MÖGLIN’S EXPLICIT CONSTRUCTION OF LOCAL $A$-PACKETS

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Abstract. This note is a proceeding of the workshop “On the Langlands Program: Endoscopy and Beyond” held in National University of Singapore from 17 Dec. 2018 to 18 Jan. 2019. The purpose is to explain Mœglin’s explicit constructions of $A$-packets both when the base field $F$ is $p$-adic and when $F$ is archimedean.

Introduction

To give a classification of discrete spectrum of automorphic forms, the notion of $A$-parameters is introduced by Arthur in 1980’s. The local $A$-parameters are thus the “local factors” of the global classification. In this article, we focus on the local situation.

Let $F$ be a local field of characteristic zero, and $WD_F$ be the Weil–Deligne group of $F$, i.e.,

$$WD_F = \begin{cases} W_F \times SL_2(\mathbb{C}) & \text{if } F \text{ is non-archimedean,} \\ W_F & \text{if } F \text{ is archimedean,} \end{cases}$$

where $W_F$ is the Weil group of $F$. For a quasi-split connected reductive algebraic group $G$ over $F$, we denote by $\hat{G}$ the complex dual group of $G$. A homomorphism $\psi : WD_F \times SL_2(\mathbb{C}) \to \hat{G} \times W_F$ is an $A$-parameter for $G$ if

1. $\psi$ commutes the two projections $WD_F \times SL_2(\mathbb{C}) \to W_F$ and $\hat{G} \times W_F \to W_F$;
2. $\psi(W_F)$ consists of semisimple elements;
3. $\psi|W_F$ is continuous;
4. $\psi(W_F)$ projects onto a relatively compact subset in $\hat{G}$;
5. $\psi|SL_2(\mathbb{C})$ is algebraic for each $SL_2(\mathbb{C}) \subset WD_F \times SL_2(\mathbb{C})$.

Two $A$-parameters are said to be equivalent if they are conjugate by an element in $\hat{G}$. We define $\Psi(G)$ to be the set of equivalence classes of $A$-parameters for $G$. We say that $\psi \in \Psi(G)$ is tempered if the restriction of $\psi$ to the last $SL_2(\mathbb{C})$ is trivial, i.e., $\psi$ factors through the projection $WD_F \times SL_2(\mathbb{C}) \to WD_F$. We denote by $\Phi_{\text{temp}}(G)$ the subset of $\Psi(G)$ consisting of tempered $A$-parameters. Associated with $\psi \in \Psi(G)$, we define the component group by $S_\psi = \pi_0(Z(\hat{G}(\text{Im}(\psi))) / Z(\hat{G})^{WD_F})$, which is a finite group. The set of equivalence classes of irreducible representations of $S_\psi$ is denoted by $\text{Irr}(S_\psi)$.

Let $\text{Irr}(G(F))$ be the set of equivalence classes of irreducible admissible representations of $G(F)$. We denote by $\text{Irr}_{\text{unit}}(G(F))$ (resp. $\text{Irr}_{\text{temp}}(G(F))$) the subset of $\text{Irr}(G(F))$ consisting of equivalence classes of unitary (resp. tempered) representations of $G(F)$. In 2013, Arthur [Ar13] has completed the magnificent work on the classification of automorphic representations of symplectic and special orthogonal groups. The local main theorem in this work is as follows.

Theorem 0.1 ([Ar13, Theorem 2.2.1]). Let $G$ be split $SO_{2n+1}$ or $Sp_{2n}$ over $F$. 

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(1) For each $\psi \in \Psi(G)$, there is a finite multiset $\Pi_\psi$ over $\Irr_{\text{unit}}(G(F))$ with a map

$$\Pi_\psi : \Irr(S_\psi), \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

enjoying certain (twisted and standard) endoscopic character identities. We call $\Pi_\psi$ the $A$-packet for $G(F)$ associated with $\psi$.

(2) When $\psi = \phi \in \Phi_{\text{temp}}(G)$, the $A$-packet $\Pi_\phi$ is in fact a subset of $\Irr_{\text{temp}}(G(F))$. Moreover the map $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle_\phi \in \Irr(S_\phi)$ is injective, which is bijective when $F$ is non-archimedean. In addition, $\Pi_\phi \cap \Pi_{\phi'} = \emptyset$ for $\phi \neq \phi'$, and

$$\Irr_{\text{temp}}(G(F)) = \bigsqcup_{\phi \in \Phi_{\text{temp}}(G)} \Pi_\phi.$$ 

Theorem 0.1 (2) says that $\phi \in \Phi_{\text{temp}}(G)$ together with $\eta \in \Irr(S_\phi)$ classifies $\Irr_{\text{temp}}(G(F))$. Let $\Phi(G)$ be the set of equivalence classes of $L$-parameters $\phi$ for $G$, i.e., homomorphisms $\phi : WD_F \rightarrow \hat{G} \rtimes W_F$ such that

(1) $\psi$ commutes the two projections $WD_F \rightarrow W_F$ and $\hat{G} \rtimes W_F \rightarrow W_F$;
(2) $\psi(W_F)$ consists of semisimple elements;
(3) $\psi|W_F$ is continuous;
(4) $\psi|\SL_2(\mathbb{C})$ is algebraic if $F$ is non-archimedean.

Using the Langlands classification, Theorem 0.1 (2) can be extended to $\phi \in \Phi(G)$, i.e., there exists a finite subset $\Pi_\phi$ of $\Irr(G(F))$ with an injective map $\Pi_\phi \ni \pi \mapsto \langle \cdot, \pi \rangle_\phi \in \Irr(S_\phi)$, which is bijective when $F$ is non-archimedean, such that

$$\Irr(G(F)) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi.$$ 

We call $\Pi_\phi$ the $L$-packet for $G(F)$ associated with $\phi$. Therefore, $L$-packets $\Pi_\phi$ classify $\Irr(G(F))$.

On the other hands, $A$-packets $\Pi_\psi$ associated with $\psi \in \Psi(G)$ are the “local components of global $A$-packets”, and they do not classify $\Irr_{\text{unit}}(G(F))$. For example,

- the map $\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \Irr(S_\psi)$ is not injective nor surjective in general;
- there are $\psi, \psi' \in \Psi(G)$ such that $\Pi_\psi \cap \Pi_{\psi'} = \emptyset$ but $\psi \neq \psi'$;
- there exists $\pi \in \Irr_{\text{unit}}(G(F))$ such that $\pi \notin \Pi_\psi$ for any $\psi \in \Psi(G)$.

Moreover, the $A$-packet $\Pi_\psi$ is determined by endoscopic character identities, so that it is difficult to describe $\Pi_\psi$ explicitly. In particular, it is an important problem to determine whether $\Pi_\psi$ is multiplicity-free, i.e., a subset of $\Irr_{\text{unit}}(G(F))$, or not.

Before Arthur, there are several works for “constructions of $A$-packets”.

- Barbasch–Vogan (1985) [BV85] constructed a packet $\Pi_\psi$ of unipotent representations when $F = \mathbb{C}$ and $\psi$ is “unipotent” (of good parity). This packet is multiplicity-free. It was proven by Mœglin–Renard [MR17] that this packet coincides with Arthur’s one.
- Adams–Johnson (1987) [AJ87] constructed a packet $\Pi_\psi$ of cohomological representations when $F = \mathbb{R}$ and $\psi$ is so-called “Adams–Johnson”. It was proven by Arancibia–Mœglin–Renard [AMR] that this packet coincides with Arthur’s one.
- Mœglin [Moe17] and Mœglin–Renard [MRa, MRb] constructed Arthur’s packet $\Pi_\psi$ generally when $F = \mathbb{R}$ by using the Howe duality correspondence, cohomological...
inductions, the translation principle, and irreducible parabolic inductions. However, since the translation functor is difficult, they have not yet obtained the multiplicity-free result.

- Using microlocal analysis of certain stratified complex varieties, Adams–Barbasch–Vogan (1992) [ABV92] constructed a packet $\Pi_{\psi}^{ABV}$ for general $F$ when $F = \mathbb{R}$. This packet is multiplicity-free, and coincides with Adams–Johnson packet $\Pi_{\psi}$ when $\psi$ is “Adams–Johnson”. However, it is an open problem that whether $\Pi_{\psi}^{ABV}$ is equal to Arthur’s packet $\Pi_{\psi}$.

- Using Jacquet modules, Mœglin constructed the $A$-packets $\Pi_{\psi}$ when $F$ is $p$-adic (up to constructions of supercuspidal representations) in her consecutive works (e.g., [Mœ06, Mœ09a], etc.). In particular, she showed in [Mœ11] that the $A$-packets are multiplicity-free. For a detailed why the $A$-packets constructed by Mœglin agree with Arthur’s ones, see also Xu’s paper [X17b] in addition to the original papers of Mœglin.

- In a recent work [CFMMX], using a vanishing cycles functor of perverse sheaves on certain stratified complex varieties, Cunningham–Fiori–Mracek–Moussaoui–Xu constructed an “ABV packet” $\Pi_{\psi}^{ABV}$ when $F$ is $p$-adic. This is a $p$-adic analogue of the work of Adams–Barbasch–Vogan [ABV92]. This packet is multiplicity-free. The main conjecture in [CFMMX] is that $\Pi_{\psi}^{ABV}$ would coincide with Arthur’s packet $\Pi_{\psi}$.

The purpose of this note is to explain Mœglin’s explicit constructions of $A$-packets both when $F$ is $p$-adic and when $F$ is archimedean. We will explain only the construction of the packet $\Pi_{\psi}$ for $G = SO_{2n+1}$ or $G = Sp_{2n}$, but will not treat the proofs, the map $\pi \mapsto \langle \cdot, \pi \rangle_{\psi}$, or other groups. For these topics, we refer to relevant references.

In Part 1, we will explain the $p$-adic case along with a series of papers of Xu [X17a, X17b, X]. In §1, we fix notations for induced representations and Jacquet modules, and recall some basic results. For Mœglin’s constructions of $A$-packets, we follow a filtration of $A$-parameters as follows:

(standard) $\subset$ (having a DDR) $\subset$ (of good parity) $\subset$ (general).

These notions are defined in §2. In §3, we treat the case where the $A$-parameter $\psi$ is elementary, together with a description of $L$-parameters for supercuspidal representations, which is one of main results in [X17a]. The case where $\psi$ has a discrete diagonal restriction (DDR) is treated in §4. In particular, we determine the cardinality of the $A$-packet $\Pi_{\psi}$ in this case. In §5, we treat the case where $\psi$ is of good parity and the general case. In the general case, the $A$-packets are constructed by irreducible parabolic inductions. As a consequence, one can check that $\Pi_{\psi}$ is multiplicity-free, which is the main conclusion in [X17b]. Unlike the case of DDR, in the case of good parity, we will construct representations which are irreducible or zero. In §6, we give an algorithm for the non-vanishing criterion of these representations, which was established in [X].

In Part 2, we will explain the archimedean case. Both in the cases where $F = \mathbb{R}$ and $F = \mathbb{C}$, the general packets $\Pi_{\psi}$ are constructed by irreducible parabolic inductions from packets for parameters of good parity as in the $p$-adic case. In §8, we explain the complex case along with Mœglin–Renard [MR17]. For $\psi$ of good parity, the packet $\Pi_{\psi} = \Pi_{\psi}^{BV}$ is constructed by assigning infinitesimal characters and wavefront sets. We also give a more explicit description.
of $\Pi_{\psi}^{BV}$ by Barbasch [B89]. In §9, we explain the real case. According to Mœglin [Mœ17] and Mœglin–Renard [MRa, MRb], one should follow a filtration of $A$-parameters as follows:

\[(\text{unipotent}) \subset (\text{very regular}) \subset (\text{of good parity}) \subset (\text{general})\]

The packets are constructed by using the theta correspondence, cohomological inductions, the translation principle, and irreducible parabolic inductions. However, since these techniques seem not to be so explicit, we only explain the case where $\psi$ is “Adams–Johnson”, which is a spatial case of very regular parameters. In this case, the packet $\Pi_\psi$ is constructed by derived functor modules $A_q(\lambda)$ with $\lambda$ in the good range.

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Part 1. The non-archimedean case

In Part 1, we explain the theory of $A$-packets in the non-archimedean case. Let $F$ be a non-archimedean local field of characteristic zero. We denote by $W_F$ the Weil group of $F$. The norm map $|\cdot|: W_F \to \mathbb{R}^\times$ is normalized so that $|\text{Frob}| = q^{-1}$, where $\text{Frob} \in W_F$ is a fixed (geometric) Frobenius element, and $q = q_F$ is the cardinality of the residual field of $F$.

