \) for p-adic reductive groups using the microlocal Euler characteristic derived from the microlocalisation functor.

In [CFMMX], the authors constructed p-adic ABV packets \( \Pi_{\psi}^{\text{ABV}} \) using vanishing cycles functors (instead of the microlocal analysis). It is conjectured (and might be known to experts) that the ABV packets coincide with Vogan’s ones, but the definition using vanishing cycles functors is more amenable than Vogan’s one. For the detail for the comparison of two definitions, see [CFMMX, §7.6].

The main conjecture in [CFMMX] is roughly stated as follows:
Conjecture 1.4. For $\psi \in \Psi(G/F)$,
$$\Pi_{\psi} = \Pi_{\psi}^{\text{ABV}}.$$ 

1.4. **Outline of the definition of $p$-adic ABV packets.** The $p$-adic ABV packets $\Pi_{\psi}^{\text{ABV}}$ will be defined using the vanishing cycles functors of equivariant perverse sheaves on Vogan varieties. Now we explain these terminologies roughly.

1. The **Vogan variety** is a moduli space of $L$-parameters. Let $\lambda \in \Lambda(G/F)$ be an infinitesimal character. This gives a complex variety $V_{\lambda}$ (Definition 2.1) with an action of a complex reductive group $H_{\lambda} = Z_{\mathbb{G}}(\text{Im}(\lambda))$. There is a canonical bijection
   $$\Phi_{\lambda}(G/F) := \{\phi \in \Phi(G/F) \mid \lambda_{\phi} = \lambda\} \to \{H_{\lambda}\text{-orbits in } V_{\lambda}\}, \phi \mapsto C_{\phi}.$$ 

See Proposition 2.2.

2. When there is a group action $H \times V \to V$ in the category of algebraic varieties (over $\mathbb{C}$), one can define a category $\text{Per}_{H}(V)$ of $H$-equivariant perverse sheaves on $V$. Every simple object in $\text{Per}_{H}(V)$ is (isomorphic to) an intersection complex $\mathcal{I}C(C, L)$, where $C$ is an $H$-orbit in $V$ and $L$ is a simple $H$-equivariant local system on $C$. Hence the simple objects in $\text{Per}_{H}(V)$ are parametrized by pairs $(C, \rho)$, where $C$ is an $H$-orbit in $V$ and $\rho$ is an isomorphism class of irreducible representations of equivariant fundamental group $A_{C}$ of $C$. Moreover, if $(V, H) = (V_{\lambda}, H_{\lambda})$ and $C = C_{\phi}$, then $A_{C} \cong A_{\phi}$ (Lemma 2.3).

3. The local Langlands correspondence together with the bijection
   $$\{(\phi, \eta) \mid \phi \in \Phi_{\lambda}(G/F), \eta \in \text{Irr}(A_{\phi})\} \quad (\phi, \eta)$$ 
   $$\{(C, \rho) \mid H_{\lambda}\text{-orbit } C \subset V_{\lambda}, \rho \in \text{Irr}(A_{C})\} \quad (C_{\phi}, \eta)$$

gives an injection
   $$\bigsqcup_{\phi \in \Phi_{\lambda}(G/F)} \Pi_{\phi} \to \text{Per}_{H_{\lambda}}(V_{\lambda})^{\text{simple}}_{/\text{iso}}, \pi \mapsto \mathcal{P}(\pi).$$

See Proposition 4.1.

4. For an $H_{\lambda}$-orbit $C$ in $V_{\lambda}$, define a subvariety $T_{C}^{s}(V_{\lambda})_{\text{reg}}$ of the conormal bundle $T_{C}^{*}(V_{\lambda})$ by
   $$T_{C}^{s}(V_{\lambda})_{\text{reg}} = T_{C}^{s}(V_{\lambda}) \setminus \bigcup_{C \subseteq \overline{C}_{1}} T_{C}^{s}_{1}(V_{\lambda}),$$
where $C_{1}$ runs over all $H_{\lambda}$-orbits in $V_{\lambda}$ such that $C \subseteq \overline{C}_{1}$. Then the vanishing cycles functor gives an exact functor
   $$\mathcal{E}v_{C} : \text{Per}_{H_{\lambda}}(V_{\lambda}) \to \text{Per}_{H_{\lambda}}(T_{C}^{s}(V_{\lambda})_{\text{reg}}).$$

See Theorem 4.2.

5. We say that $(x, \xi) \in T_{C}^{s}(V_{\lambda})$ is **strongly regular** if its $H_{\lambda}$-orbit is open and dense in $T_{C}^{s}(V_{\lambda})$. We denote by $T_{C}^{s}(V_{\lambda})_{\text{reg}}$ the strongly regular part of $T_{C}^{s}(V_{\lambda})$. Then $T_{C}^{s}(V_{\lambda})_{\text{reg}} \subset T_{C}^{s}(V_{\lambda})_{\text{reg}}$ is an $H_{\lambda}$-orbit if it is non-empty (Proposition 3.4). One can define a functor
   $$\mathcal{E}v_{C} : \text{Per}_{H_{\lambda}}(V_{\lambda}) \to \text{Loc}_{H_{\lambda}}(T_{C}^{s}(V_{\lambda})_{\text{reg}}),$$
   $$\mathcal{E}v_{C} : \text{Per}_{H_{\lambda}}(V_{\lambda}) \to \text{Loc}_{H_{\lambda}}(T_{C}^{s}(V_{\lambda})_{\text{reg}}),$$
where $\text{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\text{sreg}})$ is the category of $H_\lambda$-equivariant local systems on $T^*_C(V_\lambda)_{\text{sreg}}$, by

$$\text{Evs}_C:\mathcal{P} = p\text{ev}_C:\mathcal{P}[-\dim V_\lambda]|_{T^*_C(V_\lambda)_{\text{sreg}}}. $$

Finally, we define a normalization $\text{NEvs}_C: \text{Per}_{H_\lambda}(V_\lambda) \to \text{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\text{sreg}})$ of $\text{Evs}_C$. Its properties are stated in Theorem 4.4.

(6) For an $A$-parameter $\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \widehat{G}$, set

$$\psi_1 = \psi|_{\{1\} \times \text{SL}_2(\mathbb{C}) \times \{1\}}: \text{SL}_2(\mathbb{C}) \to \widehat{G},$$

$$\psi_2 = \psi|_{\{1\} \times \{1\} \times \text{SL}_2(\mathbb{C})}: \text{SL}_2(\mathbb{C}) \to \widehat{G},$$

and put

$$x_\psi = d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \xi_\psi = d\psi_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g} = \text{Lie}(\widehat{G}).$$

Then

$$(x_\psi, \xi_\psi) \in T^*_C(V_\lambda)_{\text{sreg}}$$

where $\lambda = \lambda_\psi$ is the infinitesimal character for $\psi$, and $C_\psi = C_{\phi_\psi}$ is the $H_\lambda$-orbit corresponding to the $L$-parameter $\phi_\psi$ for $\psi$. Moreover, the equivariant fundamental group $A_{T^*_C(V_\lambda)_{\text{sreg}}}$ of $T^*_C(V_\lambda)_{\text{sreg}}$ is isomorphic to $A_{\psi}$ (Proposition 3.5). In particular, $T^*_C(V_\lambda)_{\text{sreg}}$ is non-empty when $\lambda = \lambda_\psi$, and there is an equivalence map

$$\text{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\text{sreg}}) \to \text{Rep}(A_{T^*_C(V_\lambda)_{\text{sreg}}}) \cong \text{Rep}(A_{\psi}).$$

(7) Consequently, $\psi \in \Psi(G/F)$ with $\lambda_\psi = \lambda$ determines a functor

$$\text{Ev}_\psi: \text{Per}_{H_\lambda}(V_\lambda) \xrightarrow{\text{Evs}_C} \text{Loc}_{H_\lambda}(T^*_C(V_\lambda)_{\text{sreg}}) \to \text{Rep}(A_{\psi}),$$

and its normalization $\text{NEv}_\psi$. See §5.1.

(8) Note that there is the zero representation in $\text{Rep}(A_{\psi})$. For $\psi \in \Psi(G/F)$, the ABV packet $\Pi_{\psi}^{\text{ABV}}$ is defined in Definition 5.1 by

$$\Pi_{\psi}^{\text{ABV}} = \left\{ \pi \in \bigcup_{\phi \in \Phi_{A_{\psi}}(G/F)} \Pi_{\phi} \left| \text{Ev}_\psi(\mathcal{P}(\pi)) \neq 0 \right. \right\}. $$

In the rest of this paper, we explain the terminologies appearing in the definition of the ABV packets. The proofs of several facts are devolved to the original paper [CFMMX]. The main conjecture is stated in Conjecture 5.2.

In Appendix A, we summarize supplements: Algebraic representations of $\text{SL}_2(\mathbb{C})$; equivariant fundamental groups and local systems; and the Kazhdan–Lusztig conjecture. In Appendix B, we recall relations between representations of semisimple Lie groups, $\mathcal{D}$-modules and perverse sheaves in the archimedean case.

2. **Vogan varieties and several groups associated to infinitesimal characters**

In this section, we define Vogan varieties $V_\lambda$ together with several reductive groups $H_\lambda \subset J_\lambda \subset K_\lambda \subset \widehat{G}$ associated to infinitesimal characters $\lambda \in \Lambda(G/F)$, and give some examples.
2.1. Infinitesimal characters for $A$-parameters. Let $\psi : W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \hat{G}$ be an $A$-parameter for $G$. Recall that $\hat{G}$ is $\text{Sp}_{2n}(\mathbb{C})$ or $\text{SO}_{2n+1}(\mathbb{C})$ so that one can regard $\psi$ as a self-dual representation of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$. We decompose $\psi$ into a direct sum of representations as follows:

$$
\psi = \left( \bigoplus_{i \in I} \rho_i \otimes V_i \right) \oplus \left( \bigoplus_{j \in J} \rho_j \otimes W_j \right) \oplus \left( \bigoplus_{k \in K} (\rho_k \oplus \rho_k^\vee) \otimes U_i \right),
$$

where

- for $i \in I \cup J \cup K$, $\rho_i$ is irreducible bounded representation of $W_F$ such that $\rho_i \not\cong \rho_i'$ and $\rho_i \not\cong \rho_i^\vee$, for $i \neq i'$;
- if $i \in I$, then $\rho_i$ is self-dual of the same type as $\psi$;
- if $j \in J$, then $\rho_j$ is self-dual of the opposite type to $\psi$;
- if $k \in K$, then $\rho_k$ is not self-dual;
- $V_i$, $W_j$, and $U_k$ are representations of $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$.

For $i \in I$ (resp. $j \in J$), the structure of $\hat{G}$ gives a non-degenerate symmetric (resp. alternating) bilinear form on $V_i$ (resp. $W_j$). The centralizer $Z_{\hat{G}}(\psi(W_F))$ of $\psi(W_F)$ is the kernel of the map

$$
\prod_{i \in I} O(V_i) \times \prod_{j \in J} \text{Sp}(W_j) \times \prod_{k \in K} \text{GL}(U_k) \to \{ \pm 1 \},
$$

$$(\alpha_i)_i, (\beta_j)_j, (\gamma_k)_k \mapsto \prod_{i \in I} \det(\alpha_i)_{\dim \rho_i}.$$

2.2. Several groups associated to infinitesimal characters. Let $\lambda : W_F \to \hat{G}$ be an infinitesimal character for $G$. Recall that we fix a Frobenius element $\text{Frob}_F \in W_F$, and we normalize the norm map $| \cdot | : W_F \to \mathbb{R}^\times$ so that $|\text{Frob}_F| = q$. We may assume that $\lambda(\text{Frob}_F)$ is diagonal, say $\text{diag}(\alpha_1, \ldots, \alpha_N)$ for some $\alpha_1, \ldots, \alpha_N \in \mathbb{C}^\times$. Write $\alpha_i = r_i u_i$ with $r_i > 0$ and $|u_i| = 1$. We define $f_\lambda, s_\lambda, t_\lambda \in \hat{G}$ by

$$
f_\lambda = \lambda(\text{Frob}_F) = \text{diag}(\alpha_1, \ldots, \alpha_N),
$$

$$s_\lambda = \text{diag}(r_1, \ldots, r_N),
$$

$$t_\lambda = \text{diag}(u_1, \ldots, u_N).$$

Note that $s_\lambda t_\lambda = t_\lambda s_\lambda = f_\lambda$. We call $s_\lambda$ (resp. $t_\lambda$) the hyperbolic part (resp. elliptic part) of $f_\lambda$. Since $\alpha_i = \alpha_j \implies u_i = u_j$, we have $Z_{\hat{G}}(f_\lambda) \subset Z_{\hat{G}}(t_\lambda)$.