Each irreducible representation $\rho$ of $W_F$ of dimension $d$ is identified with the irreducible supercuspidal representation of $\text{GL}_d(F)$ via the local Langlands correspondence for $\text{GL}_d$. For each integer $d$, the unique irreducible algebraic representation of $\text{SL}_2(\mathbb{C})$ of dimension $d$ is denoted by $S_d$. We denote by $S_a \boxtimes S_b$ the outer tensor product, which is an irreducible representation of $\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$, and by $S_a \otimes S_b$ the inner tensor product, which is a representation of $\text{SL}_2(\mathbb{C})$ such that

$$S_a \otimes S_b \cong S_{a+b-1} \oplus S_{a+b-3} \oplus \cdots \oplus S_{a-b+1}.$$ 

For a representation $\Pi$ of some group, we write $\text{s.s.}(\Pi)$ for the semisimplification of $\Pi$.

1. Induced representations and Jacquet modules

In this section, we recall some results on induced representations and Jacquet modules.

1.1. The case of $\text{GL}_N$. Let $P = MN$ be a standard parabolic subgroup of $\text{GL}_N(F)$, i.e., $P$ contains the Borel subgroup consisting of upper half triangular matrices. Then the Levi subgroup $M$ is isomorphic to $\text{GL}_{d_1}(F) \times \cdots \times \text{GL}_{d_r}(F)$ with $d_1 + \cdots + d_r = N$. For smooth representations $\tau_1, \ldots, \tau_r$ of $\text{GL}_{d_1}(F), \ldots, \text{GL}_{d_r}(F)$, respectively, we denote the normalized induced representation by

$$\tau_1 \times \cdots \times \tau_r := \text{Ind}_P^{\text{GL}_N(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r).$$

A segment is a symbol $[x, y]$, where $x, y \in \mathbb{R}$ with $x \leq y \in \mathbb{Z}$. We identify $[x, y]$ with the set $\{x, x+1, \ldots, y\}$ if $x \leq y$, and $\{x, x+1, \ldots, y\}$ if $x \geq y$, so that $\#[x, y] = |x-y| + 1$. Let $\rho$ be an irreducible (unitary) supercuspidal representation of $\text{GL}_d(F)$. Then the normalized induced representation

$$\rho : [x, y] \times \cdots \times \rho : [y]$$

of $\text{GL}_{d(|x-y|+1)}(F)$ has a unique irreducible subrepresentation, which is denoted by

$$\langle \rho; x, \ldots, y \rangle.$$

If $x \geq y$, this is called a Steinberg representation and is denoted by

$$\langle \rho; x, y \rangle$$

which is a discrete series representation of $\text{GL}_{d(x-y+1)}(F)$. When $\rho = 1_{\text{GL}_1(F)}$, we write $\text{St}_d = \text{St}(1_{\text{GL}_1(F)}, d) = (d-1)/2, (d-3)/2, \ldots, -(d-1)/2$. If $x < y$, this is called a Speh representation and is denoted by

$$\langle \rho; y, x \rangle.$$

For example, if $\rho = \mu$ is a unitary character (i.e., $d = 1$) and $x < y$, then $\langle \mu; x, \ldots, y \rangle = \mu(\det_{y-x+1}((x+y)/2)$ is a character of $\text{GL}_{y-x+1}(F)$, where we denote by $\det_k$ the determinant character of $\text{GL}_k(F)$.

Definition 1.1. Let $[x, y]$ and $[x', y']$ be two segments.
(1) When \((x-y)(x-y') \geq 0\), we say that \([x,y]\) and \([x',y']\) are linked if \([x,y] \not\subset [x',y']\), \([x',y'] \not\subset [x,y]\) as sets, and \([x,y] \cup [x',y']\) is also a segment.

(2) When \((x-y)(x'-y') < 0\), we say that \([x,y]\) and \([x',y']\) are linked if \([y,x]\) and \([x',y']\) are linked, and \(x,y \not\subset [x',y']\) and \(x',y' \not\subset [x,y]\).

The linked-ness gives an irreducibility criterion for induced representations.

**Theorem 1.2** (Zelevinsky [Z80, Theorems 4.2, 9.7], Mœglin–Waldspurger [MW89]). Let \([x,y]\) and \([x',y']\) be segments, and let \(\rho\) and \(\rho'\) be irreducible unitary supercuspidal representations of \(\mathrm{GL}_d(F)\) and \(\mathrm{GL}_{d'}(F)\), respectively. Then the induced representation

\[
\langle \rho; x, \ldots, y \rangle \times \langle \rho'; x', \ldots, y' \rangle
\]

is irreducible unless \([x,y]\) are linked, and \(\rho \cong \rho'\).

For a partition \((k_1, \ldots, k_r)\) of \(k\), we denote by \(\mathrm{Jac}_{(k_1, \ldots, k_r)}\) the normalized Jacquet functor of representations of \(\mathrm{GL}_k(F)\) with respect to the standard maximal parabolic subgroup \(P = MN\) with \(M \cong \mathrm{GL}_{k_1}(F) \times \cdots \times \mathrm{GL}_{k_r}(F)\). The Jacquet module of \(\langle \rho; x, \ldots, y \rangle\) with respect to a maximal parabolic subgroup is computed by Zelevinsky.

**Proposition 1.3** ([Z80, Propositions 3.4, 9.5]). Let \(\rho\) be an irreducible (unitary) supercuspidal representation of \(\mathrm{GL}_d(F)\). Suppose that \(x \neq y\) and set \(k = d(\mid x - y \mid + 1)\). Then \(\mathrm{Jac}_{(k_1, k_2)}(\langle \rho; x, \ldots, y \rangle) = 0\) unless \(k_1 \equiv 0 \text{ mod } d\). If \(k_1 = dm\) with \(1 \leq m \leq |x - y|\), we have

\[
\mathrm{Jac}_{(k_1, k_2)}(\langle \rho; x, \ldots, y \rangle) = \langle \rho; x, \ldots, x - \epsilon(m - 1) \rangle \boxtimes \langle \rho; x - \epsilon m, \ldots, y \rangle,
\]

where \(\epsilon \in \{\pm 1\}\) is given so that \(\epsilon(x - y) > 0\).

Let \(\mathcal{R}_N\) be the Grothendieck group of the category of smooth representations of \(\mathrm{GL}_N(F)\) of finite length. By the semisimplification, we identify the objects in this category with elements in \(\mathcal{R}_N\). Equivalence classes of irreducible smooth representations of \(\mathrm{GL}_N(F)\) form a \(\mathbb{Z}\)-basis of \(\mathcal{R}_N\). Set \(\mathcal{R} = \oplus_{N \geq 0} \mathcal{R}_N\). The induction functor gives a product

\[
m: \mathcal{R} \otimes \mathcal{R} \to \mathcal{R}, \quad \tau_1 \otimes \tau_2 \mapsto \text{s.s.}(\tau_1 \times \tau_2).
\]

This product makes \(\mathcal{R}\) an associative commutative ring. On the other hand, the Jacquet functor gives a coproduct

\[
m^*: \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}
\]

which is defined by the \(\mathbb{Z}\)-linear extension of

\[
\text{Irr}(\mathrm{GL}_N(F)) \ni \tau \mapsto \sum_{k=0}^N \text{s.s.} \mathrm{Jac}_{(k,N-k)}(\tau).
\]

Then \(m\) and \(m^*\) make \(\mathcal{R}\) a graded Hopf algebra, i.e., \(m^*: \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}\) is a ring homomorphism.

1.2. **The cases of \(\mathrm{SO}_{2n+1}\) and \(\mathrm{Sp}_{2n}\)**. Next, we set \(G_n\) to be split \(\mathrm{SO}_{2n+1}\) or \(\mathrm{Sp}_{2n}\), i.e., \(G_n\) is the split algebraic group of type \(B_n\) or \(C_n\). Fix a Borel subgroup of \(G_n(F)\). Let \(P = MN\) be a standard parabolic subgroup of \(G_n(F)\). Then the Levi part \(M\) is of the form \(\mathrm{GL}_{d_1}(F) \times \cdots \times \mathrm{GL}_{d_r}(F) \times G_{n_0}(F)\) with \(d_1 + \cdots + d_r + n_0 = n\). For a smooth representation \(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0\) of \(M\), we denote the normalized induced representation by

\[
\tau_1 \times \cdots \times \tau_r \times \pi_0 = \text{Ind}_P^{G_n(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0).
\]
On the other hand, for a smooth representation \( \pi \) of \( G_n(F) \), we denote the normalized Jacquet module with respect to \( P \) by

\[
\text{Jac}_P(\pi),
\]
and its semisimplification by \( \text{s.s.}\text{Jac}_P(\pi) \). When \( r = 1 \), i.e., \( M \cong \text{GL}_d(F) \times G_{n-d}(F) \) and

\[
\text{s.s.}\text{Jac}_P(\pi) = \bigoplus_{i \in I} \tau_i \boxtimes \pi_i,
\]
for a fixed irreducible supercuspidal unitary representation \( \rho \) of \( \text{GL}_d(F) \) and for a real number \( x \), we set

\[
\text{Jac}_{\rho|\cdot|^{x}}(\pi) = \bigoplus_{i \in I} \pi_i.
\]
This is a representation of \( G_{n-d}(F) \). Also, for \( \rho_1, \ldots, \rho_r \) and for \( x_1, \ldots, x_r \in \mathbb{R} \), we set

\[
\text{Jac}_{\rho_1|\cdot|^{x_1}, \ldots, \rho_r|\cdot|^{x_r}}(\pi) = \text{Jac}_{\rho_r|\cdot|^{x_r}} \circ \cdots \circ \text{Jac}_{\rho_1|\cdot|^{x_1}}(\pi).
\]

Now suppose that an irreducible representation \( \pi \) of \( G_n(F) \) is a subrepresentation (resp. a quotient) of an induced representation \( \tau \boxtimes \pi_0 \) with irreducible representation \( \tau \boxtimes \pi_0 \) of \( M \cong \text{GL}_d(F) \times G_{n-d}(F) \). Then (using the contragredient and the MVW functors if necessary), the Frobenius reciprocity implies that \( \text{s.s.}\text{Jac}_P(\pi) \) contains \( \tau \boxtimes \pi_0 \) (resp. \( \tilde{\tau} \boxtimes \pi_0 \)). In particular:

**Lemma 1.4.** Let \( \tau \boxtimes \pi_0 \) be an irreducible representation of \( M \cong \text{GL}_d(F) \times G_{n-d}(F) \). If \( \text{s.s.}\text{Jac}_P(\tau \boxtimes \pi_0) \) contains \( \tau \boxtimes \pi_0 \) (resp. \( \tilde{\tau} \boxtimes \pi_0 \)) with multiplicity one, then the induced representation \( \tau \boxtimes \pi_0 \) has a unique irreducible subrepresentation (resp. a unique irreducible quotient).

We will use this technique (or its variant) to construct \( A \)-packets.

Let

\[
\mathcal{R}(G) = \bigoplus_{n \geq 0} \mathcal{R}(G_n)
\]
be the direct sum of the Grothendieck groups \( \mathcal{R}(G_n) \) of the categories of smooth representations of \( G_n(F) \) of finite length. The parabolic induction defines a module structure

\[
\times : \mathcal{R} \otimes \mathcal{R}(G) \to \mathcal{R}(G),
\]
and the Jacquet functor defines a comodule structure

\[
\mu^* : \mathcal{R}(G) \to \mathcal{R} \otimes \mathcal{R}(G)
\]
by

\[
\text{Irr}(G_n(F)) \ni \pi \mapsto \sum_{d=0}^n \text{s.s.}\text{Jac}_{P_d}(\pi),
\]
where \( P_d = M_dN_d \) is the standard parabolic subgroup of \( G_n(F) \) with the Levi factor \( M_d \cong \text{GL}_d(F) \times G_{n-d}(F) \).

The contragredient functor \( \tau \mapsto \tilde{\tau} \) defines an automorphism \( \sim : \mathcal{R} \to \mathcal{R} \) in a natural way. Let \( s : \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R} \) be the homomorphism defined by \( \sum_i \tau_i \otimes \tau'_i \mapsto \sum_i \tau'_i \otimes \tau_i \).

One can compute the Jacquet modules of induced representations by the following formula.
Theorem 1.5 (Tadić [T95]). Consider the composition
\[ M^* = (m \otimes \text{id}) \circ (\sim \otimes m^*) \circ s \circ m^* : \mathcal{R} \to \mathcal{R} \otimes \mathcal{R}. \]

Then for a standard parabolic subgroup \( P = MN \) of \( G_n(F) \) with \( M \cong \text{GL}_d(F) \times G_{n-d}(F) \),
and for an admissible representation \( \tau \boxtimes \pi \) of \( M \), we have
\[ \mu^*(\tau \times \pi) = M^*(\pi) \times \mu^*(\pi). \]

1.3. Aubert involution. For \( \pi \in \mathcal{R}(G_n) \), we define \( D_{G_n}(\pi) \in \mathcal{R}(G_n) \) by
\[ D_{G_n}(\pi) = \sum_{P=MN} (-1)^{\dim A_M} \text{Ind}_P^{G_n(F)}(\text{Jac}_P(\pi)) \in \mathcal{R}(G_n), \]
where \( P = MN \) runs over all standard parabolic subgroups of \( G_n(F) \), and \( A_M \) is the maximal split central torus of \( M \).

Theorem 1.6 ([Au95]). The operator \( D_{G_n} \) on \( \mathcal{R}(G_n) \) has the following properties:
1. \( D_{G_n} \circ \sim = \sim \circ D_{G_n} \);
2. \( D_{G_n}^2 = \text{id} \).
3. When \( \tau_i = (\rho_i; x_i, \ldots, y_i) \) for \( i = 1, \ldots, r \),
\[ D_{G_n}(\tau_1 \times \cdots \times \tau_r \times \pi_0) = \hat{\tau}_1 \times \cdots \times \hat{\tau}_r \times D_{G_n}(\pi_0), \]
where \( \hat{\tau}_i = (\rho_i; y_i, \ldots, x_i) \).
4. If \( P = MN \) with \( M \cong \text{GL}_d(F) \times G_{n-d}(F) \), and if \( \pi \in \mathcal{R}(G_n) \) satisfies
\[ \text{s.s.}\text{Jac}_P(\pi) \cong \sum_i \tau_i \boxtimes \pi_0, \]
then
\[ \text{s.s.}\text{Jac}_P(D_{G_n}(\pi)) \cong \sum_i \hat{\tau}_i \boxtimes D_{G_n}(\pi_0). \]

In particular,
\[ D_{G_n}(\text{Jac}_{\rho|}^\pi(\pi)) \cong \text{Jac}_{\rho|}^\varepsilon (D_{G_n}(\pi)). \]
5. If \( \pi \) is an irreducible representation of \( G_n(F) \), then there exists a sign \( \varepsilon \in \{ \pm 1 \} \) such that \( \hat{\pi} = \varepsilon \cdot D_{G_n}(\pi) \) is also an irreducible representation of \( G_n(F) \).
6. If \( \pi \) is an irreducible supercuspidal representation, then \( \hat{\pi} \cong \pi \).

For an irreducible representation \( \pi \) of \( G_n(F) \), we call the irreducible representation \( \hat{\pi} \) the Aubert involution of \( \pi \).

2. \( A \)-parameters

In this section, we review Arthur’s theory for \( A \)-packets.

2.1. The case of \( \text{GL}_N \). Fix a (geometric) Frobenius element \( \text{Frob} \in W_F \). A homomorphism
\[ \psi : W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \text{GL}_N(\mathbb{C}) \]
is called a representation of \( W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \) if
\[ \bullet \psi(\text{Frob}) \in \text{GL}_N(\mathbb{C}) \text{ is semisimple; } \]
\[ \bullet \psi|_{W_F} \text{ is smooth, i.e., has an open kernel; } \]
\[ \bullet \psi|_{\text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})} \text{ is algebraic. } \]
An $A$-parameter for $\text{GL}_N(F)$ is a representation $\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \text{GL}_N(\mathbb{C})$ such that $\psi(W_F)$ is bounded. Two $A$-parameters are said to be equivalent if they are equivalent as representations.

For an $A$-parameter $\psi$ for $\text{GL}_N(F)$, one can associate an irreducible unitary representation $\tau_\psi$ of $\text{GL}_N(F)$ as follows: When $\psi$ is irreducible, it is decomposed into a tensor product
\[
\psi = \rho \otimes S_a \otimes S_b,
\]
where $\rho$ is an irreducible bounded representation of $W_F$. Then we set $\tau_\psi$ to be the unique irreducible subrepresentation $\text{Sp}(\text{St}(\rho, a), b)$ of
\[
\left\langle \rho; \frac{a - 1}{2}, \ldots, -\frac{a - 1}{2} \right\rangle \cdot \left| -\frac{b - 1}{2} \right| \times \cdots \times \left\langle \rho; \frac{a - 1}{2}, \ldots, -\frac{a - 1}{2} \right\rangle \cdot \left| -\frac{b - 1}{2} \right|.
\]
If $b = 1$, it is the Steinberg representation $\text{St}(\rho, a) = \langle \rho; (a - 1)/2, \ldots, -(a - 1)/2 \rangle$, and if $a = 1$, it is the Speh representation $\text{Sp}(\rho, b) = \langle \rho; -(b - 1)/2, \ldots, (b - 1)/2 \rangle$. It is easy to check that $\tau_\psi$ is also the unique irreducible subrepresentation of
\[
\left\langle \rho; \frac{b - 1}{2}, \ldots, -\frac{b - 1}{2} \right\rangle \cdot \left| -\frac{a - 1}{2} \right| \times \cdots \times \left\langle \rho; \frac{b - 1}{2}, \ldots, -\frac{b - 1}{2} \right\rangle \cdot \left| -\frac{a - 1}{2} \right|.
\]
In general, $\psi$ can be decomposed into a direct sum
\[
\psi = \psi_1 \oplus \cdots \oplus \psi_r,
\]
where $\psi_1, \ldots, \psi_r$ are irreducible representations of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$. We set
\[
\tau_\psi = \tau_{\psi_1} \times \cdots \times \tau_{\psi_r},
\]
which is irreducible by Theorem 1.2.

2.2. The case of $\text{SO}_{2n+1}$ and $\text{Sp}_{2n}$. An $A$-parameter for $\text{SO}_{2n+1}$ is a symplectic representation
\[
\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \text{Sp}_{2n}(\mathbb{C})
\]
of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ such that $\psi(W_F)$ is bounded. Similarly, an $A$-parameter for $\text{Sp}_{2n}$ is an orthogonal representation
\[
\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \text{SO}_{2n+1}(\mathbb{C})
\]
of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ such that $\psi(W_F)$ is bounded. For $G_n = \text{SO}_{2n+1}$ or $G_n = \text{Sp}_{2n}$, we set $\Psi(G_n)$ to be the set of equivalence classes of $A$-parameters for $G_n$. We say that $\psi \in \Psi(G_n)$ is tempered if $\psi|\{1\} \times \{12\} \times \text{SL}_2(\mathbb{C})$ is trivial. We denote by $\Phi_{\text{temp}}(G_n)$ the subset of $\Psi(G_n)$ consisting of tempered $A$-parameters.

For $\psi \in \Psi(\text{SO}_{2n+1})$ (resp. $\psi \in \Psi(\text{Sp}_{2n})$), we can decompose
\[
\psi = m_1 \psi_1 + \cdots + m_r \psi_r + \psi' + \psi'',
\]
where $\psi_1, \ldots, \psi_r$ are distinct irreducible symplectic (resp. orthogonal) representations of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$, and $\psi'$ is a sum of irreducible representations which are not symplectic (resp. not orthogonal). We define the component group $A_\psi$ of $\psi$ by
\[
A_\psi = \bigoplus_{i=1}^{r} (\mathbb{Z}/2\mathbb{Z})\alpha_i \cong (\mathbb{Z}/2\mathbb{Z})^r.
\]
Namely, $A_\psi$ is a free $\mathbb{Z}/2\mathbb{Z}$-module of rank $r$ and $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of $A_\psi$ with $\alpha_i$ associated to $\psi_i$. We set
\[
z_\psi = \sum_{i=1}^{r} m_i \alpha_i \in A_\psi
\]
and call $z_\psi$ the central element in $A_\psi$. Then $S_\psi = \pi_0(Z_{\mathcal{G}}(\text{Im}(\psi)))/Z(\mathcal{G})^{W_F}$ is canonically isomorphic to $A_\psi/\langle z_\psi \rangle$.

As explained in Theorem 0.1, for $\psi \in \Psi(G_n)$, there is an $A$-packet $\Pi_\psi$, which is a finite multiset over $\text{Irr}_{\text{unit}}(G_n(F))$, together with a map
\[
\Pi_\psi \rightarrow \mathcal{A}_\psi, \quad \pi \mapsto \langle \cdot, \pi \rangle_{\psi}
\]
such that $\langle z_\psi, \pi \rangle_{\psi} = 1$ for any $\pi \in \Pi_\psi$. These are characterized by certain (twisted and standard) endoscopic character identities. Mœglin constructed $A$-packets explicitly, and showed that they are multiplicity-free (see, e.g., [Mœ06, Mœ09a, Mœ11], etc.). Since Mœglin’s $A$-packets satisfy the endoscopic character identities (see also [X17b]), they coincide with Arthur’s ones. Consequently, we obtain the following deep result.

**Theorem 2.1** ([Mœ11], [X17b, Theorem 8.12]). The $A$-packet $\Pi_\psi$ is multiplicity-free, i.e., $\Pi_\psi$ is a subset of $\text{Irr}_{\text{unit}}(G_n(F))$.

The purpose of Part 1 is to review Mœglin’s construction of $A$-packets. It is carried out through several stages.

Let
\[
\Delta: W_F \times \text{SL}_2(\mathbb{C}) \rightarrow W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}), \quad (w, \alpha) \mapsto (w, \alpha, \alpha)
\]
be the diagonal map. For $\psi \in \Psi(G_n)$, we set $\psi_d = \psi \circ \Delta$ to be the diagonal restriction of $\psi$.

**Definition 2.2.** Let $G_n = \text{SO}_{2n+1}$ or $G_n = \text{Sp}_{2n}$. We define a chain
\[
\Psi(G_n) \supset \Psi_{_{\text{gp}}}(G_n) \supset \Psi_{_{\text{DDR}}}(G_n) \supset \Psi_{_{\text{el}}}(G_n)
\]
as follows:

(1) $\psi \in \Psi_{_{\text{gp}}}(G_n)$ if $\psi$ is a sum of irreducible symplectic (resp. orthogonal) representations when $G_n = \text{SO}_{2n+1}$ (resp. $G_n = \text{Sp}_{2n}$). In this case, we say that $\psi$ is of good parity.

(2) $\psi \in \Psi_{_{\text{DDR}}}(G_n)$ if $\psi$ is of good parity and the diagonal restriction $\psi_d = \psi \circ \Delta$ is multiplicity-free. In this case, we say that $\psi$ has a discrete diagonal restriction (DDR).

(3) $\psi \in \Psi_{_{\text{el}}}(G_n)$ if $\psi$ has a DDR and
\[
\psi = \bigoplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}
\]
with $\min\{a_i, b_i\} = 1$ for any $i \in I$. In this case, we say that $\psi$ is elementary.

**3. The case of elementary parameters**

In this section, we construct $A$-packets $\Pi_\psi$ for elementary $A$-parameters $\psi \in \Psi_{_{\text{el}}}(G_n)$. See [X17a] and [X17b, §6] for more precision. Before the construction, we review the case of general linear groups $\text{GL}_N$. 
3.1. The case of $GL_N$. Let $\psi$ be an $A$-parameter for $GL_N(F)$. Assume in this subsection that $\psi$ is elementary, i.e., $\psi \cong \oplus_{i \in I}\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$ with $\min\{a_i, b_i\} = 1$ for any $i \in I$ and $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \neq \rho_j \boxtimes S_{a_j} \boxtimes S_{b_j}$ for $i \neq j$. Recall that one can associate an irreducible unitary representation $\tau_\psi$ of $GL_N(F)$. We construct $\tau_\psi$ once more.

If $a_i = b_i = 1$ for any $i \in I$, then $\tau_\psi = \times_{i \in I}\rho_i$. Now fix an irreducible representation $\rho$ of $W_F$ such that $\rho \subset \psi|W_F$. For $\rho \boxtimes S_a \subset \psi_d$, we set
\[
\delta_{\rho,a,\psi} = \begin{cases} 
1 & \text{if } \rho \boxtimes S_a \boxtimes 1 \subset \psi, \\
-1 & \text{otherwise}.
\end{cases}
\]

Let $a_0$ be the smallest integer such that $\rho \boxtimes S_{a_0} \subset \psi_d$. If $a_0 > 1$, then $\tau_\psi$ is the unique irreducible subrepresentation of
\[
\rho| \cdot |\delta_{a_0-1,2} \times \tau_{\psi'} \times \rho| \cdot |\delta_{a_0-1,2}^{-1},
\]
where $\psi'$ is an elementary $A$-parameter such that
\[
\psi_d' = \psi_d - \rho \boxtimes S_{a_0} + \rho \boxtimes S_{a_0-2}
\]
and
\[
\delta_0 := \delta_{\rho,a_0,\psi} = \delta_{\rho,a_0-2,\psi'}.
\]
If $a_0 = 1$ and there exists the next smallest integer $a_1$ such that $\rho \boxtimes S_{a_1} \subset \psi_d$, then $\tau_\psi$ is the unique irreducible subrepresentation of
\[
\left\langle \rho; \delta_1 \frac{a_1 - 1}{2}, \ldots, 0 \right\rangle \times \tau_{\psi'} \times \left\langle \rho; 0, \ldots, -\delta_1 \frac{a_1 - 1}{2} \right\rangle,
\]
where $\psi'$ is an elementary $A$-parameter such that
\[
\psi_d' = \psi_d - \rho \boxtimes (1 \oplus S_{a_1})
\]
and $\delta_1 := \delta_{\rho,a_1,\psi}$.