Now we define $H_\lambda \subset J_\lambda \subset K_\lambda \subset \hat{G}$ by

$$
H_\lambda = Z_{\hat{G}}(\lambda(W_F)),
$$

$$J_\lambda = Z_{\hat{G}}(\lambda(I_F)) \cap Z_{\hat{G}}(t_\lambda),
$$

$$K_\lambda = Z_{\hat{G}}(\lambda(I_F)).$$

Remark that $H_\lambda = Z_{\hat{G}}(\lambda(I_F)) \cap Z_{\hat{G}}(f_\lambda)$. These are complex reductive groups, but not necessarily connected. We note that $f_\lambda$ stabilizes $K_\lambda$ and that $s_\lambda \in J_\lambda$. 
2.3. Vogan varieties. Let $\mathfrak{h}_\lambda \subset \mathfrak{j}_\lambda \subset \mathfrak{t}_\lambda \subset \hat{\mathfrak{g}}$ be the Lie algebras of $H_\lambda \subset J_\lambda \subset K_\lambda \subset \hat{G}$, respectively. The adjoint action of $G$ on $\mathfrak{g}$ is denoted by $\text{Ad}$. 

**Definition 2.1.** For $\lambda \in \Lambda(G/F)$, we define

$$V_\lambda = \{ x \in \mathfrak{t}_\lambda \mid \text{Ad}(f_\lambda)x = qx \} ,$$

$$t^\prime V_\lambda = \{ x \in \mathfrak{t}_\lambda \mid \text{Ad}(f_\lambda)x = q^{-1}x \} ,$$

Then $H_\lambda$ acts on $V_\lambda$ and $t^\prime V_\lambda$ by $\text{Ad}$. We call $V_\lambda$ the Vogan variety for $\lambda$.

We note that

$$V_\lambda = \{ x \in \mathfrak{i}_\lambda \mid \text{Ad}(s_\lambda)x = qx \} ,$$

$$t^\prime V_\lambda = \{ x \in \mathfrak{i}_\lambda \mid \text{Ad}(s_\lambda)x = q^{-1}x \}$$

since $f_\lambda = s_\lambda t_\lambda = t_\lambda s_\lambda$ and any eigenvalue of $t_\lambda$ has complex norm 1.

There is a concrete description of $V_\lambda$ ([CFMMX, §4.6]). Notice that $s_\lambda \in J_\lambda$ by definition. Let $S$ be a maximal torus of $J_\lambda$ such that $s_\lambda \in S$. We denote the set of roots of $S$ in $J_\lambda$ by $R(S, J_\lambda)$. Then

$$V_\lambda \cong \mathbb{A}^d, \quad \text{for } d = |\{ \alpha \in R(S, J_\lambda) \mid \alpha(s_\lambda) = q \}| .$$

The Lie algebra $\mathfrak{i}_\lambda$ has a decomposition

$$\mathfrak{i}_\lambda = Z(\mathfrak{i}_\lambda) \oplus [\mathfrak{j}_\lambda, \mathfrak{i}_\lambda],$$

where $Z(\mathfrak{i}_\lambda)$ is the center of $\mathfrak{i}_\lambda$, and $[\mathfrak{j}_\lambda, \mathfrak{i}_\lambda]$ is a semisimple Lie algebra. They are given explicitly by

$$[\mathfrak{j}_\lambda, \mathfrak{i}_\lambda] = \bigoplus_{i \in I} \mathfrak{so}(V_i) \oplus \bigoplus_{j \in J} \mathfrak{sp}(W_j) \oplus \bigoplus_{k \in K} \mathfrak{sl}(U_k), \quad Z(\mathfrak{i}_\lambda) \cong \mathbb{C}^{\oplus |K|}.$$

We choose any non-degenerate symmetric bilinear form on $Z(\mathfrak{i}_\lambda)$. Using it, we extend the Killing form on $[\mathfrak{j}_\lambda, \mathfrak{i}_\lambda]$ to $\mathfrak{i}_\lambda$ so that $[\mathfrak{j}_\lambda, \mathfrak{i}_\lambda]$ and $Z(\mathfrak{i}_\lambda)$ are orthogonal to each other. Then we obtain a non-degenerate symmetric $\mathfrak{j}_\lambda$-invariant bilinear pairing

$$(\_ | \_) : \mathfrak{i}_\lambda \times \mathfrak{i}_\lambda \to \mathbb{C} .$$

This pairing allows us to identify $t^\prime V_\lambda$ with the linear dual of $V_\lambda$. It is known that the Killing forms on $\mathfrak{so}(V_i)$, $\mathfrak{sp}(W_j)$, and $\mathfrak{sl}(U_k)$ are given as follows.

$$\mathfrak{so}(V_i) \times \mathfrak{so}(V_i) \to \mathbb{C}, \quad (X, Y) \mapsto (\dim V_i - 2) \cdot \text{tr}(XY),$$

$$\mathfrak{sp}(W_j) \times \mathfrak{sp}(W_j) \to \mathbb{C}, \quad (X, Y) \mapsto (\dim W_j + 2) \cdot \text{tr}(XY),$$

$$\mathfrak{sl}(U_k) \times \mathfrak{sl}(U_k) \to \mathbb{C}, \quad (X, Y) \mapsto 2 \dim U_k \cdot \text{tr}(XY).$$

2.4. $L$-parameters and $H_\lambda$-orbits in $V_\lambda$. Let $\Phi_\lambda(G/F)$ be the subset of $\Phi(G/F)$ consisting of $L$-parameters $\phi : W_F \times \text{SL}_2(\mathbb{C}) \to \hat{G}$ such that $\lambda_\phi = \lambda$. For $\phi \in \Phi_\lambda(G/F)$, set

$$\varphi = \phi|_{\{1\} \times \text{SL}_2(\mathbb{C})} : \text{SL}_2(\mathbb{C}) \to \hat{G},$$

and put

$$x_\phi = d\varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \hat{\mathfrak{g}} .$$

Then $x_\phi \in V_\lambda$. The $H_\lambda$-orbit of $x_\phi$ is denoted by $C_\phi$. 


Lemma 2.3 ([CFMMX, Lemma 3.6.1]). For any \( \phi \in \Phi_\lambda(G/F) \), we have
\[
A_{C_\phi} \cong A_\phi.
\]

Now we set
\[
\Pi_\lambda = \bigsqcup_{\phi \in \Phi_\lambda(G/F)} \Pi_\phi
\]
to be the union of the \( L \)-packets associated with \( \phi \in \Phi_\lambda(G/F) \). The LLC together with Proposition 2.2 and Lemma 2.3 gives the following:

Proposition 2.4. There is a canonical injection
\[
\Pi_\lambda \rightarrow \{(C, \rho) \mid H_\lambda \text{-orbit } C \subset V_\lambda, \rho \in \operatorname{Irr}(A_C)\}.
\]

2.5. Examples of \( \Phi_\lambda(G/F) \) and \( V_\lambda \). In this subsection, we describe \( V_\lambda \), \( ^tV_\lambda \) and \( \Phi_\lambda(G/F) \) more explicitly.

Let \( \lambda = \lambda_\psi \) be the infinitesimal character for the \( A \)-parameter
\[
\psi = \left( \bigoplus_{i \in I} \rho_i \boxtimes V_i \right) \oplus \left( \bigoplus_{j \in J} \rho_j \boxtimes W_j \right)
\]
as in §2.1 (with \( K = \emptyset \)). Suppose that \( \psi \) is of good parity, i.e., each irreducible constituent of \( \psi \) is self-dual of the same type as \( \psi \). Then for each \( i \in I \) and \( j \in J \), there are decompositions
\[
\begin{align*}
V_i &= (V_{i,t_i} \oplus \cdots \oplus V_{i,1}) \oplus (V_{i,0}^* \oplus \cdots \oplus V_{i,t_i}^*), \\
W_j &= (W_{j,t_j} \oplus \cdots \oplus W_{j,1}) \oplus (W_{j,0}^* \oplus \cdots \oplus W_{j,t_j}^*),
\end{align*}
\]
where
- \( V_{i,k} \) (resp. \( W_{j,k} \)) is a totally isotropic subspace of \( V_i \) (resp. \( W_j \)) for \( i = 1, \ldots, t_i \) (resp. \( j = 1, \ldots, t_j \));
- \( V_{i,k} \oplus V_{i,k}^* \) (resp. \( W_{j,k} \oplus W_{j,k}^* \)) is non-degenerate for \( i = 1, \ldots, t_i \) (resp. \( j = 1, \ldots, t_j \));
- \( V_{i,k'} \oplus V_{i,k'}^* \) (resp. \( W_{j,k'} \oplus W_{j,k'}^* \)) is orthogonal to \( V_{i,k} \oplus V_{i,k}^* \) (resp. \( W_{j,k} \oplus W_{j,k}^* \)) for \( k' \neq k \);
- \( V_0 \) is the orthogonal complement of \( (V_{i,t_i} \oplus \cdots \oplus V_{i,1}) \oplus (V_{i,0}^* \oplus \cdots \oplus V_{i,t_i}^*) \)
such that \( s_\lambda \) stabilizes \( V_{i,k} \) (resp. \( W_{j,k} \)) and acts on it by the scalar \( q^k \) (resp. \( q^{k - \frac{1}{2}} \)). Moreover, \( \dim(V_{i,k+1}) \leq \dim(V_{i,k}) \) and \( \dim(W_{j,k+1}) \leq \dim(W_{j,k}) \) for \( k \geq 1 \).

We have
\[
J_\lambda \subset \prod_{i \in I} \operatorname{O}(V_i) \times \prod_{j \in J} \operatorname{Sp}(W_j), \quad j_\lambda = \bigoplus_{i \in I} \operatorname{so}(V_i) \oplus \bigoplus_{j \in J} \operatorname{sp}(W_j),
\]
and
\[
H_\lambda \subset \prod_{i \in I} \left( \operatorname{O}(V_{i,0}) \times \prod_{k=1}^{t_i} \operatorname{GL}(V_{i,k}) \right) \times \prod_{j \in J} \prod_{k=1}^{t_j} \operatorname{GL}(W_{j,k}).
\]
More explicitly, $J_\lambda$ is the kernel of

$$\prod_{i \in I} O(V_i) \times \prod_{j \in J} \text{Sp}(W_j) \to \{\pm 1\}, \quad ((\alpha_i)_i, (\beta_j)_j) \mapsto \prod_{i \in I} \det(\alpha_i)^{\dim \rho_i},$$

and $H_\lambda$ is the kernel of

$$\prod_{i \in I} \left( O(V_i,0) \times \prod_{k=1}^{t_i} \text{GL}(V_i,k) \right) \times \prod_{j \in J} \prod_{k=1}^{t_j} \text{GL}(W_j,k) \to \{\pm 1\},$$

$$((\alpha_i, (a_i,k)_k)_i, (b_j,k)_j)_j \mapsto \prod_{i \in I} \det(\alpha_i)^{\dim \rho_i}.$$