3.2. L-parameters for supercuspidal representations. Let $G_n$ be split $SO_{2n+1}$ or $Sp_{2n}$. Fix an elementary $A$-parameter $\psi \in \Psi_{el}(G_n)$ and a character $\varepsilon \in \hat{A}_\psi$ such that $\varepsilon(z_\psi) = 1$. When $\rho \boxtimes S_a \subset \psi_d$, we set $\varepsilon(\rho, a) = \varepsilon(\alpha_{\rho,a}) \in \{\pm 1\}$, where $\alpha_{\rho,a} \in A_\psi$ is the element associated to $\rho \boxtimes S_a \boxtimes S_1$ or $\rho \boxtimes S_1 \boxtimes S_a$.

**Definition 3.1.**

1. If $\rho \boxtimes S_a \subset \psi_d$ with $a > 1$, set
\[
\delta_{\rho,a,\psi} = \begin{cases} 
1 & \text{if } \rho \boxtimes S_a \boxtimes 1 \subset \psi, \\
-1 & \text{if } \rho \boxtimes 1 \boxtimes S_a \subset \psi.
\end{cases}
\]
When $a = 1$, we set $\delta_{\rho,a,\psi} = 1$.

2. Define $\mathcal{T}_{\rho,\psi,\varepsilon}$ to be the set containing 0 and all integers $a > 0$ satisfying the following conditions:

   (chain condition): $\rho \boxtimes S_k \subset \psi_d$ for any $k \leq a$ with $k \equiv a \bmod 2$;

   (alternating condition): $\varepsilon(\rho, k) = -\varepsilon(\rho, k + 2)$ for $0 < k < a$ with $k \equiv a \bmod 2$;

   (initial condition): if $a \equiv 0 \bmod 2$ so that $\rho \boxtimes S_2 \subset \psi_d$, then $\varepsilon(\rho, 2) = -1$.

3. Set
\[
\begin{align*}
    b_{\rho,\psi,\varepsilon} &= \max \mathcal{T}_{\rho,\psi,\varepsilon}, \\
    a_{\rho,\psi,\varepsilon} &= \min\{a > b_{\rho,\psi,\varepsilon} \mid \rho \boxtimes S_a \subset \psi_d\}, \\
    \delta_{\rho,\psi,\varepsilon} &= \delta_{\rho,a_{\rho,\psi,\varepsilon},\psi}.
\end{align*}
\]
Remark that if \( a, a' \in \mathcal{T}_{\rho, \psi} \) with \( a, a' \neq 0 \), then \( a \equiv a' \mod 2 \). When \( \{ a > b_{\rho, \psi} : \rho \boxtimes S_a \subset \psi_d \} = \emptyset \), we understand \( a_{\rho, \psi} = \infty \).

Now we set \( \Phi_2(G_n) = \Psi_g(G_n) \cap \Phi_{\text{temp}}(G_n) \). For \( \phi \in \Phi_2(G_n) \) and \( \varepsilon \in \hat{A}_\phi \) with \( \varepsilon(z_\phi) = 1 \), we denote by \( \pi(\phi, \varepsilon) \) the unique element in \( \Pi_\phi \) corresponding to \( \varepsilon \) via the bijection \( \Pi_\phi \rightarrow (A_\phi / \langle z_\phi \rangle)^\vee \). It is known that \( \pi(\phi, \varepsilon) \) is a discrete series representation of \( G_n(F) \). The following theorem is a criterion when \( \pi(\phi, \varepsilon) \) is supercuspidal.

**Theorem 3.2** (Mœglin, Xu [X17a]). Let \( \phi \in \Phi_2(G_n) \) and \( \varepsilon \in \hat{A}_\phi \) with \( \varepsilon(z_\phi) = 1 \). Then \( \pi(\phi, \varepsilon) \in \Pi_\phi \) is supercuspidal if and only if \( a_{\rho, \phi, \varepsilon} = \infty \) for any \( \rho \subset \phi|W_F \).

This is one of main results in [X17a]. In the rest of this subsection, we give a sketch of the proof of this theorem.

First, we explain the following proposition.

**Proposition 3.3** ([X17a, Proposition 3.1]). Suppose that \( \pi(\phi, \varepsilon) \in \Pi_\phi \) is supercuspidal. Fix an irreducible bounded representation \( \rho \) of \( W_F \). If \( \phi \supset \rho \boxtimes S_k \) with \( k > 2 \), then \( \phi \supset \rho \boxtimes S_{k-2} \).

**Proof.** Recall that \( \rho \) is identified with the corresponding irreducible supercuspidal unitary representation of \( \text{GL}_d(F) \). For \( s \in \mathbb{C} \), we consider the usual unnormalized intertwining operators

\[
J_{\mathcal{P}|\mathcal{P}}(s) : \text{Ind}_{\mathcal{P}}^{G_{n+d}(F)}(\rho \cdot |^s \boxtimes \pi) \rightarrow \text{Ind}_{\mathcal{P}}^{G_{n+d}(F)}(\rho \cdot |^s \boxtimes \pi),
\]

\[
J_{\mathcal{P}|\mathcal{P}}(s) : \text{Ind}_{\mathcal{P}}^{G_{n+d}(F)}(\rho \cdot |^s \boxtimes \pi) \rightarrow \text{Ind}_{\mathcal{P}}^{G_{n+d}(F)}(\rho \cdot |^s \boxtimes \pi),
\]

where \( \mathcal{P} = MN \) is the standard parabolic subgroup of \( G_{n+d}(F) \) with Levi part \( M \cong \text{GL}_d(F) \times G_n(F) \), and \( \mathcal{P} = MN \) is the opposite parabolic subgroup to \( \mathcal{P} \). These operators are defined by the meromorphic continuations of Jacquet integrals which converge when \( s \) belongs to a certain open set in \( \mathbb{C} \). Since \( \rho \boxtimes \pi \) is supercuspidal, it is known as a theorem of Harish-Chandra that both \( J_{\mathcal{P}|\mathcal{P}}(s) \) and \( J_{\mathcal{P}|\mathcal{P}}(s) \) are holomorphic for \( \text{Re}(s) \neq 0 \).

On the other hand, there is a rational function \( \mu(s) \) such that

\[
J_{\mathcal{P}|\mathcal{P}}(s) \circ J_{\mathcal{P}|\mathcal{P}}(s) = \mu(s)^{-1}.
\]

The function \( \mu(s) \) is called the **Plancherel measure**. By results of Arthur [Ar13] and Shahidi [S90], it is known that the Plancherel measure \( \mu(s) \) is given by the product of gamma factors:

\[
\mu(s) = \gamma(s, \rho \otimes \bar{\phi}, \psi_F) \gamma(-s, \bar{\rho} \otimes \phi, \psi_F^{-1}) \gamma(2s, R \circ \rho, \psi_F) \gamma(-2s, R \circ \bar{\rho}, \psi_F^{-1})
\]

up to a positive constant. Here, \( \psi_F \) is a fixed non-trivial additive character of \( F \), and

\[
R = \begin{cases} 
\text{Sym}^2 & \text{if } G_n = \text{SO}_{2n+1}, \\
\wedge^2 & \text{if } G_n = \text{Sp}_{2n}.
\end{cases}
\]

The ambiguity comes from the choices of Haar measures on \( N \) and \( \overline{N} \) to define Jacquet integrals. In fact, one can choose these Haar measures using \( \psi_F \) so that the above equation actually holds.

Suppose now that \( \phi \supset \rho \boxtimes S_k \) with \( k > 2 \). Then in \( s \in \mathbb{R} \) with \( s > 1/2 \),

- \( \gamma(2s, R \circ \rho, \psi_F) \) and \( \gamma(-2s, R \circ \bar{\rho}, \psi_F^{-1}) \) are holomorphic and nonzero;
- \( \gamma(s, \rho \otimes \bar{\phi}, \psi_F) \) is nonzero;
- \( \gamma(-s, \bar{\rho} \otimes \phi, \psi_F^{-1}) \) is holomorphic and has a zero at \( s = (k - 1)/2 \).
Since $\mu(s)$ is nonzero at $s = (k - 1)/2$, the gamma factor $\gamma(s, \rho \otimes \psi_F)$ must have a pole at $s = (k - 1)/2$. This occurs only when $\phi \ni \rho \boxtimes S_{k-2}$.  

Now fix an irreducible bounded representation $\rho$ of $W_F$, and a real number $x$. If $\phi \ni \rho \boxtimes S_{2x+1}$ and $x > 0$, define $\phi_- \in \Phi_2(G_{n-d})$ with $d = \dim(\rho)$ by

$$\phi_- = \phi - \rho \boxtimes S_{2x+1} \oplus \rho \boxtimes S_{2x-1}.$$ 

The following lemma follows from a compatibility of twisted endoscopic character identities and Jacquet modules.

**Lemma 3.4** ([X17a, Lemma 7.2]). We have

$$\text{Jac}_{\rho,|t|} \left( \bigoplus_{\pi \in \Pi_{\phi}} \pi \right) = \begin{cases} \bigoplus_{\pi_- \in \Pi_{\phi_-}} \pi_- & \text{if } \phi \ni \rho \boxtimes S_{2x+1} \text{ and } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, a compatibility of standard endoscopic character identities and Jacquet modules gives the following more precise result:

**Lemma 3.5** ([X17a, Lemma 7.3]). Suppose that $\phi \in \Phi_2(G_n)$ satisfies that $\phi \ni \rho \boxtimes S_{2x+1}$. Set $\phi_- = \phi - \rho \boxtimes S_{2x+1} \oplus \rho \boxtimes S_{2x-1}$.

1. If $x > 1/2$ and $\rho \boxtimes S_{2x-1} \not\subset \phi$, then $\pi(\phi_-, \varepsilon) = \text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon)$ for any $\varepsilon \in \widehat{A}_{\phi_-} \cong \widehat{A}_\phi$.
2. If $x > 1/2$ and $\rho \boxtimes S_{2x-1} \subset \phi$, then $A_\phi = A_{\phi_-} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{\rho,2x+1}$, where $\alpha_{\rho,a}$ is the element in $A_\phi$ corresponding to $\rho \boxtimes S_a$. Define a surjection $A_\phi \to A_{\phi_-}$ by $\alpha_{\rho,2x+1} \mapsto \alpha_{\rho,2x-1}$ and by identity on $A_{\phi_-}$. This gives an injection $\widehat{A}_{\phi_-} \hookrightarrow \widehat{A}_\phi$.

Then $\text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{A}_{\phi_-}$, i.e.,

$$\varepsilon(\rho, 2x+1)\varepsilon(\rho, 2x-1) = 1.$$ 

In this case, $\pi(\phi_-, \varepsilon) = \text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon)$.

3. If $x = 1/2$, then $A_\phi = A_{\phi_-} \oplus (\mathbb{Z}/2\mathbb{Z})\alpha_{\rho,2}$. Let $A_\phi \to A_{\phi_-}$ be the projection, and $A_{\phi_-} \hookrightarrow \widehat{A}_\phi$ be the induced injection. Then $\text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon) = 0$ unless $\varepsilon \in \widehat{A}_{\phi_-}$, i.e.,

$$\varepsilon(\rho, 2) = 1.$$ 

In this case, $\pi(\phi_-, \varepsilon) = \text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon)$.

Let $\phi \in \Phi_2(G_n)$ and $\varepsilon \in \widehat{A}_\phi$ such that $\varepsilon(z_\phi) = 1$. By Lemmas 3.4 and 3.5, $a_{\rho,\phi,\varepsilon} = \infty$ for any $\rho \subset \phi|W_F$ if and only if $\text{Jac}_{\rho,|t|} \pi(\phi, \varepsilon) = 0$ for any $\rho$ and $x \in \mathbb{R}$. This condition is equivalent that $\pi(\phi, \varepsilon)$ is supercuspidal. Hence we obtain Theorem 3.2.