The Vogan variety $V_\lambda$ is the subspace of $\mathfrak{j}_\lambda$ given by

$$\bigoplus_{i \in I} \bigoplus_{k=1}^{t_i} \text{Hom}(V_{i,k-1}, V_{i,k}) \oplus \bigoplus_{j \in J} \left( \text{Sym}(W_{j,1}^*, W_{j,1}) \oplus \bigoplus_{k=2}^{t_j} \text{Hom}(W_{j,k-1}, W_{j,k}) \right),$$

where $\text{Sym}(W_{j,1}^*, W_{j,1}) = \{ c \in \text{Hom}(W_{j,1}^*, W_{j,1}) \mid c^* = -c \}$ with $c^* \in \text{Hom}(W_{j,1}^*, W_{j,1})$ being defined so that

$$\langle cw_1, w_2 \rangle_{W_j} = \langle w_1, c^*w_2 \rangle_{W_j}$$

for $w_1, w_2 \in W_{j,1}^*$. More precisely, $\bigoplus_{k=1}^{t_i} \text{Hom}(V_{i,k-1}, V_{i,k})$ and $\text{Sym}(W_{j,1}^*, W_{j,1}) \oplus \bigoplus_{k=2}^{t_j} \text{Hom}(W_{j,k-1}, W_{j,k})$ are regarded as subspaces in $\mathfrak{so}(V_i)$ and $\mathfrak{sp}(W_j)$ by

$$(A_{i,k})_k \mapsto \begin{pmatrix} 0_{V_{i,0}} & A_{i,1} & \cdots & \cdots & 0_{V_{i,t_i}} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_{V_{i,1}} & A_{i,1}^* & \cdots & \cdots & 0_{V_{i,t_i}} \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 0_{V_{i,t_i}} & 0_{V_{i,1}} & \cdots & \cdots & A_{i,t_i}^* \end{pmatrix} \in \mathfrak{so}(V_i),$$

$$(B_{j,k})_k \mapsto \begin{pmatrix} 0_{W_{j,0}} & B_{j,1} & \cdots & \cdots & 0_{W_{j,t_j}} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_{W_{j,1}} & B_{j,1} & \cdots & \cdots & 0_{W_{j,t_j}} \\ \cdots & \cdots & \ddots & \ddots & \cdots \\ 0_{W_{j,t_j}} & 0_{W_{j,1}} & \cdots & \cdots & B_{j,t_j}^* & \cdots & \cdots & \vdots & \cdots & 0_{W_{j,t_j}} \end{pmatrix} \in \mathfrak{sp}(W_j),$$
where $A^*_{i,k} \in \text{Hom}(V^*_{i,k}, V^*_{i,k-1})$ (with $V^*_{i,0} := V_{i,0}$) and $B^*_{j,k} \in \text{Hom}(W^*_{j,k}, W^*_{j,k-1})$ are defined so that
\[
\langle A^*_{i,k}v, v^* \rangle_{V_i} + \langle v, A^*_{i,k}v^* \rangle_{V_i} = 0,
\]
\[
\langle B^*_{j,k}w, w^* \rangle_{W_j} + \langle w, B^*_{j,k}w^* \rangle_{W_j} = 0
\]
for $v \in V_{i,k-1}$, $v^* \in V^*_{i,k}$ and $w \in W_{j,k-1}$, $w^* \in W^*_{j,k}$, respectively. Choosing bases of $V_{i,k}$ and $W_{j,k}$, we have an identification
\[
V_\lambda = \bigoplus_{i \in I} \bigoplus_{k = 1}^{t_i} \text{Mat}_{d_{i,k}, d_{i,k-1}}(\mathbb{C}) \oplus \bigoplus_{j \in J} \left( \text{Sym}_{d_{j,1}}(\mathbb{C}) \oplus \bigoplus_{k = 2}^{t_j} \text{Mat}_{d_{j,k}, d_{j,k-1}}(\mathbb{C}) \right),
\]
where we set $d_{i,k} = \dim(V_{i,k})$, $d_{j,k} = \dim(W_{j,k})$, and $\text{Sym}_{d_{j,1}}(\mathbb{C})$ is the subspace of $\text{Mat}_{d_{j,1}}(\mathbb{C})$ consisting of symmetric matrices. Therefore, as an algebraic variety, $V_\lambda$ is the affine space $\mathbb{A}^d$ with
\[
d = \sum_{i \in I} \sum_{k = 1}^{t_i} d_{i,k}d_{i,k-1} + \sum_{j \in J} \left( \frac{d_{j,1}(d_{j,1} + 1)}{2} + \sum_{k = 2}^{t_j} d_{j,k}d_{j,k-1} \right).
\]
Similarly, $tV_\lambda$ is the subspace of $j_\lambda$ given by
\[
\bigoplus_{i \in I} \bigoplus_{k = 1}^{t_i} \text{Hom}(V_{i,k}, V_{i,k-1}) \oplus \bigoplus_{j \in J} \left( \text{Sym}(W_{j,1}, W^*_{j,1}) \oplus \bigoplus_{k = 2}^{t_j} \text{Hom}(W_{j,k}, W_{j,k-1}) \right),
\]
which is isomorphic to
\[
\bigoplus_{i \in I} \bigoplus_{k = 1}^{t_i} \text{Mat}_{d_{i,k-1}, d_{i,k}}(\mathbb{C}) \oplus \bigoplus_{j \in J} \left( \text{Sym}_{d_{j,1}}(\mathbb{C}) \oplus \bigoplus_{k = 2}^{t_j} \text{Mat}_{d_{j,k-1}, d_{j,k}}(\mathbb{C}) \right).
\]
Since $J_\lambda$ is semisimple, the bilinear pairing $( \mid ) : j_\lambda \times j_\lambda \to \mathbb{C}$ is defined by the Killing form. The restriction $( \mid ) : V_\lambda \times tV_\lambda \to \mathbb{C}$ is given explicitly by
\[
\left( \langle A_{i,k}, (B_{j,k}) \rangle \mid \langle A'_{i,k}, (B'_{j,k}) \rangle \right)
= \sum_{i \in I} 2(\dim V_i - 2) \sum_{k = 1}^{t_i} \text{tr}(A_{i,k}A'_{i,k}) + \sum_{j \in J} (\dim W_j + 2) \left( \text{tr}(B_{j,1}B'_{j,1}) + 2 \sum_{k = 2}^{t_j} \text{tr}(B_{j,k}B'_{j,k}) \right).
\]
The sets $\Phi_\lambda(G/F)$ and $V_\lambda/H_\lambda$ can be complicated. For a positive integer $k$, we denote by $S_k$ the unique irreducible algebraic representation of $\text{SL}_2(\mathbb{C})$ of dimension $k$. For its realization, see Appendix A.1 below.

**Example 2.5.** Suppose that $\lambda = \lambda_\phi$ with
\[
\phi = \rho \boxtimes (S_3 \oplus S_5),
\]
where $\rho$ is irreducible and self-dual of the same type as $\phi$. Then $d_0 = 2$, $d_1 = 2$ and $d_2 = 1$ so that
\[
H_\lambda = \begin{cases} 
O(2, \mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) & \text{if } \dim(\rho) \equiv 0 \mod 2, \\
\text{SO}(2, \mathbb{C}) \times \text{GL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C}) & \text{if } \dim(\rho) \equiv 1 \mod 2,
\end{cases}
\]
and
\[ V_\lambda = \text{Mat}_{2,2}(\mathbb{C}) \oplus \text{Mat}_{1,2}(\mathbb{C}). \]

The action of \( H_\lambda \) on \( V_\lambda \) is given by
\[ (\alpha, \gamma_1, \gamma_2) \cdot (X_1, X_2) = (\gamma_1 X_1 \alpha^{-1}, \gamma_2 X_2 \gamma_1^{-1}) \]
for \((\alpha, \gamma_1, \gamma_2) \in H_\lambda \) and \((X_1, X_2) \in V_\lambda \).

For simplicity, we assume that \( \dim \rho \) is even. Then the elements \( \phi \) in \( \Phi_\lambda(G/F) \) and the associated groups \( A_\phi \), the vectors \( x_\phi \in V_\lambda \), the \( H_\lambda \)-orbits \( C_\phi \) are listed as follows:

- \( \phi_1 = \rho \otimes (| \cdot |^2 \oplus | \cdot |^1 \oplus | \cdot |^1 \oplus 1 \oplus 1 \oplus | \cdot |^{-1} \oplus | \cdot |^{-1} \oplus | \cdot |^{-2}) \), \( A_{\phi_1} \cong \mathbb{Z}/2\mathbb{Z} \), and
\[ x_{\phi_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,0) \in C_{\phi_1} = \begin{Bmatrix} (0,0) \end{Bmatrix} \times \{0,0\}; \]

- \( \phi_2 = \rho \otimes (| \cdot |^2 \oplus | \cdot |^1 \oplus | \cdot |^1 \oplus \frac{3}{2}S_2 \oplus | \cdot |^{-1} \oplus | \cdot |^{-1} \oplus | \cdot |^{-2}) \), \( A_{\phi_2} \cong 1 \), and
\[ x_{\phi_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 1 & -\sqrt{-1} \end{pmatrix}, (0,0) \in C_{\phi_2} = \begin{Bmatrix} (x,y,z,w) \mid (x,y,z,w) \neq (0,0,0,0), xw-zy=x^2+y^2-z^2+w^2=0 \end{Bmatrix} \times \{0,0\}; \]

- \( \phi_3 = \rho \otimes (| \cdot |^2 \oplus S_3 \oplus S_3 \oplus | \cdot |^{-1} \oplus | \cdot |^{-2}) \), \( A_{\phi_3} \cong (\mathbb{Z}/2\mathbb{Z})^\oplus 2 \), and
\[ x_{\phi_3} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \in C_{\phi_3} = \begin{Bmatrix} (x,y) \mid x^2+y^2 \neq 0 \text{ or } z^2+w^2 \neq 0 \end{Bmatrix} \times \{0,0\}; \]

- \( \phi_4 = \rho \otimes (| \cdot |^2 \oplus S_3 \oplus S_3 \oplus | \cdot |^{-2}) \), \( A_{\phi_4} \cong \mathbb{Z}/2\mathbb{Z} \), and
\[ x_{\phi_4} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \in C_{\phi_4} = \text{GL}(\mathbb{C}) \times \{0,0\}; \]

- \( \phi_5 = \rho \otimes (| \cdot |^{\frac{3}{2}}S_2 \oplus | \cdot |^1 \oplus 1 \oplus 1 \oplus | \cdot |^{-1} \oplus | \cdot |^{-\frac{3}{2}}S_2) \), \( A_{\phi_5} \cong \mathbb{Z}/2\mathbb{Z} \), and
\[ x_{\phi_5} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \in C_{\phi_5} = \begin{Bmatrix} (a,b) \mid (a,b) \neq (0,0) \end{Bmatrix} \times \{0,0\}; \]

- \( \phi_6 = \rho \otimes (| \cdot |^{\frac{3}{2}}S_2 \oplus | \cdot |^{\frac{1}{2}}S_2 \oplus | \cdot |^{-\frac{1}{2}}S_2 \oplus | \cdot |^{-\frac{3}{2}}S_2) \), \( A_{\phi_6} \cong 1 \), and
\[ x_{\phi_6} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \frac{1}{\sqrt{2}}(1, -\sqrt{-1}) \in C_{\phi_6} = \begin{Bmatrix} (x,y) \mid (x,y,z,w) \neq (0,0,0,0), (a,b) \neq (0,0), xw-zy=0, (ax+bz,ay+bw)=(0,0), (aw-by)^2+(aw-by)^2=0 \end{Bmatrix}; \]

- \( \phi_7 = \rho \otimes (| \cdot |^{\frac{3}{2}}S_2 \oplus 1 \oplus S_3 \oplus | \cdot |^{-\frac{3}{2}}S_2) \), \( A_{\phi_7} \cong (\mathbb{Z}/2\mathbb{Z})^\oplus 2 \), and
\[ x_{\phi_7} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, (1,0) \in C_{\phi_7} = \begin{Bmatrix} (x,y) \mid (a,b) \neq (0,0), xw-zy=0, (ax+bz,ay+bw)=(0,0), (aw-by)^2+(aw-by)^2 \neq 0 \end{Bmatrix}; \]
• \( \phi_8 = \rho \otimes (| \cdot |^{1} S_3 \oplus | \cdot |^{1} \oplus | \cdot |^{-1} \oplus | \cdot |^{-1} S_3), A_{\phi_8} \cong 1, \text{ and} \)
  \[
x_{\phi_8} = \left( \frac{1}{\sqrt{2}}, \begin{pmatrix} 1 & -\sqrt{-1} \\ 0 & 0 \end{pmatrix}, (1, 0) \right)
\]
  \( \in C_{\phi_8} = \left\{ \left( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, (a, b) \right) \mid \begin{array}{l}
(a, b) \neq (0, 0), \quad xw-yz=0, \\
ax+bz \neq 0, \quad (ax+bz)^2+(ay+bw)^2=0
\end{array} \right\}; \)

• \( \phi_9 = \rho \otimes (| \cdot |^{1} \oplus S_5 \oplus | \cdot |^{-1}, A_{\phi_9} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, \text{ and} \)
  \[
x_{\phi_9} = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1, 0) \right)
\]
  \( \in C_{\phi_9} = \left\{ \left( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, (a, b) \right) \mid \begin{array}{l}
(a, b) \neq (0, 0), \quad xw-yz=0, \\
(ax+bz)^2+(ay+bw)^2=0 \text{ or } (\bar{a}z-\bar{b}x)^2+(\bar{a}w-\bar{b}y)^2 \neq 0
\end{array} \right\}; \)

• \( \phi_{10} = \rho \otimes (| \cdot |^{1} S_4 \oplus | \cdot |^{-1} S_4), A_{\phi_{10}} \cong 1, \text{ and} \)
  \[
x_{\phi_{10}} = \left( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, (1, \sqrt{-1}) \right)
\]
  \( \in C_{\phi_{10}} = \left\{ \left( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, (a, b) \right) \mid \begin{array}{l}
(a, b) \neq (0, 0), \quad xw-yz \neq 0, \\
(ax+bz)^2+(ay+bw)^2=0
\end{array} \right\}; \)

• \( \phi_{11} = \rho \otimes (S_3 \oplus S_5), A_{\phi_{11}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, \text{ and} \)
  \[
x_{\phi_{11}} = \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, (1, 0) \right)
\]
  \( \in C_{\phi_{11}} = \left\{ \left( \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, (a, b) \right) \mid \begin{array}{l}
(a, b) \neq (0, 0), \quad xw-yz \neq 0, \\
(ax+bz)^2+(ay+bw)^2 \neq 0
\end{array} \right\}; \)

Note that \( C_{\phi_1} \) is a unique closed orbit and \( C_{\phi_{11}} \) is a unique open dense orbit. The closure relations of \( H_\lambda \)-orbits are given as follows:

\[
C_{\phi_1} = \overline{C_{\phi_1}} \subset \overline{C_{\phi_2}} \subset \overline{C_{\phi_3}} \subset \overline{C_{\phi_4}} \subset \overline{C_{\phi_{10}}} \subset \overline{C_{\phi_{11}}}
\]

\[
\bigcap \overline{C_{\phi_5}} \subset \bigcap \overline{C_{\phi_6}} \subset \bigcap \overline{C_{\phi_7}} \subset \bigcap \overline{C_{\phi_8}} \subset \bigcap \overline{C_{\phi_9}}.
\]

3. \( A \)-parameters and strongly regular elements in the conormal bundle

In this section, we define the regular part of the conormal bundle \( T_C^*(V_\lambda) \), and we introduce the notion of strongly regular elements.

3.1. Cotangent space and conormal bundle to the Vogan variety. Let \( T^*(V_\lambda) \) be the cotangent space to the Vogan variety \( V_\lambda \). For an \( H_\lambda \)-orbit \( C \) in \( V_\lambda \), we denote the conormal bundle by \( T_C^*(V_\lambda) \). Recall that \( V_\lambda \) is an affine space.

**Proposition 3.1** ([CFMMX, Propositions 5.2.1, 5.3.1]). There is an \( H_\lambda \)-equivariant isomorphism

\[
T^*(V_\lambda) \cong V_\lambda \times \mathbb{I} V_\lambda.
\]

Similarly,

\[
T_C^*(V_\lambda) = \{ (x, \xi) \in T^*(V_\lambda) \mid x \in C, \ [x, \xi] = 0 \},
\]
Lemma 3.2. Proposition 3.4 ([CFMMX, Proposition 5.5.1])

which we call **transposition**. As in Proposition 3.1, for an $H_\lambda$-orbit $B$ in $V_\lambda$, we have

$$T_B^*(tV_\lambda) = \{(\xi, x) \in T^*tV_\lambda \mid \xi \in B, \ [\xi, x] = 0\}.$$ 

**Lemma 3.2.** For any $H_\lambda$-orbit $C$ in $V_\lambda$, there is a unique $H_\lambda$-orbit $C^*$ in $tV_\lambda$ such that the restriction of the transposition gives an isomorphism

$$\overline{T_C^*(V_\lambda)} \cong \overline{T_{C^*}^*(tV_\lambda)}.$$ 

The map $C \mapsto C^*$ is a bijection

$$H_\lambda \setminus V_\lambda \to H_\lambda \setminus tV_\lambda.$$ 

Define $T_C^*(V_\lambda)_{reg} \subset T_C^*(V_\lambda)$ by

$$T_C^*(V_\lambda)_{reg} = T_C^*(V_\lambda) \setminus \bigcup_{C_1 \neq C \subseteq C_1} \overline{T_{C_1}^*(V_\lambda)},$$

where $C_1$ runs over all $H_\lambda$-orbits in $V_\lambda$ such that $C \subseteq \overline{C_1}$ and $C_1 \neq C$.

**Proposition 3.3 ([CFMMX, Proposition 5.4.3]).** If $(x, \xi) \in T_C^*(V_\lambda)_{reg}$, then $(x, \xi) \in C \times C^*$, $[x, \xi] = 0$, and $(x \mid \xi) = 0$.

We say that $(x, \xi) \in T_C^*(V_\lambda)$ is **strongly regular** if its $H_\lambda$-orbit is open and dense in $T_C^*(V_\lambda)$. We write $T_C^*(V_\lambda)_{reg}$ for the strongly regular part of $T_C^*(V_\lambda)$.

**Proposition 3.4 ([CFMMX, Proposition 5.5.1]).** We have

$$T_C^*(V_\lambda)_{reg} \subset T_C^*(V_\lambda)_{reg}.$$ 

Moreover, $T_C^*(V_\lambda)_{reg}$ is an $H_\lambda$-orbit if it is non-empty.

Let $\psi$ be an $A$-parameter for $G$ such that $\lambda = \lambda_\psi$. We set

$$\psi_1 = \psi\mid_{\{1\} \times SL_2(\mathbb{C}) \times \{1\}} : SL_2(\mathbb{C}) \to \hat{G};$$

$$\psi_2 = \psi\mid_{\{1\} \times \{1\} \times SL_2(\mathbb{C})} : SL_2(\mathbb{C}) \to \hat{G},$$

and we put

$$x_\psi = d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \xi_\psi = d\psi_2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \hat{g} = \text{Lie}(\hat{G}).$$

It is easy to see that

$$(x_\psi, \xi_\psi) \in T_{C_\psi}^*(V_\lambda),$$

where $C_\psi = C_{\phi_\psi}$ is the $H_\lambda$-orbit corresponding to the $L$-parameter $\phi_\psi$ for $\psi$.

**Proposition 3.5 ([CFMMX, Propositions 5.6.1, 5.7.1]).** We have

$$T_{C_\psi}^*(V_\lambda)_{reg} \subset T_{C_\psi}^*(V_\lambda)_{reg};$$

Moreover, the equivariant fundamental group $A_{T_{C_\psi}^*(V_\lambda)_{reg}}$ of $T_{C_\psi}^*(V_\lambda)_{reg}$ is isomorphic to $A_\psi$. 

Where $[\cdot, \cdot]$ denotes the Lie bracket on $j_\lambda$, and we identify $V_\lambda$ and $tV_\lambda$ with subspaces of $j_\lambda$.

Since $V_\lambda^* \cong tV_\lambda$, we can define an $H_\lambda$-equivariant isomorphism

$$T^*(V_\lambda) \to T^*(tV_\lambda), \ (x, \xi) \mapsto (\xi, x),$$

which we call **transposition**. As in Proposition 3.1, for an $H_\lambda$-orbit $B$ in $tV_\lambda$, we have

$$T_B^*(tV_\lambda) = \{(\xi, x) \in T^*tV_\lambda \mid \xi \in B, \ [\xi, x] = 0\}.$$
Example 3.6. Consider 
\[ \lambda = \rho \otimes (| \cdot |^2 \otimes | \cdot |^1 \otimes | \cdot |^1 \otimes | \cdot |^1 \otimes | \cdot |^{-1} \otimes | \cdot |^{-1} \otimes | \cdot |^{-2}), \]
which is the same as in Example 2.5. Then the A-parameters \( \psi \in \Psi(G/F) \) such that \( \lambda = \lambda_\psi \) and the associated vectors \((x_\psi, \xi_\psi)\) are listed as follows:

- \( \psi_1 = \rho \otimes 1 \otimes (S_3 \otimes S_5) \) and 
  \[ x_{\psi_1} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, (0,0) \right) \in C_{\phi_1}, \quad \xi_{\psi_1} = \left( \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right) \in \mathbb{i}V_\lambda; \]
- \( \psi_3 = \rho \otimes (S_3 \otimes 1 \otimes 1 \otimes S_5) \) and 
  \[ x_{\psi_3} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, (0,0) \right) \in C_{\phi_3}, \quad \xi_{\psi_3} = \left( \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \end{pmatrix} \right) \in \mathbb{i}V_\lambda; \]
- \( \psi_6 = \rho \otimes S_2 \otimes S_4 \) and 
  \[ x_{\psi_6} = \left( \begin{pmatrix} 1/2 & 1/2 \\ -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} (1, -\sqrt{-1}) \right) \in C_{\phi_6}, \]
  \[ \xi_{\psi_6} = \left( \begin{pmatrix} 1/2 & 1/2 \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix}, \begin{pmatrix} 3/\sqrt{2} \\ -3\sqrt{-1} \end{pmatrix} \right) \in \mathbb{i}V_\lambda; \]
- \( \psi_9 = \rho \otimes (S_5 \otimes 1 \otimes 1 \otimes S_3) \) and 
  \[ x_{\psi_9} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right) \in C_{\phi_9}, \quad \xi_{\psi_9} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in \mathbb{i}V_\lambda; \]
- \( \psi_{10} = \rho \otimes S_4 \otimes S_2 \) and 
  \[ x_{\psi_{10}} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ \sqrt{2} \end{pmatrix} (1, \sqrt{-1}) \right) \in C_{\phi_{10}}, \]
  \[ \xi_{\psi_{10}} = \left( \begin{pmatrix} 1/2 & 1/2 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ \sqrt{-1} \end{pmatrix} \right) \in \mathbb{i}V_\lambda; \]
- \( \psi_{11} = \rho \otimes (S_3 \otimes S_5) \otimes 1 \) and 
  \[ x_{\psi_{11}} = \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (1,0) \right) \in C_{\phi_{11}}, \quad \xi_{\psi_{11}} = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) \in \mathbb{i}V_\lambda. \]

For a realization of \( S_k \), see Appendix A.1.

Let \( \text{Loc}_{H_\lambda}(T^\ast_{C_\psi}(V_\lambda)_\text{sreg}) \) be the category of \( H_\lambda \)-equivariant local systems on \( T^\ast_{C_\psi}(V_\lambda)_\text{sreg} \). Then \((x_\psi, \xi_\psi) \in T^\ast_{C_\psi}(V_\lambda)_\text{sreg}\) determines an equivalence 
\[ \text{Loc}_{H_\lambda}(T^\ast_{C_\psi}(V_\lambda)_\text{sreg}) \to \text{Rep}(A_{T^\ast_{C_\psi}(V_\lambda)_\text{sreg}}) \cong \text{Rep}(A_\psi). \]

4. Vanishing cycles functor of perverse sheaves on Vogan varieties

In this section, we define microlocal vanishing cycles functors, which we will use to define the ABV packets.
4.1. *L*-packets and equivariant perverse sheaves on $V_\lambda$. For an algebraic variety $V$, one can consider the category $\text{Per}(V)$ of perverse sheaves, which is a subcategory of the derived category $D(V) = D^b_c(V)$ of bounded (constructible) $\overline{\mathbb{Q}}_\ell$-sheaves on $V$. See [BBD82].

When $H \times V \rightarrow V$ is a group action in the category of algebraic varieties, one can define the category $\text{Per}_H(V)$ of $H$-equivariant perverse sheaves on $V$, equipped with a forgetful functor $\text{Per}_H(V) \rightarrow \text{Per}(V)$.

For more precision, see [CFMMX, §3.4–3.6]. Every simple object in $\text{Per}_H(V)$ is the intersection complex $\mathcal{I}C(C, \mathcal{L})$, where $C$ is an $H$-orbit in $V$, and $\mathcal{L}$ is a simple $H$-equivariant local system of $C$. It is defined by

$$\mathcal{I}C(C, \mathcal{L}) = \text{Im}(pH^0(j_!\mathcal{L}) \rightarrow pH^0(Rj_*\mathcal{L})), $$

where $j$ is the embedding $C \hookrightarrow V$. Let $\text{Loc}_H(C)$ denote the category of $H$-equivariant local systems of $C$. Hence there exists a canonical bijection:

$$\text{Per}_H(V)_{/\text{iso}}^{\text{simple}} \leftrightarrow \{(C, \mathcal{L}) \mid H\text{-orbit } C \subset V, \mathcal{L} \in \text{Loc}_H(C)_{/\text{iso}}^{\text{simple}}\} \leftrightarrow \{(C, \eta) \mid H\text{-orbit } C \subset V, \eta \in \text{Irr}(A_C)\},$$

where $A_C$ is the $H$-equivariant fundamental group of $C$.