3.3. The case of $\text{SO}_{2n+1}$ and $\text{Sp}_{2n}$. Now let $\psi \in \Psi_{el}(G_n)$. We construct the elements in the $A$-packet $\Pi_\psi$ by induction on $n$. The following is the main theorem in the elementary case.

**Theorem 3.6** (Mœglin). For $\psi \in \Psi_{el}(G_n)$ and $\varepsilon \in \widehat{A}_\psi$ with $\varepsilon(z_\psi) = 1$, we can construct a representation $\pi(\psi, \varepsilon)$ of $G_n(F)$ as the following manner. Then $\pi(\psi, \varepsilon)$ is irreducible, $\pi(\psi, \varepsilon) \not\cong \pi(\psi, \varepsilon')$ if $\varepsilon \neq \varepsilon'$, and $\pi(\psi, \varepsilon)$ satisfies the following basic properties:

- **(Jacquet module):** If $\text{Jac}_{\rho,|t|}(\pi) \neq 0$ for some $x \in \mathbb{R}$, then there exists $a > b_{\rho,\psi,\varepsilon}$ with $\rho \boxtimes S_a \subset \psi_d$ such that

$$x = \delta_{\rho,a,\psi} \frac{a - 1}{2}.$$
(Non-unitary irreducibility): For \( x \in (1/2) \mathbb{Z} \) with \( x \geq 1/2 \), if \( x \neq 1/2 \) and \( \rho \boxtimes S_{2x-1} \not\subset \psi_d \), or if \( 0 \leq 2x - 1 < b_{\rho,\psi,\varepsilon} \), then \( \rho | \psi \times \pi(\psi, \varepsilon) \) is irreducible.

(Backwards reducibility): Suppose that there exists an odd integer \( a \) such that \( a \not\subset \psi_d \). Then \( \rho \times \pi(\psi, \varepsilon) \) is irreducible if \( \rho \subset \psi_d \), and is semisimple of length 2 without multiplicities otherwise. Moreover, if \( \sigma \) is an irreducible subrepresentation of \( \rho \times \pi(\psi, \varepsilon) \) in the both cases, then \( \rho \times \sigma \times \rho \times \sigma \) is irreducible.

Construction: Let \( \psi \in \Psi_{el}(G_n) \) and \( \varepsilon \in \hat{A}_\psi \) with \( \varepsilon(z_\psi) = 1 \).

1. If \( a_{\rho,\psi,\varepsilon} = \infty \) for all \( \rho \), then we define \( \pi(\psi, \varepsilon) = \pi(\psi_d, \varepsilon) \), which is supercuspidal by Theorem 3.2.

2. If \( a_{\rho,\psi,\varepsilon} > b_{\rho,\psi,\varepsilon} + 2 \) or \( b_{\rho,\psi,\varepsilon} = 0 \), we define \( \pi(\psi, \varepsilon) \) to be the unique irreducible subrepresentation of

\[
\rho | \delta \psi \times \pi(\psi', \varepsilon'),
\]

where \( a = a_{\rho,\psi,\varepsilon} \), \( \delta = \delta_{\rho,\psi,\varepsilon} \), and \( (\psi', \varepsilon') \) is given so that

\[
\psi' = \psi_d - \rho \boxtimes S_a \quad \text{and} \quad \delta_{\rho,\psi,\varepsilon} = \delta_{\rho,\alpha,\psi},
\]

and \( c'(\rho, a - 2) = c(\rho, a) \).

3. If \( a_{\rho,\psi,\varepsilon} = b_{\rho,\psi,\varepsilon} + 2 \), we divide into three cases.

(a) If \( b_{\rho,\psi,\varepsilon} \) is even and nonzero, then we define \( \pi(\psi, \varepsilon) \) to be the unique irreducible subrepresentation of

\[
\left\langle \rho; \delta \frac{a-1}{2}, \ldots, \delta \frac{1}{2} \right\rangle \times \pi(\psi_- \varepsilon_-),
\]

where \( a = a_{\rho,\psi,\varepsilon} \), \( \delta = \delta_{\rho,\psi,\varepsilon} \), and \( (\psi_-, \varepsilon_-) \) is given so that

\[
(\psi_-, \delta) = \psi_d - \rho \boxtimes S_a, \quad \delta_{\rho,\alpha,\psi} = \begin{cases} -\delta & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \delta_{\rho,\alpha,\psi} & \text{otherwise}, \end{cases}
\]

and

\[
\varepsilon_-(\rho, \alpha) = \begin{cases} -\varepsilon(\rho, \alpha) & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \varepsilon(\rho, \alpha) & \text{otherwise}. \end{cases}
\]

(b) If \( b_{\rho,\psi,\varepsilon} \) is odd and \( b_{\rho,\psi,\varepsilon} \neq 1 \), then we define \( \pi(\psi, \varepsilon) \) to be the unique common irreducible subrepresentation of

\[
\left\langle \rho; \delta \frac{a-1}{2}, \ldots, 0 \right\rangle \times \pi(\psi_- \varepsilon_-)
\]

and

\[
\left\langle \rho; \delta \frac{a-1}{2}, \ldots, -\delta \frac{b-1}{2} \right\rangle \times \pi(\psi' \varepsilon'),
\]

where \( a = a_{\rho,\psi,\varepsilon} \), \( b = b_{\rho,\psi,\varepsilon} \), \( \delta = \delta_{\rho,\psi,\varepsilon} \), and \( (\psi_-, \varepsilon_-), (\psi', \varepsilon') \) are given so that

\[
(\psi')_d = \psi_d - \rho \boxtimes S_a - \rho \boxtimes S_b, \quad \delta_{\rho,\alpha,\psi'} = \delta_{\rho,\alpha,\psi},
\]

\[
(\psi_-)_d = \psi_d - \rho \boxtimes S_a - \rho, \quad \delta_{\rho,\alpha,\psi} = \begin{cases} -\delta & \text{if } \alpha \leq b_{\rho,\psi,\varepsilon}, \\ \delta_{\rho,\alpha,\psi} & \text{otherwise}. \end{cases}
\]
and \( \varepsilon' = \varepsilon|A_{\psi} \), and

\[
\varepsilon_-(\rho, \alpha) = \begin{cases} -\varepsilon(\rho, \alpha) & \text{if } \alpha \leq b_{\rho, \psi, \varepsilon}, \\ \varepsilon(\rho, \alpha) & \text{otherwise}. \end{cases}
\]

(c) If \( a_{\rho, \psi, \varepsilon} = 3 \) and \( b_{\rho, \psi, \varepsilon} = 1 \), we have \((\psi_-, \varepsilon_-) = (\psi', \varepsilon')\) in the notation (b). By (Unitary reducibility), \( \sigma = \rho \rtimes \pi(\psi', \varepsilon') \) is semisimple of length 2, and hence we can write \( \sigma = \pi_+ \oplus \pi_- \) according to the following two cases.

(i) When \( \psi_d \not\supset \rho \boxtimes S_\alpha \) for any \( \alpha > 3 \), we fix arbitrary parametrization in \( \sigma \), and we define \( \pi(\psi, \varepsilon) \) to be the unique irreducible subrepresentation of

\[
\rho | : [\delta_{\rho, 3, \psi} \rtimes \pi_\zeta
\]

with \( \zeta = \varepsilon(\rho, 3)\delta_{\rho, 3, \psi} \).

(ii) When \( \psi_d \supset \rho \boxtimes S_\alpha \) for some \( \alpha > 3 \), we can specify the parametrization in \( \sigma \) as follows. Set \((\psi''_d, \varepsilon'')\) so that

\[
(\psi''_d) = (\psi') - \rho \boxtimes S_{a'}. \quad \delta_{\rho, 1, \psi''} = \delta' = \delta_{\rho, a', \psi}
\]

with \( a' = a_{\rho, \psi', \varepsilon'} \), and \( \varepsilon''(\rho, 1) = \varepsilon'(\rho, a') \). Put

\[
\Pi = \rho \times \left\langle \rho, \frac{a' - 1}{2}, \ldots, \delta' \right\rangle \rtimes \pi(\psi'', \varepsilon''),
\]

\[
\sigma_q = \left\langle \rho, \frac{a' - 1}{2}, \ldots, 0 \right\rangle \rtimes \pi(\psi'', \varepsilon''),
\]

\[
\sigma_s = \left\langle \rho \times \left\langle \rho, \frac{a' - 1}{2}, \ldots, \delta' \right\rangle \right\rangle \rtimes \pi(\psi'', \varepsilon''),
\]

where \( \langle \rho \times \langle \rho, \delta'(a' - 1)/2, \ldots, \delta' \rangle \rangle \) is the unique irreducible subrepresentation of \( \rho \times \langle \rho, \delta'(a' - 1)/2, \ldots, \delta' \rangle \). Hence there is an exact sequence

\[
0 \longrightarrow \sigma_s \longrightarrow \Pi \longrightarrow \sigma_q \longrightarrow 0.
\]

We set \( \pi_+ = \sigma \cap (s.s.\sigma_q) \) and \( \pi_- = \sigma \cap (s.s.\sigma_s) \). Then we define \( \pi(\psi, \varepsilon) \) to be the unique irreducible subrepresentation of

\[
\rho | : [\delta_{\rho, 3, \psi} \rtimes \pi_\zeta
\]

with \( \zeta = \varepsilon(\rho, a_{\rho, \psi', \varepsilon'})\delta_{\rho, \psi', \varepsilon'}\delta_{\rho, 3, \psi} \).

**Remark 3.7.** (1) The representation \( \pi(\psi, \varepsilon) \) can be constructed by using the **generalized Aubert involution**. This is defined by a combination of induction functors and Jacquet functors (see [X17b, §6.2]). By proving a compatibility of the generalized Aubert involution and twisted endoscopic character identities, Xu [X17b, §6] showed that

\[
\Pi_\psi = \left\{ \pi(\psi, \varepsilon) \mid \varepsilon \in \hat{A}_\psi, \varepsilon(z_\psi) = 1 \right\}.
\]
Example 3.8. Suppose that \( \psi \in \Psi_\text{el}(G_n) \) such that \( \hat{\psi} = S_4 \oplus S_2 \). Fix \( \varepsilon \in \widehat{A}_\psi \) with \( \varepsilon(z_\psi) = 1 \). For \( \rho = 1_{GL_1(F)} \), set \( \delta_\rho = \delta_\rho, \lambda, \psi \) and \( \varepsilon_\rho = \varepsilon(\rho, a) \). Then \( \pi(\psi, \varepsilon) \) is given as follows.

1. If \( (\delta_4, \delta_2) = (+, +) \) and \( (\varepsilon_4, \varepsilon_2) = (+, +) \), then \( \pi_1 = \pi(\psi, \varepsilon) \) is discrete series but not supercuspidal. It is a subrepresentation of
   \[
   | \cdot |^{1/2} \times \pi(S_4) \hookrightarrow | \cdot |^{1/2} \times | \cdot |^{1/2} \times | \cdot |^{1/2} \times 1_{SO_4(F)},
   \]
   where \( \pi(S_4) \in \Pi_{S_4} \) is a discrete series representation of \( SO_5(F) \).

2. If \( (\delta_4, \delta_2) = (+, +) \) and \( (\varepsilon_4, \varepsilon_2) = (+, +) \), then \( \pi_2 = \pi(\psi, \varepsilon) \) is discrete series but not supercuspidal. It is a subrepresentation of
   \[
   \left\langle \begin{array}{cc}
   3 & 1 \\
   2 & 2
   \end{array} \right\rangle \times 1_{SO_3(F)} \hookrightarrow \left\langle \begin{array}{cc}
   3 & 1 \\
   2 & 2
   \end{array} \right\rangle \times \cdot \cdot |^{-1/2} \times 1_{SO_1(F)}.
   \]

3. If \( (\delta_4, \delta_2) = (+, +) \) and \( (\varepsilon_4, \varepsilon_2) = (+, +) \), then \( \pi_3 = \pi(\psi, \varepsilon) \) is non-tempered and is the unique subrepresentation of
   \[
   | \cdot |^{1/2} \times \pi(S_4) \hookrightarrow | \cdot |^{1/2} \times | \cdot |^{1/2} \times | \cdot |^{1/2} \times 1_{SO_1(F)},
   \]
   where \( \pi(S_4) \in \Pi_{S_4} \) is a discrete series representation of \( SO_5(F) \). Hence \( \pi_3 \in \Pi_{\phi_\psi} \) with \( \phi_\psi = | \cdot |^{1/2} \oplus S_4 \oplus | \cdot |^{-1/2} \).