Now we consider the Vogan variety $V = V_\lambda$ equipped with the action of the reductive group $H = H_\lambda$. Proposition 2.4 gives the following:

**Proposition 4.1** ([CFMMX, Proposition 3.6.2]). There is a canonical injection

$$\Pi_\lambda \rightarrow \text{Per}_{H_\lambda}(V_\lambda)_{/\text{iso}}^{\text{simple}}, \; \pi \mapsto \mathcal{P}(\pi).$$

4.2. Vanishing cycles functors. In this subsection, we fix the notation for vanishing cycles functors. Let

- $R = \mathbb{C}[[t]], \; K = \mathbb{C}((t))$;
- $\overline{K}$ be a separable closure of $K$, and $\overline{R}$ be the integral closure of $R$ in $\overline{K}$;
- $S = \text{Spec}(R), \; \eta = \text{Spec}(\overline{K}), \; s = \text{Spec}(\overline{C})$, and $\overline{S} = \text{Spec}(\overline{R})$.

Note that $S$ is a trait, i.e., the spectrum of a henselian discrete valuation ring with generic point $\eta$ and special point $s$. The morphism $s \rightarrow S$ has a canonical section corresponding to $\mathbb{C} \rightarrow \mathbb{C}[[t]]$. Hence we have

$$\eta \xrightarrow{j} S \xleftarrow{i} s.$$

Let $\overline{\eta}$ be a geometric point of $S$ localized at $\eta$, i.e., $\overline{\eta}: \text{Spec}(\overline{K}) \rightarrow \eta \rightarrow S$. Note that $\text{Gal}(\overline{\eta}/\eta) \cong \mathbb{Z}$. For any morphism $X \rightarrow S$, we have the cartesian diagram:
where $X = X \times_S \overline{S}$, $X_\eta = \overline{X} \times_\overline{S} \eta$, and $X_s = \overline{X} \times_\overline{S} \overline{s}$.

Recall that

- “a sheaf on $X_s \times_s \eta$” means a sheaf on $X_s$ equipped with a Gal($\eta/\eta$)-action;
- $D(X_s \times_s \eta) = D^b_c(X_s \times_s \eta)$ is the bounded derived category of constructible $\overline{Q}_\ell$-sheaves on $X_s \times_s \eta$;
- “a sheaf on $X_s \times_s S$” means a triple $(F, F_s, F_s ! F)$, where $F$ and $F_s$ are sheaves on $X_s \times_s \eta$ and on $X_s$, respectively, and $F_s ! F$ is a Gal($\eta/\eta$)-equivariant morphism, with the action on $F_s$ being trivial;
- $D(X_s \times_s S) = D^b_c(X_s \times_s S)$ is the bounded derived category of constructible $\overline{Q}_\ell$-sheaves on $X_s \times_s S$.

There exists a functor $R\Psi_{X_\eta}: D(X_\eta) \to D(X_s \times_s \eta)$ called the **nearby cycles functor** [DK73]. This is defined by

$$R\Psi_{X_\eta}F = (i_{X_\eta})^*(j_{X_\eta})_*(b_{X_\eta})^*F$$

for $F \in D(X_\eta)$, which is a sheaf on $X_s$ equipped with an action of Gal($\eta/\eta$) obtained from the canonical action on $(b_{X_\eta})^*F$. Finally, there exists a functor

$$R\Phi_X: D(X) \to D(X_s \times_s S)$$

called the **vanishing cycles functor**. The triangle

$$\xymatrix{ & & \ar[d] \ar[d] \ar[dl] \ar[l] \ar[r] & & \\
 & & & & \ar[d] \ar[d] \ar[dl] \ar[l] \ar[r] & & \\
i^*_{X_B} b^*_X & \ar[r] & R\Psi_{X_\eta} j^*_{X_\eta} & \ar[l] \ar[r] & R\Phi_X F & \ar[l] \ar[r] & H^q(i^*_{X_B} b^*_X F) & \ar[l] \ar[r] & \cdots}
$$

is a distinguished triangle in $D(X_s \times_s S)$. Thus, for $F \in D(X)$, we obtain a long exact sequence

$$\cdots \to H^q(i^*_{X_B} b^*_X F) \to H^q(R\Psi_{X_\eta} j^*_{X_\eta} F) \to H^q(R\Phi_X F) \to H^{q+1}(i^*_{X_B} b^*_X F) \to \cdots.$$

### 4.3. Definition of functors.

Now fix an infinitesimal character $\lambda \in \Lambda(G/F)$, and consider the Vogan variety $V_\lambda$ with an action of a reductive group $H_\lambda$. Recall in §2.3 that we fixed a non-degenerate symmetric $J_\lambda$-invariant bilinear pairing $(\ | \ ) : j_\lambda \times j_\lambda \to \mathbb{C}$. By restriction, we obtain an $s$-morphism

$$(\ | \ ) : T^s(V_\lambda) \to A^1_C = \text{Spec}(\mathbb{C}[t]).$$

For an $H_\lambda$-orbit $B \subset tV_\lambda$, we denote by

$$X_B = (V_\lambda \times B) \times_A \mathbb{A}^1_C \xrightarrow{f_B} S$$

the base change of the restriction of $(\ | \ )$ to $V_\lambda \times B$ along the canonical map $g: S \to A^1_C$. Namely, we have a cartesian diagram

$$\begin{array}{ccc}
X_B & \longrightarrow & V_\lambda \times B \\
\downarrow f_B & & \downarrow (\ | \ ) \\
S & \longrightarrow & A^1_C.
\end{array}$$
The structure sheaf of $X_B$ is

$$\mathcal{O}_{X_B} = R \otimes_{\mathbb{C}[t]} (\mathcal{O}_{V_\lambda} \otimes \mathcal{O}_B).$$

The special fibre of $X_B$ is the $s$-scheme

$$X_{B,s} = f_B^{-1}(s) = f_B^{-1}(0) = \{(x, \xi) \in V_\lambda \times B \mid (x \mid \xi) = 0\}.$$ We obtain the vanishing cycles functor

$$\mathcal{R}\Phi_{X_B} : \mathcal{D}(X_B) \to \mathcal{D}(f_B^{-1}(0) \times_s S).$$

As an $s$-scheme, $H_\lambda$ acts on $V_\lambda \times B$. Its base change along $S \to s$ gives an action of $H_\lambda \times_s S$ on $(V_\lambda \times B) \times_s S$. Since $(\mid)$ is $H_\lambda$-invariant, this action preserves

$$\{(x, \xi, t) \in (V_\lambda \times B) \times_s S \mid (x|\xi) = g(t)\}.$$ This is precisely $X_B = (V_\lambda \times B) \times_{\mathbb{A}_s^1} S$. Hence $H_\lambda \times_s S$ acts on $X_B$ in the category of $S$-schemes.

We denote the $H_\lambda$-equivariant derived category on $V_\lambda$ by $\mathcal{D}_{H_\lambda}(V_\lambda)$. For any $H_\lambda$-orbit $C \subset V_\lambda$, we define a functor

$$\mathcal{E}v_C : \mathcal{D}_{H_\lambda}(V_\lambda) \to \mathcal{D}_{H_\lambda}(T^*_C(V_\lambda) \times_{\mathbb{A}_s^1} S)$$

by the diagram

$$\begin{array}{ccc}
\mathcal{D}_{H_\lambda}(V_\lambda) & \xrightarrow{\mathcal{E}v_C} & \mathcal{D}_{H_\lambda}(T^*_C(V_\lambda) \times_{\mathbb{A}_s^1} S) \\
\mathcal{H}_\lambda(V_\lambda \times C^*) & \xrightarrow{\mathcal{B}C} & \mathcal{D}_{H_\lambda \times_{\mathbb{A}_s^1} S}(X_{C^*}) & \xrightarrow{\mathcal{R}\Phi_{X_{C^*}}} & \mathcal{D}_{H_\lambda}(f_C^{-1}(0) \times_s S),
\end{array}$$

where

1. $\mathcal{H}_\lambda (V_\lambda \times C^*) : \mathcal{D}_{H_\lambda}(V_\lambda) \to \mathcal{D}_{H_\lambda}(V_\lambda \times C^*)$ is the pullback along the projection $V_\lambda \times C^* \to V_\lambda$;
2. $\mathcal{B}C : \mathcal{D}_{H_\lambda}(V_\lambda \times C^*) \to \mathcal{D}_{H_\lambda \times_s S}(X_{C^*})$ is the base change functor, which is an exact functor;
3. $\mathcal{R}\Phi_{X_{C^*}} : \mathcal{D}_{H_\lambda \times_{\mathbb{A}_s^1} S}(X_{C^*}) \to \mathcal{D}_{H_\lambda}(f_C^{-1}(0) \times_s S)$ is the vanishing cycles functor;
4. $\mathcal{R} \mathcal{S} : \mathcal{D}_{H_\lambda}(f_C^{-1}(0) \times_s S) \to \mathcal{D}_{H_\lambda}(T^*_C(V_\lambda) \times_{\mathbb{A}_s^1} S)$ is given by the pullback along the inclusion $T^*_C(V_\lambda) \times_{\mathbb{A}_s^1} S$ induced by Proposition 3.3.

For $\xi_0 \in V_\lambda$, define $f_{\xi_0} : X_{\xi_0} \to S$ by the base change of $V_\lambda \ni x \mapsto (x|\xi_0) \in \mathbb{A}_s^1$, i.e., we have a cartesian diagram

$$\begin{array}{ccc}
X_{\xi_0} & \xrightarrow{f_{\xi_0}} & V_\lambda \\
\downarrow g & & \downarrow (\mid \xi_0) \\
S & \xrightarrow{g} & \mathbb{A}_s^1.
\end{array}$$

The structure sheaf of $X_{\xi_0}$ is

$$\mathcal{O}_{X_{\xi_0}} = R \otimes_{\mathbb{C}[t]} \mathcal{O}_{V_\lambda},$$

where $\mathcal{O}_{V_\lambda}(A) = A[x]$ for any $\mathbb{C}$-algebra $A$, on which $A[t]$ acts by $t \mapsto (x|\xi_0)$. The special fibre of $X_{\xi_0}$ is the $s$-scheme

$$X_{\xi_0,s} = f_{\xi_0}^{-1}(s) = f_{\xi_0}^{-1}(0) = \{x \in V_\lambda \mid (x|\xi_0) = 0\}.$$
Define
\[ R\Phi_{f_0} : \mathcal{D}_{H\lambda}(V_\lambda) \to \mathcal{D}_{Z_{H\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times S) \]
by
\[
\begin{array}{c}
\mathcal{D}_{H\lambda}(V_\lambda) \xrightarrow{R\Phi_{f_0}} \mathcal{D}_{Z_{H\lambda}(\xi_0)}(f_{\xi_0}^{-1}(0) \times S) \\
\text{forget} \downarrow \quad \text{forget} \downarrow \\
\mathcal{D}_{Z_{H\lambda}(\xi_0)}(V_\lambda) \xrightarrow{\text{BC}} \mathcal{D}_{Z_{H\lambda}(\xi_0) \times S}(X_{\xi_0})
\end{array}
\]
where \( \text{BC} : \mathcal{D}_{Z_{H\lambda}(\xi_0)}(V_\lambda) \to \mathcal{D}_{Z_{H\lambda}(\xi_0) \times S}(X_{\xi_0}) \) is the pullback along \( X_{\xi_0} \to V_\lambda \).

The following is a summary of properties of \( \mathcal{E}v_\lambda \) ([CFMMX, Propositions 6.4.1, 6.5.1, 6.6.2, 6.8.1, 6.9.1]).