4. If \( (\delta_4, \delta_2) = (+, +) \) and \( (\varepsilon_4, \varepsilon_2) = (+, +) \), then by definition, \( \pi_4 \cong \pi_2 \) so that \( \pi_4 \in \Pi_{\psi_d} \) is discrete series.

5. If \( (\delta_4, \delta_2) = (+, +) \) and \( (\varepsilon_4, \varepsilon_2) = (+, +) \), then we claim that \( \pi_5 = \pi(\psi, \varepsilon) \) is non-tempered and is the unique subrepresentation of
   \[
   | \cdot |^{-1/2} \times \sigma,
   \]
   where \( \sigma \) is tempered and is the direct summand of \( St_2 \times 1_{SO_1(F)} \) such that \( J_{1/2}(\sigma) \) does not contain \( \pi(S_2) \), where \( \pi(S_2) \in \Pi_{S_2} \) is a discrete series representation of \( SO_3(F) \). Hence the \( L \)-parameter of \( \pi_5 \) is
   \[
   | \cdot |^{-3/2} + S_2 + S_2 + | \cdot |^{-3/2}.
   \]
   Indeed, by the construction,
   \[
   \pi_5 \hookrightarrow \cdot |^{3/2} \times | \cdot |^{3/2} \times 1_{SO_3(F)} \cong | \cdot |^{-3/2} \times | \cdot |^{3/2} \times 1_{SO_3(F)}.
   \]
Consider the following diagram of two exact sequences

\[
\begin{array}{ccccccc}
0 & \\
& 1_{\text{GL}_2(F)} \times 1_{\text{SO}_1(F)} & \\
& \\
& 0 & \\
& \\
& \\
& 0 & \\
& \\
& \\
& \\
0 & \\
& |\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_3(F)} & \\
& \\
& \\
& \\
& |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}} \times 1_{\text{SO}_1(F)} & \\
& \\
& \\
& \\
& |\cdot|^{\frac{1}{2}} \times \pi(S_2) & \\
& \\
& \\
& \\
& \text{St}_2 \times 1_{\text{SO}_1(F)} & \\
& \\
& \\
& \\
& 0 & \\
\end{array}
\]

Since \( \pi_5 \hookrightarrow |\cdot|^{\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}} \times 1_{\text{SO}_3(F)} \), we have \( \text{Jac}_{|\cdot|^{\frac{1}{2}}}(\pi_5) \neq 0 \). This implies that there is no nonzero homomorphism \( \pi_5 \to |\cdot|^{-\frac{1}{2}} \times 1_{\text{GL}_2(F)} \times 1_{\text{SO}_1(F)} \) since \( \text{Jac}_{|\cdot|^{\frac{1}{2}}}(1_{\text{GL}_2(F)} \times 1_{\text{SO}_1(F)}) = 0 \). Hence there is a representation \( \sigma \) of \( \text{SO}_5(F) \) which is a common subrepresentation of \( |\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_3(F)} \) and \( \text{St}_2 \times 1_{\text{SO}_1(F)} \) such that \( \pi_5 \hookrightarrow |\cdot|^{-\frac{1}{2}} \times \sigma \). Computing Jacquet modules, we see that

\[
\begin{align*}
\text{s.s.}(|\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_3(F)}) \cap (\text{St}_2 \times 1_{\text{SO}_1(F)}) & \neq 0, \\
\text{s.s.}(|\cdot|^{\frac{1}{2}} \times \pi(S_2)) \cap (\text{St}_2 \times 1_{\text{SO}_1(F)}) & \neq 0.
\end{align*}
\]

Since \( \text{St}_2 \times 1_{\text{SO}_1(F)} \) is a direct sum of two irreducible tempered representations, \( \text{s.s.}(|\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_3(F)}) \cap (\text{St}_2 \times 1_{\text{SO}_1(F)}) \) is irreducible. Hence \( \sigma \) must be irreducible. Moreover, since \( \sigma \hookrightarrow |\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_3(F)} \), we see that \( \text{Jac}_{|\cdot|^{\frac{1}{2}}} (\sigma) \) does not contain \( \pi(S_2) \).

(6) If \((\delta_4, \delta_2) = (-, +)\) and \((\varepsilon_4, \varepsilon_2) = (-, -)\), then \( \pi_6 = \pi(\psi, \varepsilon) \) is non-tempered and is the unique subrepresentation of

\[
|\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}} \times \pi(S_2) \hookrightarrow |\cdot|^{-\frac{1}{2}} \times |\cdot|^{-\frac{1}{2}} \times |\cdot|^{\frac{1}{2}} \times 1_{\text{SO}_1(F)},
\]

where \( \pi(S_2) \in \Pi_{S_2} \) is a discrete series representation of \( \text{SO}_3(F) \). Hence \( \pi_6 \in \Pi_{\phi_\psi} \) with \( \phi_\psi = |\cdot|^\frac{3}{2} \oplus |\cdot|^\frac{1}{2} \oplus S_2 \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}} \).

(7) If \((\delta_4, \delta_2) = (-, -)\) and \((\varepsilon_4, \varepsilon_2) = (+, +)\), then \( \pi_7 = \pi(\psi, \varepsilon) \) is non-tempered and is the unique subrepresentation of

\[
|\cdot|^{-\frac{1}{2}} \times 1_{\text{SO}_5(F)} \hookrightarrow \left( -\frac{3}{2}, -\frac{1}{2} \right) \times |\cdot|^{-\frac{1}{2}} \times 1_{\text{SO}_1(F)}.
\]

Hence \( \pi_7 \in \Pi_{\phi_\psi} \) with \( \phi_\psi = |\cdot|^\frac{3}{2} \oplus |\cdot|^\frac{1}{2} \oplus |\cdot|^\frac{1}{2} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}} \).

(8) If \((\delta_4, \delta_2) = (-, -)\) and \((\varepsilon_4, \varepsilon_2) = (−, −)\), then by definition, \( \pi_8 \cong \pi_6 \) so that \( \pi_8 \) is non-tempered but \( \pi_8 \notin \Pi_{\phi_\psi} \) with \( \phi_\psi = |\cdot|^\frac{3}{2} \oplus |\cdot|^\frac{1}{2} \oplus |\cdot|^\frac{1}{2} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}} \oplus |\cdot|^{-\frac{3}{2}} \).
4. The case of discrete diagonal restriction

In this section, we construct $A$-packets $\Pi_\psi$ for $A$-parameters $\psi \in \Psi_{\text{DDR}}(G_n)$ with discrete diagonal restrictions. See [X17b, §7] for more precision. Before the construction, we review the case of general linear groups $GL_N$.

4.1. The case of $GL_N$. Let $\psi$ be an $A$-parameter for $GL_N(F)$. Assume in this subsection that the diagonal restriction $\psi_d = \psi \circ \Delta$ is multiplicity-free. Recall that one can associate an irreducible unitary representation $\tau_\psi$ of $GL_N(F)$. When $\psi \supset \rho \boxtimes S_a \boxtimes S_b$ with $\min\{a, b\} > 1$, we see that $\tau_\psi$ is the unique irreducible subrepresentation of

$$\left\langle \rho; \frac{a-b}{2}, \ldots, -\frac{a+b-2}{2} \right\rangle \times \tau_\psi \times \left\langle \rho; \frac{a-b}{2}, \ldots, -\frac{a+b-2}{2} \right\rangle,$$

where $\zeta \in \{\pm 1\}$ such that $\zeta(a-b) \geq 0$, and

$$\psi' = \begin{cases} \psi - \rho \boxtimes S_a \boxtimes S_b \oplus \rho \boxtimes S_a \boxtimes S_{b-2} & \text{if } \zeta = +1, \\ \psi - \rho \boxtimes S_a \boxtimes S_b \oplus \rho \boxtimes S_{a-2} \boxtimes S_b & \text{if } \zeta = -1. \end{cases}$$

4.2. The case of $SO_{2n+1}$ and $Sp_{2n}$. Let $G_n$ be split $SO_{2n+1}$ or $Sp_{2n}$. Recall that $\psi \in \Psi(G_n)$ has a discrete diagonal restriction if $\psi$ is of good parity and $\psi_d = \psi \circ \Delta$ is multiplicity-free. Note that

$$S_a \otimes S_b = S_{a+b-1} \oplus S_{a+b-3} \oplus \cdots \oplus S_{|a-b|+1}.$$

Write

$$\psi = \bigoplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}.$$  

For $i = 1, \ldots, r$, we set

$$d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i. \end{cases}$$

When $a_i = b_i$, we choose $\zeta_i \in \{\pm 1\}$ arbitrarily.

**Definition 4.1.** Let $\psi = \oplus_{i=1}^r \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\text{DDR}}(G_n)$ be an $A$-parameter with DDR, and $d_i$ and $\zeta_i$ be as above.

1. Define $\Sigma_\psi$ to be the set of pairs $(\underline{l}, \underline{\eta})$ such that
   - $\underline{l} = (l_1, \ldots, l_r) \in \mathbb{Z}^r$ such that $0 \leq l_i \leq d_i/2$;
   - $\underline{\eta} = (\eta_1, \ldots, \eta_r) \in \{\pm 1\}^r$ such that
     $$\prod_{i=1}^r \eta_i^{d_i} (-1)^{[d_i/2]+l_i} = 1.$$

   Here, $[x]$ denotes the greatest integer which is not larger than $x$.

2. Define an equivalence relation $\sim_\psi$ on $\Sigma_\psi$ by

   $$(\underline{l}, \underline{\eta}) \sim_\psi (\underline{l}', \underline{\eta}') \iff \underline{l} = \underline{l}' \quad \text{and} \quad \eta_i = \eta_i' \text{ unless } l_i = d_i/2.$$  

3. Define $\varepsilon_{\underline{l}, \underline{\eta}} \in \hat{A}_\psi$ by

   $$\varepsilon_{\underline{l}, \underline{\eta}}(\alpha_i) = \eta_i^{d_i} (-1)^{[d_i/2]+l_i},$$

   where $\alpha_i \in A_\psi$ is the element corresponding to $\rho_i \boxtimes S_{a_i} \boxtimes S_{b_i}$.  

Lemma 4.2. The cardinality of the set \( \Sigma_\psi / \sim \psi \) of equivalence classes is equal to

\[
\begin{cases}
\frac{1}{2} \prod_{i=1}^{r} (d_i + 1) & \text{if } d_i \text{ is odd for some } i, \\
\frac{1}{2} \left( \prod_{i=1}^{r} (d_i + 1) + (-1)^{\frac{1}{2}(d_1+\cdots+d_r)} \right) & \text{if } d_i \text{ is even for any } i.
\end{cases}
\]

Proof. For \( \epsilon \in \{ \pm 1 \} \), we define \( \Sigma_\psi^\epsilon \) by a similar set to \( \Sigma_\psi \) changing the second condition with

\[\prod_{i=1}^{r} \eta_i^{d_i} (-1)^{[d_i/2]+l_i} = \epsilon,\]

and set

\[a^\epsilon_{(d_1, \ldots, d_r)} = \# \Sigma_\psi^\epsilon / \sim \psi.\]

Hence \( a^+_{(d_1, \ldots, d_r)} = \# \Sigma_\psi / \sim \psi. \)

If \( d_i \) is odd for some \( i \), then \( a^+_{(d_1, \ldots, d_r)} = a^-_{(d_1, \ldots, d_r)} \) so that

\[a^\epsilon_{(d_1, \ldots, d_r)} = \frac{1}{2} \left( a^+_{(d_1, \ldots, d_r)} + a^-_{(d_1, \ldots, d_r)} \right) = \frac{1}{2} \prod_{i=1}^{r} (d_i + 1).\]

Suppose that \( d_i \) is even for any \( i \). Set \( a^+ = 1 \) and \( a^- = 0 \). Then for \( r \geq 1 \), we have

\[
\left( \begin{array}{c}
\frac{d_r + 1}{2} \\
\frac{d_r + 1}{2}
\end{array} \right)
\times \left( \begin{array}{c}
\frac{d_r + 1}{2} \\
\frac{d_r + 1}{2}
\end{array} \right)
= \left( \begin{array}{c}
\frac{1}{2} \left( d_r + 1 \right) \\
\frac{1}{2} \left( d_r + 1 \right)
\end{array} \right)
\times \left( \begin{array}{c}
\frac{1}{2} \left( d_r + 1 \right) \\
\frac{1}{2} \left( d_r + 1 \right)
\end{array} \right).
\]

Thus,

\[
P \left( \begin{array}{c}
\alpha \\
\beta
\end{array} \right) P^{-1} = \left( \begin{array}{c}
\alpha + \beta \\
0
\end{array} \right).
\]

Therefore,

\[
\left( \begin{array}{c}
a^+_{(d_1, \ldots, d_r)} \\
-1
\end{array} \right)
\times \left( \begin{array}{c}
-1
\end{array} \right)
= \frac{1}{2} \left( \prod_{i=1}^{r} (d_i + 1) + (-1)^{\frac{1}{2}(d_1+\cdots+d_r)} \right) \left( \begin{array}{c}
1 \\
0
\end{array} \right).
\]

We obtain the lemma. \( \square \)

Now for \( (\ell, \eta) \in \Sigma_\psi \), we define \( \pi(\psi, \ell, \eta) \) to be the unique irreducible subrepresentation of

\[
\prod_{i=1}^{r} \left( | \cdot |^{-\frac{1}{2} \zeta_i} d_i^{-l_i} \right) \text{Sp} \left( \text{St} \left( \rho_i, \frac{a_i + l_i}{2} + \frac{\zeta_i}{2}, \frac{b_i + l_i}{2} - \frac{\zeta_i}{2} \right) \right) \times \pi(\psi', \zeta'),
\]

where
• \( \psi' \) is an elementary parameter given so that
\[
(\psi')_d = \bigoplus_{i=1}^{r} \bigoplus_{c_i} \rho_i \boxtimes S_{c_i}
\]
with \( c_i \) running over all integers such that
\[
|a_i - b_i| + 2l_i + 1 \leq c_i \leq a_i + b_i - 2l_i - 1 \quad \text{and} \quad c_i \equiv a_i + b_i - 1 \mod 2,
\]
and \( \delta_{\rho_i,c_i,\psi'} = \zeta_i; \)
• \( \varepsilon' \in A_{\psi'} \) is given so that
\[
\varepsilon'(\rho_i, c_i) = \eta_i \cdot (-1)^{\frac{c_i - 1}{2} - \frac{|a_i - b_i| - l_i}.}
\]
We recall that \( \text{Sp}(\text{St}(\rho,a),b) \) denotes the irreducible unitary representation \( \tau_{\rho,a,b} \). Also, we remark that when \( \psi \) is elementary, any \((\ell, \eta) \in \Sigma_\psi \) satisfies that \( l_i = 0 \), and \( \pi(\psi, \ell, \eta) = \pi(\psi, \ell_2^0) \) with \( \varepsilon_{L_2}^{\ell_2}(\alpha_i) = \eta_i \).

**Theorem 4.3.** Let \( \psi \in \Psi_{\text{DDR}}(G_n) \) be an \( A \)-parameter having a DDR. Then for \((\ell, \eta), (\ell', \eta') \in \Sigma_\psi \),
\[
\pi(\psi, \ell, \eta) \cong \pi(\psi, \ell', \eta') \iff (\ell, \eta) \sim_\psi (\ell', \eta').
\]
The \( A \)-packet \( \Pi_\psi \) is given by
\[
\Pi_\psi = \{ \pi(\psi, \ell, \eta) \mid (\ell, \eta) \in \Sigma_\psi / \sim_\psi \}.
\]
In particular, \#\( \Pi_\psi = \#(\Sigma_\psi / \sim_\psi) \), which is given explicitly in Lemma 4.2.

**Remark 4.4.** Let \( \psi \in \Psi_{\text{DDR}}(G_n) \). One can define a character \( \varepsilon_{M/MW}^{\psi} \in \tilde{A}_\psi \) such that
\[
\langle \cdot, \pi(\psi, \ell, \eta) \rangle_\psi = \varepsilon_{L_2}^{\ell} \cdot \varepsilon_{M/MW}^{\psi}.
\]

**Example 4.5.** (1) Consider \( \psi = S_3 \boxtimes S_2 \in \Psi_{\text{DDR}}(\text{SO}_7) \). Then \( \Sigma_\psi / \sim_\psi = \{(1, \pm 1)\} \) is a singleton. The representation \( \pi(\psi, (1, \pm 1)) \) is the unique irreducible subrepresentation of
\[
| \cdot |^{-\frac{1}{2}} (1, 0, -1) \rtimes 1_{\text{SO}_4(F)}.
\]
Hence it is non-tempered, and belongs to \( \Pi_{\phi_\psi} \) with \( \phi_\psi = | \cdot |^{\frac{1}{2}} S_3 \oplus | \cdot |^{-\frac{1}{2}} S_3 \).

(2) Consider \( \psi = S_2 \boxtimes S_3 \in \Psi_{\text{DDR}}(\text{SO}_7) \). Then \( \Sigma_\psi / \sim_\psi = \{(1, \pm 1)\} \) is a singleton. The representation \( \pi(\psi, (1, \pm 1)) \) is the unique irreducible subrepresentation of
\[
| \cdot |^{\frac{3}{2}} (-1, 0, 1) \rtimes 1_{\text{SO}_4(F)}.
\]
Since \( | \cdot |^{\frac{3}{2}} \rtimes 1_{\text{SO}_4(F)} \) is irreducible, we have
\[
\pi(\psi, (1, \pm 1)) \leftrightarrow | \cdot |^{\frac{1}{2}} (-1, 0, 1) \rtimes 1_{\text{SO}_4(F)}
\]
\[
\leftrightarrow | \cdot |^{-\frac{1}{2}} \times | \cdot |^{\frac{3}{2}} \times | \cdot |^{\frac{3}{2}} \rtimes 1_{\text{SO}_4(F)}
\]
\[
\cong | \cdot |^{-\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} \times | \cdot |^{-\frac{1}{2}} \rtimes 1_{\text{SO}_4(F)}
\]
\[
\cong | \cdot |^{-\frac{1}{2}} \times | \cdot |^{-\frac{1}{2}} \times | \cdot |^{\frac{1}{2}} \rtimes 1_{\text{SO}_4(F)}.
\]
Consider the following diagram of exact sequences:

\[
\begin{array}{c}
0 \xrightarrow{\langle -\frac{3}{2}, -\frac{1}{2} \rangle} \pi(S_2) \xrightarrow{\langle -\frac{3}{2}, -\frac{1}{2} \rangle} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times | \cdot |^{\frac{1}{2}} \times 1_{SO_1(F)} \xrightarrow{\langle -\frac{3}{2}, -\frac{1}{2} \rangle} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_3(F)} \xrightarrow{\langle -\frac{3}{2}, -\frac{1}{2} \rangle} 0 \\
0 \xrightarrow{| \cdot |^{\frac{1}{2}}} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times \pi(S_2) \xrightarrow{| \cdot |^{\frac{1}{2}}} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times | \cdot |^{\frac{1}{2}} \times 1_{SO_1(F)} \xrightarrow{| \cdot |^{\frac{1}{2}}} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_3(F)} \xrightarrow{| \cdot |^{\frac{1}{2}}} 0
\end{array}
\]

Since \( \pi(\psi, (1, \pm 1)) \rightarrow | \cdot |^{\frac{1}{2}} \times \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_1(F)} \), we see that \( \pi(\psi, (1, \pm 1)) \) is a subrepresentation of one of \( \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times \pi(S_2), \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_3(F)}, \langle -\frac{1}{2}, -\frac{3}{2} \rangle \times \pi(S_2) \), or \( \langle -\frac{1}{2}, -\frac{3}{2} \rangle \times 1_{SO_3(F)} \). Since Jac_{\langle -\frac{1}{2}, -\frac{3}{2} \rangle} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times \pi(S_2) \) and \( \pi(\psi, (1, \pm 1)) \) must be the unique irreducible subrepresentation, \( \pi(\psi, (1, \pm 1)) \) must be the unique irreducible subrepresentation of \( | \cdot |^{\frac{1}{2}} \times \sigma \). Since \( \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_1(F)} \) is reducible and

\[
\text{Jac}_{| \cdot |^{\frac{1}{2}}} \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_1(F)} = \langle -\frac{3}{2}, -\frac{1}{2} \rangle \times 1_{SO_1(F)}
\]

is irreducible, we see that Jac_{\langle -\frac{1}{2}, -\frac{3}{2} \rangle}(\sigma) = 0. Also, we note that Jac_{\langle -\frac{1}{2}, -\frac{3}{2} \rangle}(\pi(\psi, (1, \pm 1))) = 0. This contradicts that \( \pi(\psi, (1, \pm 1)) \rightarrow \langle -\frac{1}{2}, -\frac{3}{2} \rangle \times 1_{SO_1(F)} \). Therefore we conclude that

\[
\pi(\psi, (1, \pm 1)) \leftrightarrow \langle -\frac{1}{2}, -\frac{3}{2} \rangle \times \pi(S_2) = | \cdot |^{-1}St_2 \times \pi(S_2).
\]

Hence \( \pi(\psi, (1, \pm 1)) \) is non-tempered, and belongs to \( \Pi_{\phi_\psi} \) with \( \phi_\psi = | \cdot |^{-1}S_2 \oplus S_2 \oplus | \cdot |^{-1}S_2 \).

5. The case of good parity and the general case

In this section, we construct \( \Pi_\psi \) for general \( \Lambda \)-parameters \( \psi \). See [X17b, §8] for more precision. First, we consider the case where \( \psi \) is of good parity. Before the construction, we introduce the notion of admissible orders and explain their roles in the case of general linear groups \( GL_N \).
5.1. **Admissible order.** Recall that \( \psi \in \Psi(G_n) \) is of good parity if \( \psi \) is a sum of irreducible self-dual representations of the same type as \( \psi \). When \( \psi = \bigoplus_{\rho \in I} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \), we set \( I_\rho = \{ i \in I \mid \rho_i \equiv \rho \} \), and

\[
\psi_\rho = \bigoplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \subset \psi.
\]

As in the previous section, we set

\[
d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i, \end{cases}
\]

and we choose \( \zeta_i \in \{ \pm 1 \} \) arbitrarily when \( a_i = b_i \). We fix a total order \( >_\psi \) on \( I_\rho \) satisfying the following condition:

\((P)\): For \( i, j \in I_\rho \), if \( a_i + b_i > a_j + b_j \), \( |a_i - b_i| > |a_j - b_j| \), and \( \zeta_i = \zeta_j \), then \( i >_\psi j \).

We call such an order \( >_\psi \) an admissible order on \( I_\rho \). Taking \( >_\psi \) on \( I_\rho \) for each \( \rho \), we extend \( >_\psi \) to a partial order on \( I = \sqcup_\rho I_\rho \). Note that there are many such orders and there is no canonical choice of them in general. When \( \psi \) has a DDR, we can always choose such an order satisfying that for \( i, j \in I_\rho \),

\[
i >_\psi j \iff a_i + b_i > a_j + b_j.
\]

We call such orders the **natural orders** for parameters with DDR.

We say that \( \psi_\rho \in \Psi(G_{n'}) \) dominates \( \psi \) with respect to \( >_\psi \) if

\[
\psi_\rho = \bigoplus_{i \in I} \rho_i \boxtimes S_{a'_i} \boxtimes S_{b'_i}
\]

such that for each \( i \in I \),

\[
(a'_i, b'_i) = \begin{cases} (a_i + 2T_i, b_i) & \text{if } \zeta_i = +1, \\ (a_i, b_i + 2T_i) & \text{if } \zeta_i = -1, \end{cases}
\]

for some integer \( T_i \geq 0 \). When \( a'_i = b'_i \), we set \( \zeta'_i = \zeta_i \). Note that for any \( \psi \in \Psi_{gp}(G_n) \) with admissible order \( >_\psi \), one can take \( \psi_\rho \in \Psi_{DDR}(G_{n'}) \) such that \( \psi_\rho \) dominates \( \psi \) with respect to \( >_\psi \), and such that the order \( >_\psi \) is a natural order for \( \psi_\rho \).

5.2. **The case of \( GL_N \).** Let \( \psi \in \Psi_{gp}(G_n) \). Fix an admissible order \( >_\psi \), and an \( A \)-parameter \( \psi_\rho \in \Psi_{DDR}(G_{n'}) \) which dominates \( \psi \) with respect to \( >_\psi \). In this subsection, we construct \( \tau_\psi \) using \( \tau_{\psi_\rho} \).

Let \( \tau \) be an irreducible representation of \( GL_N(F) \). For \( d \leq N/2 \), we write

\[
s.s.\Jac_{(d,N-2d,d)}(\tau) = \bigoplus_{i \in I} \tau_i^{(1)} \boxtimes \tau_i^{(0)} \boxtimes \tau_i^{(2)},
\]

where \( \tau_i^{(j)} \) is an irreducible representation of \( GL_d(F) \) if \( j = 1, 2 \), and of \( GL_{N-2d}(F) \) if \( j = 0 \). For a fixed irreducible supercuspidal unitary representation \( \rho \) of \( GL_d(F) \) and for a real number \( x \), we set

\[
\Jac_{|\cdot|^{-x}}^{\rho}(\tau) = \bigoplus_{i \in I} \tau_i^{(0)},
\]

where \( \tau_i^{(1)} \equiv \rho \), \( \tau_i^{(2)} \equiv \rho \), \( \tau_i^{(-x)} \equiv \rho \).