**Theorem 4.2.** Let \( C \subset V_\lambda \) be an \( H_\lambda \)-orbit.

(a) The functor \( \mathcal{E}v_\lambda : \mathcal{D}_{H\lambda}(V_\lambda) \to \mathcal{D}_{H\lambda}(T^*_C(V_\lambda)_{\text{reg}} \times S) \)
\[ \text{is exact.} \]
(b) For every \( \mathcal{F} \in \mathcal{D}_{H\lambda}(V_\lambda) \) and every \( (x_0, \xi_0) \in T^*_C(V_\lambda)_{\text{reg}} \), there is a canonical isomorphism
\[ (\mathcal{E}v_\lambda \mathcal{F})(x_0, \xi_0) \cong (R\Phi_{f_0}\mathcal{F})_{x_0}. \]
(c) If \( \mathcal{F} \in \mathcal{D}_{H\lambda}(V_\lambda) \), then
\[ \mathcal{E}v_\lambda \mathcal{F} = 0 \text{ unless } C \subset \text{supp } \mathcal{F} = \bigcup_i \text{supp } H^i(\mathcal{F}). \]
(d) For any \( H \)-equivariant local system \( \mathcal{L} \) on \( C \),
\[ \mathcal{E}v_\lambda \mathcal{I}\mathcal{C}(C, \mathcal{L}) = \mathcal{E}v_\lambda \mathcal{I}\mathcal{C}(C, \mathcal{1}_C) \otimes (\mathcal{L} \boxtimes \mathcal{1}_C^{*})|_{T^*_C(V_\lambda)_{\text{reg}}}, \]
where \( \mathcal{1}_C \) is the constant sheaf on \( C \).
(e) If \( \mathcal{P} \in \text{Per}_{H\lambda}(V_\lambda) \), then
\[ \mathcal{E}v_\lambda \mathcal{P}[\dim C^* - 1] \in \text{Per}_{H\lambda}(T^*_C(V_\lambda)_{\text{reg}}). \]
We set
\[ p\mathcal{E}v_\lambda := \mathcal{E}v_\lambda[\dim C^* - 1] : \text{Per}_{H\lambda}(V_\lambda) \to \text{Per}_{H\lambda}(T^*_C(V_\lambda)_{\text{reg}}). \]
(f) Suppose that \( T^*_C(V_\lambda)_{\text{reg}} \) is non-empty. If \( \mathcal{P} \in \text{Per}_{H\lambda}(V_\lambda) \), then the restriction of \( p\mathcal{E}v_\lambda \mathcal{P} \) to \( T^*_C(V_\lambda)_{\text{reg}} \) is a local system concentrated in degree \( \dim V_\lambda \). We set
\[ \mathcal{E}v_{\lambda C} := p\mathcal{E}v_\lambda[\dim V_\lambda] : \text{Per}_{H\lambda}(V_\lambda) \to \text{Loc}_{H\lambda}(T^*_C(V_\lambda)_{\text{reg}}). \]

For a description of the stalk of \( \mathcal{E}v_\lambda \mathcal{I}\mathcal{C}(C, \mathcal{1}_C) \) at \( (x, \xi) \in T^*_C(V_\lambda)_{\text{reg}} \), see [CFMMX, Theorem 6.7.5]. In particular, we note that
\[ \text{rank } \mathcal{E}v_\lambda \mathcal{I}\mathcal{C}(C, \mathcal{1}_C) = 1. \]
If we put \( T_C = \mathcal{E}v_{\lambda C} \mathcal{I}\mathcal{C}(C, \mathcal{1}_C) \in \text{Loc}_{H\lambda}(T^*_C(V_\lambda)_{\text{reg}}) \), by Theorem 4.2 (d), we have \( \mathcal{E}v_{\lambda C} \mathcal{I}\mathcal{C}(C, \mathcal{L}) = T_C \otimes (\mathcal{L} \boxtimes \mathcal{1}_C^{*})|_{T^*_C(V_\lambda)_{\text{reg}}} \) for every \( \mathcal{L} \in \text{Loc}_{H\lambda}(C) \).

Now we normalize \( \mathcal{E}v_\lambda \) as follows.
Definition 4.3. We define a functor $\text{NEv}_C : D_{H_A}(V_\lambda) \to D_{H_A}(T^*_C(V_{\lambda, \text{reg}}))$ by

$$\text{NEv}_C = (\text{Ev}_C \text{IC}(C, 1_C))^\vee \otimes \text{Ev}_C,$$

where we put $(\text{Ev}_C \text{IC}(C, 1_C))^\vee = \mathcal{H}om(\text{Ev}_C \text{IC}(C, 1_C), 1_{T^*_C(V_{\lambda, \text{reg}})})$, and we use the left derived tensor product. We refer to $\text{NEv}_C$ as the normalized microlocal vanishing cycles functor.

When $T^*_C(V_{\lambda, \text{reg}})$ is non-empty, we set

$$\text{NEvs}_C \mathcal{F} = (\text{NEv}_C \mathcal{F} |(\dim C^* - 1 - \dim V_\lambda))|_{T^*_C(V_{\lambda, \text{reg}})}$$

for $\mathcal{F} \in D_{H_A}(V_\lambda)$. Then

$$\text{NEvs}_C = \mathcal{T}_C^\vee \otimes \text{Ev}_C,$$

where $\mathcal{T}_C^\vee$ is the dual local system of $\mathcal{T}_C$.

Theorem 4.4 ([CFMMX, Theorem 6.10.1]). Suppose that $T^*_C(V_{\lambda, \text{reg}})$ is non-empty.

(a) The functor $\text{NEvs}_C : \text{Per}_{H_A}(V_\lambda) \to \text{Loc}_{H_A}(T^*_C(V_{\lambda, \text{reg}}))$ is exact.

(b) If $\mathcal{P} \in \text{Per}_{H_A}(V_\lambda)$, then $\text{NEvs}_C \mathcal{P} = 0$ unless $C \subset \text{supp} \mathcal{P}$.

(c) If $\mathcal{P} \in \text{Per}_{H_A}(V_\lambda)$, then

$$\text{rank}(\text{NEvs}_C \mathcal{P}) = \text{rank}(\mathcal{Rf}_x \mathcal{P})_x$$

for every $(x, \xi) \in T^*_C(V_\lambda)_{\text{reg}}$.

(d) For every $\mathcal{L} \in \text{Loc}_{H_A}(C)$,

$$\text{NEvs}_C \text{IC}(C, \mathcal{L}) = (\mathcal{L} \boxtimes 1_{C^*})|_{T^*_C(V_{\lambda, \text{reg}})}$$

In particular,

$$\text{rank}(\text{NEvs}_C \text{IC}(C, \mathcal{L})) = \text{rank} \mathcal{L}.$$

5. $A$-packets v.s. ABV packets

In this section, we define the ABV packets, and state a conjecture.

5.1. Definition of ABV packets and the main conjecture. Now we can define the Adams–Barbasch–Vogan packets. Let $\psi \in \Psi(G/F)$ and set $\lambda = \lambda_\psi$. Consider the $H_\lambda$-orbit $C_\psi = C_{\phi_\psi}$ in $V_\lambda$ corresponding to the $L$-parameter $\phi_\psi$ for $\psi$ via Proposition 2.2. Recall in Proposition 4.1 that there is a canonical injection

$$\Pi_\lambda = \bigcup_{\phi \in \Phi_\lambda(G/F)} \Pi_\phi \to \text{Per}_{H_\lambda}(V_\lambda)^{\text{simple}}_{/\text{iso}}, \pi \mapsto \mathcal{P}(\pi).$$

On the other hand, $\psi$ gives a strongly regular point $(x_\psi, \xi_\psi) \in T^*_C(V_{\lambda, \text{reg}})$ (Proposition 3.5). It determines an equivalence

$$\text{Loc}_{H_\lambda}(T^*_C(V_{\lambda, \text{reg}})) \to \text{Rep}(A_{T^*_C(V_{\lambda, \text{reg}})}) \cong \text{Rep}(A_\psi).$$

Define

$$\text{Ev}_\psi : \text{Per}_{H_\lambda}(V_\lambda) \to \text{Rep}(A_\psi)$$

by the composition of $\text{Ev}_{C_\psi}$ and this equivalence, and set

$$\text{NEv}_\psi = \mathcal{T}_\psi^\vee \otimes \text{Ev}_\psi,$$

where $\mathcal{T}_\psi = \text{Ev}_\psi \text{IC}(C_\psi, 1_{C_\psi})$ is the representation of $A_\psi$ corresponding to $\mathcal{T}_{C_\psi}$. 

Definition 5.1. For \( \psi \in \Psi(G/F) \), we define the Adams–Barbasch–Vogan packet (shortly, the ABV packet) \( \Pi^{\text{ABV}}_\psi \) of \( G(F) \) associated with \( \psi \) by

\[
\Pi^{\text{ABV}}_\psi = \{ \pi \in \Pi_\lambda \mid \text{Evc}_\psi \mathcal{P}(\pi) \neq 0 \}.
\]

Recall that there is Arthur’s A-packet \( \Pi_\psi \subset \Pi(G(F)) \). The following is the main conjecture in [CFMMX].

Conjecture 5.2. Let \( G \) be split \( \text{SO}_{2n+1} \) or \( \text{Sp}_{2n} \). For an A-parameter \( \psi \in \Psi(G/F) \), we would have

\[
\Pi_\psi = \Pi^{\text{ABV}}_\psi.
\]

Moreover, the map

\[
\Pi_\psi \to \text{Irr}(A_\psi), \quad \pi \mapsto \langle \cdot, \pi \rangle_\psi
\]

would be given by

\[
\langle a_\psi a_s, \pi \rangle_\psi = (-1)^{\dim C_\psi - \dim \text{supp} \mathcal{P}(\pi)} \text{tr}(\text{NEv}_\psi \mathcal{P}(\pi))(a_s)
\]

for any \( a_s \in A_\psi \), where \( a_\psi \) is the image of \( \psi(1, 12, -12) \) in \( A_\psi \).

Let \( K\Pi_\lambda \) and \( K\text{Per}_{H_\lambda}(V_\lambda) \) be the Grothendieck groups of \( \Pi_\lambda \) and \( \text{Per}_{H_\lambda}(V_\lambda) \), i.e., the free abelian groups for which \( \Pi_\lambda \) and \( \text{Per}_{H_\lambda}(V_\lambda)_{\text{simple}} \) are bases, respectively. Define a bilinear pairing

\[
\langle \cdot, \cdot \rangle : K\Pi_\lambda \times K\text{Per}_{H_\lambda}(V_\lambda) \to \mathbb{Z}
\]

by

\[
\langle \pi, \mathcal{P} \rangle = \begin{cases} (-1)^{\dim \text{supp} \mathcal{P}} & \text{if } \mathcal{P} = \mathcal{P}(\pi), \\ 0 & \text{otherwise}. \end{cases}
\]

If we set

\[
\eta_{\psi, s} = \sum_{\pi \in \Pi_\psi} \langle a_\psi a_s, \pi \rangle_\psi \pi,
\]

then Conjecture 5.2 is equivalent that

\[
\langle \eta_{\psi, s}, \mathcal{P} \rangle = (-1)^{\dim C_\psi} \text{tr}(\text{NEv}_\psi \mathcal{P})(a_s)
\]

for any \( a_s \in A_\psi \) and any \( \mathcal{P} \in K\text{Per}_{H_\lambda}(V_\lambda) \).

In [CFMMX, PART II], several examples were given.

Proposition 5.3. Conjecture 5.2 holds for the following cases:

1. \( q \) is odd, \( G = \text{SL}_2 = \text{Sp}_2 \) and \( \psi \in \Psi(G/F) \) with \( \lambda_\psi = \chi_1 \oplus \chi_2 \oplus \chi_3 \), where \( \chi_1, \chi_2, \chi_3 \) are the three distinct non-trivial quadratic characters of \( F^\times \);
2. \( G = \text{SO}_3 \) and \( \psi \in \Psi(G/F) \) with \( \lambda_\psi = | \cdot |^{\frac{3}{2}} \oplus | \cdot |^{-\frac{3}{2}} \);
3. \( G = \text{SO}_5 \) and \( \psi \in \Psi(G/F) \) with \( \lambda_\psi = | \cdot |^{\frac{3}{2}} \oplus | \cdot |^{\frac{1}{2}} \oplus | \cdot |^{-\frac{1}{2}} \oplus | \cdot |^{-\frac{3}{2}} \);
4. \( G = \text{SO}_5 \) and \( \psi \in \Psi(G/F) \) with \( \lambda_\psi = | \cdot |^{\frac{3}{2}} \oplus | \cdot |^{\frac{1}{2}} \oplus | \cdot |^{-\frac{3}{2}} \oplus | \cdot |^{-\frac{1}{2}} \);
5. \( G = \text{SO}_7 \) and \( \psi \in \Psi(G/F) \) with \( \lambda_\psi = | \cdot |^{\frac{3}{2}} \oplus | \cdot |^{\frac{1}{2}} \oplus | \cdot |^{\frac{1}{2}} \oplus | \cdot |^{-\frac{1}{2}} \oplus | \cdot |^{-\frac{3}{2}} \).

Appendix A. Supplements

In this appendix, we explain several supplementary topics.
A.1. **Algebraic representations of SL₂(ℂ)**. For each positive integer \( k \), there is a unique (up to isomorphism) irreducible algebraic representation \( S_k \) of \( SL₂(ℂ) \) of dimension \( k \). Note that the representation \( w \mapsto S_k(d_w) \) of \( W_F \) is isomorphic to

\[
| \cdot |^{\frac{1-k}{2}} \oplus \cdots \oplus | \cdot |^{-\frac{k-1}{2}}.
\]

The representation \( S_k \) is realized as the \((k - 1)\)-th symmetric power \( Sym^{k-1}ℂ^2 \) of the standard representation \( S_2 = ℂ^2 \). It has a non-degenerate \( SL₂(ℂ) \)-invariant bilinear form

\[
[\cdot, \cdot]: Sym^{k-1}ℂ^2 \times Sym^{k-1}ℂ^2 \to ℂ,
\]

\[
\begin{pmatrix}
(a_1) & \cdots & (a_{k-1}) \\
(b_1) & \cdots & (b_{k-1})
\end{pmatrix}
\begin{pmatrix}
(c_1) & \cdots & (c_{k-1}) \\
(d_1) & \cdots & (d_{k-1})
\end{pmatrix}
\mapsto \prod_{\sigma \in S_{k-1}} \det \begin{pmatrix}
(a_{\sigma(i)}) \\
(b_{\sigma(i)}
\end{pmatrix}.
\]

In particular, \( S_k \) is self-dual of sign \((-1)^{k-1}\).

If we put

\[
e_p = \frac{1}{p!} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-p-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^p \in Sym^{k-1}ℂ^2
\]

for \( 0 \leq p \leq k - 1 \), then we have

\[
[e_p, e_q] = \begin{cases} (-1)^p & \text{if } q = k - p - 1, \\ 0 & \text{otherwise}. \end{cases}
\]

We identify \( Sym^{k-1}ℂ^2 \) with \( ℂ^k \) by \( e_0, \ldots, e_{k-1} \). Since

\[
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e_p = \sum_{l=0}^{p} \frac{t^{p-l}}{(p-l)!} e_l, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} e_p = \sum_{l=p}^{k-1} \frac{t!(k-p-1)!}{p!(l-p)!(k-l-1)!} t^{l-p} e_l,
\]

the actions of \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in sl₂(ℂ) \) on \( Sym^{k-1}ℂ^2 \cong ℂ^k \) are given by

\[
\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (k-1) & 0 & \cdots & 0 \end{pmatrix},
\]

respectively, where they are in \( so(k, ℂ) \) or \( sp(k, ℂ) \) with respect to \( \begin{pmatrix} 1 & & & \cdots & \cdots & & \cdots & 0 \\ & -1 & & \cdots & \cdots & \cdots & \cdots & 1 \\ & & \ddots \cdots & & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \) according to \( k \equiv 1 \mod 2 \) or \( k \equiv 0 \mod 2 \).
A.2. Equivariant fundamental groups and local systems. Here, we define equivariant fundamental groups and equivariant local systems.

Let $C$ and $X$ be topological spaces equipped with continuous actions of a topological group $H$. A subset $U$ of $X$ is stable if $h \cdot u \in U$ for any $u \in U$ and $h \in H$. We call $X$ $H$-equivariantly connected if only $\emptyset$ and $X$ are the open and closed stable subsets of $X$. A map $f: X \to C$ is said to be $H$-equivariant if $f(h \cdot x) = h \cdot f(x)$ for any $x \in X$ and $h \in H$. Moreover, it is called an $H$-equivariant covering map of $C$ if $f$ is a covering in the usual sense, i.e., $f$ is surjective and for any $p \in C$, there exists an open neighborhood $U$ of $p$ in $C$ such that $f^{-1}(U)$ is a disjoint union of open subsets $V_i$ of $X$ satisfying that the restriction $f|V_i$ is a homeomorphism $V_i \xrightarrow{\sim} U$. An $H$-equivariant base point of $X$ is an $H$-equivariant map $b: H \to X$. Fix an $H$-equivariant base point $c: H \to C$ of $C$. We say that an $H$-equivariant map $f: X \to C$ is base point-preserving if $f(b(h)) = c(h)$ for any $h \in H$. Such a map is denoted by $f: (X, b) \to (C, c)$. A universal $H$-equivariant covering of $C$ is an $H$-equivariant covering map $\pi: (\widetilde{C}, \widetilde{c}) \to (C, c)$ with $\widetilde{C}$ being $H$-equivariantly connected such that for any $H$-equivariant covering $f: (X, b) \to (C, c)$ with $X$ being $H$-equivariantly connected, there exists a unique $H$-equivariant continuous map $\tilde{f}: (\widetilde{C}, \widetilde{c}) \to (X, b)$ such that the diagram

$$
\begin{array}{ccc}
\widetilde{C} & \xrightarrow{\tilde{f}} & X \\
\downarrow{\pi} & & \downarrow{f} \\
C & &
\end{array}
$$

commutes. We define the $H$-equivariant fundamental group $\pi_1(C)_H$ by the group of $H$-equivariant homeomorphisms $\tilde{f}: \widetilde{C} \to \widetilde{C}$ such that the diagram

$$
\begin{array}{ccc}
\widetilde{C} & \xrightarrow{\tilde{f}} & \widetilde{C} \\
\downarrow{\pi} & & \downarrow{\pi} \\
C & &
\end{array}
$$

commutes. By the universality, $\pi_1(C)_H$ is uniquely determined by $C$, up to a unique isomorphism.

**Example A.1.** Let $C = \{0\}$ be the set of a point, equipped with the trivial action of the orthogonal group $H = O(m, \mathbb{C})$. Then the covering map is a map from a discrete topological space $X$ equipped with a continuous action of $O(m, \mathbb{C})$. Hence this action factors through the quotient $O(m, \mathbb{C})/SO(m, \mathbb{C}) \cong \{\pm 1\}$. The discrete space $X$ is $O(m, \mathbb{C})$-equivariantly connected if and only if the action of $O(m, \mathbb{C})$ is transitive. In particular, an $O(m, \mathbb{C})$-equivariant universal cover of $C$ is given by the set $\widetilde{C} = \{p_+, p_-\}$ of two points with the action $\alpha \cdot p_\epsilon = p_{\epsilon \det \alpha}$ for $\alpha \in O(m, \mathbb{C})$ and $\epsilon \in \{\pm 1\}$. Therefore,

$$
\pi_1(\{0\})_{O(m, \mathbb{C})} = \{\pm 1\}.
$$

More generally, when $C$ is a quotient $H/Z$ for some subgroup $Z$ of $H$ with the canonical base point $c: H \to C$, for any $H$-equivariant covering $f: (X, x) \to (C, c)$ of $C$, we have $hx = x$ for $h \in Z^0$. In particular, a universal $H$-equivariant covering of $C$ is given by $\widetilde{C} = H/Z^0$, and
the $H$-equivariant fundamental group is given by

$$\pi_1(X)_H \cong \pi_0(Z) = Z/Z^0. $$

Let $m: H \times X \to X$ be an group action in the category of algebraic varieties (over $\mathbb{C}$). We also consider the projection $p: H \times X \to X$. An $H$-equivariant sheaf on $X$ is a sheaf $\mathcal{F}$ on $X$ equipped with an isomorphism of sheaves on $H \times X$

$$\varphi: m^{-1}\mathcal{F} \cong p^{-1}\mathcal{F}$$

such that $\varphi$ satisfies the usual cocycle condition on $H \times H \times X$. On the stalk level, $\varphi$ implies an isomorphism $\mathcal{F}_{hx} \cong \mathcal{F}_x$ for $h \in H$ and $x \in X$. The cocycle condition says that the isomorphism $\mathcal{F}_{ghx} \cong \mathcal{F}_x$ is the same as the composition $\mathcal{F}_{ghx} \cong \mathcal{F}_{hx} \cong \mathcal{F}_x$ for $g, h \in H$ and $x \in X$.

Recall that a local system on $X$ is a locally constant sheaf on $X$. We denote the category of $H$-equivariant local systems on $X$ by $\text{Loc}_H(X)$. There is a relation between $\text{Loc}_H(X)$ and representations of $\pi_1(X)_H$. Recall that $\pi_1(X)_H$ is the group of $H$-equivariant homeomorphisms on the universal $H$-equivariant cover $\tilde{X}$ of $X$ which commute with the projection $\pi: \tilde{X} \to X$. Given a representation $\rho: \pi_1(X)_H \to \text{GL}(V)$, we consider the sections of the bundle $(\tilde{X} \times V)/\pi_1(X)_H \to X$. More precisely, for an open set $U$ of $X$, we set $\mathcal{L}_\rho(U)$ to be the space of locally constant functions $f: \pi^{-1}(U) \to V$ satisfying

$$f(\gamma x) = \rho(\gamma)f(x)$$

for $\gamma \in \pi_1(X)_H$. Then $\mathcal{L}_\rho$ is an $H$-equivariant local system on $X$. The map $\rho \mapsto \mathcal{L}_\rho$ gives an identification $\text{Rep}(\pi_1(X)_H) \to \text{Loc}_H(X)$.

A.3. Kazhdan–Lusztig conjecture. Let $\lambda \in \Lambda(G/F)$. Recall that $\text{KIP}_\lambda$ and $\text{KPer}_{H,\lambda}(V)$ are the free abelian groups with the canonical bases

$$\{\pi(\phi, \eta) \mid \phi \in \Phi_\lambda(G/F), \, \eta \in \widehat{A}_\phi, \, \eta(-1) = 1 \text{ if } G = \text{SO}_{2n+1}\},$$

$$\left\{\text{IC}(C, \mathcal{L}) \mid H\text{-orbit } C \subset V, \, \mathcal{L} \in \text{Loc}_{H,\lambda}\left(C^{\text{simple}} / \text{Iso}\right)\right\},$$

respectively. They also have other bases, consisting of standard modules and standard sheaves.

For given $\phi \in \Phi_\lambda(G/F)$, one can obtain a parabolic subgroup $P = MN$ of $G$ such that $\phi$ factors through the embedding $\widehat{M} \hookrightarrow \widehat{G}$ so that $\phi$ can be regarded as an $L$-parameter for $M$, which is essentially tempered. Then $\eta \in \widehat{A}_\phi$ (with $\eta(-1) = 1$) gives an irreducible essentially tempered representation $\pi_M(\phi, \eta)$ of $M(F)$. Moreover, one can assume that the exponents of $\pi_M(\phi, \eta)$ are in the positive Weyl chamber with respect to $P$. In this case, the parabolic induction

$$M(\phi, \eta) = \text{Ind}^{G(F)}_{\mathcal{P}(F)}(\pi_M(\phi, \eta))$$

is called a **standard module**. The representation $\pi(\phi, \eta)$ of $G(F)$ is the unique irreducible quotient of the standard module $M(\phi, \eta)$. Moreover,

$$\{M(\phi, \eta) \mid \phi \in \Phi_\lambda(G/F), \, \eta \in \widehat{A}_\phi, \, \eta(-1) = 1 \text{ if } G = \text{SO}_{2n+1}\}$$

forms a basis of $\text{KIP}_\lambda$. In particular, for two pairs $(\phi, \eta)$ and $(\phi', \eta')$, there exists a non-negative integer $m_{\text{rep}}((\phi, \eta), (\phi', \eta'))$ such that

$$M(\phi', \eta') = \sum_{(\phi, \eta)} m_{\text{rep}}((\phi, \eta), (\phi', \eta')) \cdot \pi(\phi, \eta)$$
in $\Pi_\lambda$. We call $(m_{\text{rep}}((\phi, \eta), (\phi', \eta'))_{(\phi, \eta), (\phi', \eta')})$ the multiplicity matrix. It is known that this matrix is an “upper-triangular unipotent matrix” in a certain sense (see e.g., [Ar13, §2.2]).

Similarly, for an $H_\lambda$-orbit $C \subset V_\lambda$ and a simple equivariant local system $L \in \text{Loc}_{H_\lambda}(C)_{/\text{iso}}^{\text{simple}}$, we consider the shifted standard sheaf

$$S(C, L) = j_C^* L[\dim C],$$

where $j_C : C \hookrightarrow V_\lambda$ is the inclusion. Then

$$\{ S(C, L) \mid H\text{-orbit } C \subset V_\lambda, \ L \in \text{Loc}_{H_\lambda}(C)_{/\text{iso}}^{\text{simple}} \}$$

forms a basis of $\text{KPer}_{H_\lambda}(V_\lambda)$. In particular, for two pairs $(C, L)$ and $(C', L')$, there exists an integer $m_{\text{geo}}((C', L'), (C, L))$ such that

$$\mathcal{I}C(C, L) = \sum \limits_{(C', L')} m_{\text{geo}}((C', L'), (C, L)) \cdot S(C', L')$$

in $\text{KPer}_{H_\lambda}(V_\lambda)$. We call $(m_{\text{geo}}((C', L'), (C, L)))_{(C', L'), (C, L)}$ the geometric multiplicity matrix. It is known that $m_{\text{geo}}((C, L), (C, L)) = 1$ and $m_{\text{geo}}((C', L'), (C, L)) = 0$ unless $C' \subset C$. Moreover, if we set

$$m'_{\text{geo}}((C', L'), (C, L)) = (-1)^{\dim C - \dim C'} m_{\text{geo}}((C', L'), (C, L)),$$

then it is a non-negative integer. We call $(m'_{\text{geo}}((C', L'), (C, L)))_{(C', L'), (C, L)}$ the normalized geometric multiplicity matrix.

Recall that a pair $(\phi, \eta)$ gives a pair $(C_\phi, L_\rho)$, and there exists a bilinear form

$$\langle \cdot, \cdot \rangle : \Pi_\lambda \times \text{KPer}_{H_\lambda}(V_\lambda) \rightarrow \mathbb{Z}$$

given by

$$\langle \pi(\phi, \eta), \mathcal{I}C(C, L) \rangle = \begin{cases} (-1)^{\dim C} & \text{if } (C, L) = (C_\phi, L_\rho), \\ 0 & \text{otherwise}. \end{cases}$$

The Kazhdan–Lusztig conjecture predicts that

$$\langle M(\phi, \eta), S(C, L) \rangle = \begin{cases} (-1)^{\dim C} & \text{if } (C, L) = (C_\phi, L_\rho), \\ 0 & \text{otherwise}. \end{cases}$$

When $G$ and $\lambda = \lambda_\psi$ are in the cases in Proposition 5.3, this conjecture is proven by showing

$$m_{\text{rep}}((\phi, \eta), (\phi', \eta')) = m'_{\text{geo}}((C_\phi, L_\rho), (C'_\phi, L'_\rho'))$$

for every pairs $(\phi, \eta)$ and $(\phi', \eta')$.

**Appendix B. Representations, $\mathcal{D}$-modules, and perverse sheaves**

To define $p$-adic ABV packets, one uses a relation between irreducible representations of $G(F)$ with simple objects of equivariant perverse sheaves (Proposition 2.2). For real reductive groups, this relation is a conclusion of a deep story. In this appendix, we try to explain this story. Note that it is hard to say that this appendix is mathematically accurate. The readers should refer to relevant references for details.

Let $G$ be a quasi-split connected semisimple group over $\mathbb{R}$, and $K$ be a maximal compact subgroup of $G(\mathbb{R})$. Assume that $G(\mathbb{R})$ is connected. Fix a rational Borel subgroup $B = TU$ of
B.1. Casselman–Wallach globalization. Let $\text{Rep}(G(\mathbb{R}))$ be the category of smooth admissible Fréchet representations of moderate growth, and $\text{Mod}_{\text{adm}}(\mathfrak{g}_C, K)$ be the category of admissible Harish-Chandra ($\mathfrak{g}_C, K$)-modules. The isomorphism classes of irreducible objects in these categories are denoted by $\text{Irr}(G(\mathbb{R}))$ and $\text{Irr}(\mathfrak{g}_C, K)$, respectively. The category $\text{Rep}(G(\mathbb{R}))$ seems to be difficult for topological reasons, whereas $\text{Mod}_{\text{adm}}(\mathfrak{g}_C, K)$ seems to be easier because it is purely algebraic.

Taking $K$-finite vectors, we obtain a functor

$$HC : \text{Rep}(G(\mathbb{R})) \rightarrow \text{Mod}_{\text{adm}}(\mathfrak{g}_C, K).$$

**Theorem B.1** (Casselman–Wallach [C89], [W92], [BK14]). There exists a quasi-inverse functor $\text{Mod}_{\text{adm}}(\mathfrak{g}_C, K) \rightarrow \text{Rep}(G(\mathbb{R}))$ of $HC$, called the Casselman–Wallach globalization functor. Hence the functor $HC : \text{Rep}(G(\mathbb{R})) \rightarrow \text{Mod}_{\text{adm}}(\mathfrak{g}_C, K)$ is an equivalence of categories.

By the Casselman–Wallach globalization, one can consider $\text{Irr}(\mathfrak{g}_C, K)$ instead of $\text{Irr}(G(\mathbb{R}))$.

B.2. Casselman’s subrepresentation theorem. A character $\chi$ of $T(\mathbb{R})$ gives a principal series representation $I(\chi) = \text{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}(\chi)$. Let $I(\chi)_K$ be the $K$-finite part of $I(\chi)$.

**Theorem B.2** (Casselman’s subrepresentation theorem [C78]). For any $\pi \in \text{Mod}_{\text{adm}}(\mathfrak{g}_C, K)$, there exists a character $\chi$ of $T(\mathbb{R})$ such that

$$\pi \mapsto I(\chi)_K.$$

There is a $p$-adic analogue of this theorem, but it asserts that any irreducible smooth representation of $G(F)$, where $F$ is $p$-adic, is a subrepresentation of the parabolic induction from a supercuspidal representation. One may understand that Casselman’s subrepresentation theorem says that real reductive Lie groups have few supercuspidal representations.

B.3. Beilinson–Bernstein correspondence. Recall that the principal series representation $I(\chi)$ is a space of sections of the $G(\mathbb{R})$-equivariant vector bundle $G(\mathbb{R}) \times_{B(\mathbb{R})} \chi \rightarrow G(\mathbb{R})/B(\mathbb{R})$. Now let us consider the $D$-modules on the complete flag variety $B = G(\mathbb{C})/B(\mathbb{C})$.

For a reference, see [HTT08, §11].

Fix $\lambda \in t^*_C/W$. For a representative $\lambda \in t^*_C$, the character $\chi_{\lambda + \rho}$ of $T(\mathbb{C})$ corresponding to $\lambda + \rho$ gives a $G(\mathbb{C})$-equivariant line bundle $\mathcal{L}(\lambda + \rho)$ on $B$. We consider the sheaf $D_\lambda$ of twisted differential operators acting on $\mathcal{L}(\lambda + \rho)$. Let $\text{Mod}_{qc}(D_\lambda)$ be the abelian category of $D_\lambda$-modules which are quasi-coherent over $\mathcal{O}_B$, and $\text{Mod}(\mathfrak{g}_C)_\lambda$ be the category of $U(\mathfrak{g}_C)$-modules with infinitesimal character $\lambda$, where $U(\mathfrak{g}_C)$ is the universal enveloping algebra of $\mathfrak{g}_C$. The global section functor gives a functor

$$\Gamma(B, \cdot) : \text{Mod}_{qc}(D_\lambda) \rightarrow \text{Mod}(\mathfrak{g}_C)_\lambda.$$

The set of roots of $T$ and its positive system with respect to $B$ are denoted by $\Delta$ and $\Delta^+$, respectively. We put

$$P = \{ \lambda \in t^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ (\alpha \in \Delta) \}.$$

It is called the weight lattice in $t^*$. 


Theorem B.3 (Beilinson–Bernstein correspondence [HTT08, Corollary 11.2.6]). When \( \lambda \in P \) satisfies that
\[
\langle \lambda, \alpha^\vee \rangle < 0, \quad \alpha \in \Delta^+,
\]
then the functor \( \Gamma(\mathcal{B}, \cdot) \) induces equivalences

\[
\operatorname{Mod}_{qc}(D\lambda) \cong \operatorname{Mod}(\mathfrak{g}_C)_{\lambda}
\]
of abelian categories. The inverse functor is given by \( D\lambda \otimes_U(\mathfrak{g}_C) (\cdot) \).

However, admissible representations of \( G(\mathbb{R}) \) are related with not just \( \mathfrak{g}_C \)-modules, but \( (\mathfrak{g}_C, K) \)-modules. Let \( \operatorname{Rep}(G(\mathbb{R}))_\lambda \) and \( \operatorname{Mod}(\mathfrak{g}_C, K)_\lambda \) be the subcategories of \( \operatorname{Rep}(G(\mathbb{R})) \) and \( \operatorname{Mod}(\mathfrak{g}_C, K) \) consisting of objects with infinitesimal character \( \lambda \), respectively. To relate the category \( \operatorname{Mod}(\mathfrak{g}_C, K)_\lambda \) of \( (\mathfrak{g}_C, K) \)-modules with \( D \)-modules, we need to consider the category of \( K \)-equivariant \( D \)-modules \( \operatorname{Mod}_{qc}(D\lambda; K) \). Then we have

\[
\operatorname{Mod}_{qc}(D\lambda; K) \cong \operatorname{Mod}(\mathfrak{g}_C, K)_\lambda.
\]

For more precision, see, e.g., [HTT08, Theorem 11.5.3, Remark 11.5.4] and its references. We notice that an object in \( \operatorname{Mod}(\mathfrak{g}_C, K)_\lambda \) is not necessarily admissible.

B.4. Riemann–Hilbert correspondence. Next, we recall Hilbert’s twenty-first problem or the Riemann–Hilbert problem. Before stating this problem, let us consider the following example.

Fix a complex number \( a \in \mathbb{C} \), and consider the differential equation
\[
\frac{df}{dz} = \frac{a}{z} f
\]
on \( \mathbb{C} \setminus \{0\} \). This equation has regular singularities at 0 and \( \infty \) in the projective line \( \mathbb{P}^1_{\mathbb{C}} \). The local solutions of the equation are of the form \( f(z) = c \cdot z^a \) for constants \( c \in \mathbb{C} \). If \( a \notin \mathbb{Z} \), then the function \( z^a \) cannot be made well-defined on all of \( \mathbb{C} \setminus \{0\} \). This means that the vector bundle
\[
E_a = \{ c \cdot z^a \mid c \in \mathbb{C} \} \ni c \cdot z^a \mapsto z \in \mathbb{C} \setminus \{0\}
\]
is a non-trivial line bundle (local system) on \( \mathbb{C} \setminus \{0\} \). In other words, the differential equation has non-trivial monodromy. Explicitly, this monodromy is the 1-dimensional representation of the fundamental group \( \pi_1(\mathbb{C} \setminus \{0\}) \cong \mathbb{Z} \) in which a generator (a loop around the origin) acts by multiplication by \( e^{2\pi \sqrt{-1} a} \).

This is a typical example that local solutions of regular differential equations give local systems
\[
\left( \frac{df}{dz} = \frac{a}{z} f \right) \xrightarrow{\text{local solutions}} E_a = \{ c \cdot z^a \mid c \in \mathbb{C} \} \in \operatorname{Loc}(\mathbb{C} \setminus \{0\}).
\]
The converse of this observation, namely, the existence of linear differential equations having a prescribed monodromy is called Hilbert’s twenty-first problem, or more commonly, the Riemann–Hilbert problem.

Now, roughly speaking, \( D \)-modules are “gluing of differential equations”, whereas perverse sheaves are “gluing of local systems”. The above observation is generalized to a correspondence between \( D \)-modules and perverse sheaves.
Theorem B.4 (Riemann–Hilbert correspondence [HTT08, Theorem 7.2.5]). Let $X$ be a complex manifold or a smooth algebraic variety (over $\mathbb{C}$). We denote by $\text{Mod}_{rh} (\mathcal{D}_X)$ the category of “regular holonomic” $\mathcal{D}$-modules on $X$, and by $\text{Per}(\mathcal{C}_X)$ the category of perverse sheaves on $X$. Then there exists an equivalence of categories

$$\text{DR}_X : \text{Mod}_{rh} (\mathcal{D}_X) \to \text{Per}(\mathcal{C}_X).$$

The functor $\text{DR}_X$ is called the de Rham functor.

In fact, one can define the category of equivariant perverse sheaves on $X$, and can relate it with the one of equivariant $\mathcal{D}$-modules. See also (the proof of) [HTT08, Theorem 11.6.1].

B.5. Conclusion. We conclude that one can relate irreducible representations of $G(\mathbb{R})$ with equivariant perverse sheaves via the following correspondences in the following rough diagram:

Irreducible representations of $G(\mathbb{R})$

$\Downarrow$ $K$-finite part, Casselman–Wallach globalization

Simple $(\mathfrak{g}_C, K)$-modules

$\Downarrow$ the Beilinson–Bernstein correspondence

Simple $K$-equivariant $\mathcal{D}$-modules on the flag variety $\mathcal{B}$

$\Downarrow$ the Riemann–Hilbert correspondence

Simple $K$-equivariant perverse sheaves on the flag variety $\mathcal{B}$.

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References


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