For example, \( \Jac_{|\cdot|^{-1/2}}^{\theta}(St_d) = St_{d-2} \).
Lemma 5.1. Let $\psi = \bigoplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{\text{gp}}(G_n)$. Fix an admissible order $>_\psi$, and an $A$-parameter $\psi >_\psi = \bigoplus_{i \in I} \rho_i \boxtimes S_{a'_i} \boxtimes S_{b'_i} \in \Psi_{\text{DR}}(G_n)$ which dominates $\psi$ with respect to $>_\psi$. Then

$$\tau_\psi = \circ_{i \in I} \left( \text{Jac}^\theta_{\rho_i \mid | \alpha_i - \beta_i| \xi_1 \cdots \xi_n \rho_i \mid | \alpha_i + \beta_i| \xi_1 \cdots \xi_n \rho_i \mid | \gamma_i (\alpha_i + \beta_i) + 1) \right) \circ \cdots \circ \text{Jac}^\theta_{\rho_i \mid | \alpha'_i - \beta'_i| \xi_1 \cdots \xi_n \rho_i \mid | \gamma'_i (\alpha'_i + \beta'_i) - 1) \right) (\tau_{\psi >_\psi}),$$

where $a_i \in I$ is taken in the decreasing order with respect to $>_\psi$.

For the meaning of the phrase “$a_i \in I$ is taken in the decreasing order with respect to $>_\psi$”, see the following examples.

Example 5.2. (1) Consider $\psi = (S_d \boxtimes 1)^{\oplus n}$ with $d > 1$ so that $\tau_\psi = \text{St}_d \times \cdots \times \text{St}_d$ ($n$-times). Take $I = \{1, \ldots, n\}$ with the order $>_\psi$ given by $i >_\psi i - 1$. Then any $A$-parameter $\psi >_\psi$ which has a DDR and dominates $\psi$, and such that $>_\psi$ is a natural order is of the form

$$\psi >_\psi = \psi(t_1, \ldots, t_n) = S_{d + t_1} \boxtimes 1 \oplus \cdots \oplus S_{d + t_n} \boxtimes 1$$

with $t_1, \ldots, t_n \in \mathbb{Z}$ and $t_n > t_{n-1} > \cdots > t_1 \geq 0$. The representation $\tau_{\psi >_\psi}$ is given by

$$\tau_{\psi(t_1, \ldots, t_n)} = \text{St}_{d + t_1} \times \cdots \times \text{St}_{d + t_n}.$$ 

When $t_1 > 0$, we have

$$\text{Jac}^\theta_{| |- \frac{d + t_1 - 1}{2}, \ldots, | |- \frac{d + t_n - 1}{2}} (\tau_{\psi(t_1, \ldots, t_n)}) = \tau_{\psi(t_1 - 1, \ldots, t_n)}.$$ 

Hence

$$\text{Jac}^\theta_{| |- \frac{d + t_1 - 1}{2}, \ldots, | |- \frac{d + t_n - 1}{2}} (\tau_{\psi(t_1, \ldots, t_n)}) = \tau_{\psi(0, t_2, \ldots, t_n)}.$$ 

Therefore we have

$$\text{Jac}^\theta_{| |- \frac{d + t_1 - 1}{2}, \ldots, | |- \frac{d + t_n - 1}{2}} \circ \cdots \circ \text{Jac}^\theta_{| |- \frac{d + t_1 - 1}{2}, \ldots, | |- \frac{d + t_n - 1}{2}} (\tau_{\psi(t_1, \ldots, t_n)}) = \tau_\psi.$$ 

(2) Consider $\psi = S_2 \boxtimes S_3 \oplus S_5 \boxtimes S_2$. If we put $(a_1, b_1) = (2, 3)$ and $(a_2, b_2) = (5, 2)$ with $I = \{1, 2\}$, then by the condition (P), the admissible order $>_\psi$ must satisfy $2 >_\psi 1$. An $A$-parameter $\psi >_\psi$ which dominants $\psi$ is given by $\psi >_\psi = S_2 \boxtimes S_4 \oplus S_7 \boxtimes S_2$. Then

$$\tau_\psi = \text{Sp}(\text{St}_3, 3) \times \text{Sp}(\text{St}_5, 2),$$

$$\tau_{\psi >_\psi} = \text{Sp}(\text{St}_2, 3) \times \text{Sp}(\text{St}_7, 2).$$ 

Since

$$\text{Sp}(\text{St}_4, 2) \leftrightarrow \begin{pmatrix} 5 & 7 \\ 2 & 2 \end{pmatrix} \times \text{Sp}(\text{St}_5, 2) \times \begin{pmatrix} -7 & 5 \\ 2 & 2 \end{pmatrix},$$

we have

$$\text{Jac}^\theta_{\frac{2}{2}, \frac{1}{2}} (\tau_{\psi >_\psi}) = \tau_\psi.$$
5.3. The case of $SO_{2n+1}$ and $Sp_{2n}$. For $\psi = \oplus_{i \in I} \rho_i \boxtimes S_{a_i} \boxtimes S_{b_i} \in \Psi_{gp}(G_n)$, we define $d_i$, $\zeta_i$, $\Sigma_\psi$, $\sim$, and $\varepsilon_{\psi}$ as in §4.2. We fix an admissible order $>_{\psi}$ on $I_\rho$ for each $\rho$, and an $A$-parameter $\psi' \in \overline{\Psi}_{DDR}(G'_n)$ such that $\psi'$ dominates $\psi$ with respect to $>_{\psi}$, and such that the order $>_{\psi}$ is a natural order for $\psi'$. For $(l, \eta) \in \Sigma_\psi$, we define $\pi_{>_{\psi}}(\psi, l, \eta)$ by

$$\pi_{>_{\psi}}(\psi, l, \eta) = \circ_{i \in I} \left( \underset{\rho_i \mid |\zeta_i - b_i|}{\text{Jac}} \circ_{i \in I} \left( \underset{\rho_i \mid |\zeta_i - a_i + b_i|}{\text{Jac}} \circ_{i \in I} \left( \underset{\rho_i \mid |\zeta_i - a_i + b_i + 1|}{\text{Jac}} \circ_{i \in I} \left( \underset{\rho_i \mid |\zeta_i - a_i + b_i + 2|}{\text{Jac}} \circ_{i \in I} \left( \underset{\rho_i \mid |\zeta_i - a_i + b_i + 3|}{\text{Jac}} \left( \pi(\psi', l, \eta) \right) \right) \right) \right) \right) \right),$$

where $\circ_{i \in I}$ is taken in the decreasing order with respect to $>_{\psi}$.

**Theorem 5.3.** Let $\psi \in \Psi_{gp}(G_n)$ with an admissible order $>_{\psi}$. Then, for $(l, \eta) \in \Sigma_\psi$, the representation $\pi_{>_{\psi}}(\psi, l, \eta)$ does not depend on the choice of $\psi'$. Moreover, it is either zero or irreducible. If $\pi_{>_{\psi}}(\psi, l, \eta) \not\equiv \pi_{>_{\psi}}(\psi', l', \eta') \neq 0$, then $(l, \eta) \sim_{\psi} (l', \eta')$. The $A$-packet $\Pi_\psi$ is given by

$$\Pi_\psi = \{ \pi_{>_{\psi}}(\psi, l, \eta) \mid (l, \eta) \in \Sigma_\psi / \sim_{\psi} \} \setminus \{0\}.$$

In particular, $\#\Pi_\psi \leq \#(\Sigma_\psi / \sim_{\psi})$.

**Example 5.4.** Suppose that $\psi \in \Psi_{gp}(SO_7)$ such that $\psi_d = S_2^{\oplus 3}$. Then

$$\Sigma_\psi = \{(0, \eta) \mid \eta = (\eta_1, \eta_2, \eta_3) \in \{\pm 1\}^3, \eta_1\eta_2\eta_3 = 1 \}$$

with trivial equivalence relation $\sim$, where $0 = (0, 0, 0)$. We always take the admissible order $>_{\psi}$ given by $3 >_{\psi} 2 >_{\psi} 1$. If we write $\psi = \oplus_{i=1}^3 S_{a_i} \boxtimes S_{b_i}$ with $(a_i, b_i) \in \{(2, 1), (1, 2)\}$, the sign $\zeta_i \in \{\pm 1\}$ is determined by $\zeta_i = \text{sgn}(a_i - b_i)$. Define $\psi' = \oplus_{i=1}^3 S_{a'_i} \boxtimes S_{b'_i}$ so that $(a'_i, b'_i) \in \{(2i, 1), (1, 2i)\}$ and $(a'_i - b'_i)(a_i - b_i) > 0$. Then $\psi'$ dominates $\psi$ with respect to $>_{\psi}$, and has a DDR with $>_{\psi}$ being a natural order. In fact, $\psi'$ is elementary. Given $\eta = (\eta_1, \eta_2, \eta_3) \in \{\pm 1\}$ such that $\eta_1\eta_2\eta_3 = 1$, define $\varepsilon \in \overline{A}_{\psi'}$ by $\varepsilon(1, 2i) = \eta_i$. Then

$$\pi_{>_{\psi}}(\psi, 0, \eta) = \text{Jac}_{|\zeta_1^2, \zeta_2^2, \zeta_3^2} \circ \text{Jac}_{|\zeta_1^2, \zeta_2^2} \circ \text{Jac}_{|\zeta_1^2} \left( \pi(\psi', \varepsilon) \right).$$

(1) Suppose that $\zeta_1 = \zeta_2 = \zeta_3 = \zeta$. By Lemma 3.5 (together with using the Aubert involution if necessary), $\text{Jac}_{|\zeta^2} \left( \pi(\psi', \varepsilon) \right)$ is nonzero if and only if $\eta_1 = \eta_2$. In this case, it is irreducible and we have

$$\text{Jac}_{|\zeta^2} \left( \pi(\psi', (+, +, +)) \right) \oplus \text{Jac}_{|\zeta^2} \left( \pi(\psi', (\varepsilon, -, +)) \right) \cong \begin{cases} \text{St}_2 \times \pi(S_6) & \text{if } \zeta = +1, \\ \text{det}_2 \times 1_{SO_5(F)} & \text{if } \zeta = -1. \end{cases}$$

Here, $\pi(S_6) \in \Pi_{S_6}$ is a discrete series representation of $SO_7(F)$, and $\text{det}_2$ is the determinant character of $GL_2(F)$. Hence

$$\pi_{>_{\psi}}(\psi, 0, (+, +, +)) \oplus \pi_{>_{\psi}}(\psi, 0, (\varepsilon, -, +)) \cong \begin{cases} \text{St}_2 \times \pi(S_2) & \text{if } \zeta = +1, \\ \text{det}_2 \times 1_{SO_5(F)} & \text{if } \zeta = -1. \end{cases}$$
Since the right hand side is irreducible, exactly one of $\pi_{>\psi}(\psi, 0, (+, +, +))$ or $\pi_{>\psi}(\psi, 0, (−, −, +))$ is nonzero. Therefore, we have

$$\Pi_\psi = \begin{cases} \{\text{St}_2 \times \pi(S_2)\} & \text{if } \zeta = +1, \\ \{\text{det}_2 \times 1_{SO_1(F)}\} & \text{if } \zeta = -1. \end{cases}$$

In particular, $\#\Pi_\psi = 1$ while $\#(\Sigma_\psi / \sim_\psi) = 4$. Note that $\Pi_\psi$ is equal to $\Pi_{\phi_\psi}$ with

$$\phi_\psi = \begin{cases} S_2^{\oplus 3} & \text{if } \zeta = +1, \\ \left(\left|\cdot\right|^{1/2} + \left|\cdot\right|^{-1/2}\right)^{\oplus 3} & \text{if } \zeta = -1. \end{cases}$$

In fact, one can show that $\pi_{>\psi}(\psi, 0, (+, +, +)) \neq 0$ and $\pi_{>\psi}(\psi, 0, (−, −, +)) = 0$.

(2) Suppose that $\{\zeta_1, \zeta_2, \zeta_3\} = \{+1, -1\}$. We can assume without loss of generality that $\zeta_2 = \zeta_3 = \zeta \neq \zeta_1$.

- If $\eta = (+, +, +)$, since $\pi(\psi, \varepsilon)$ is a subrepresentation of

$$\left|\cdot\right|^{-\zeta_1/2} \times \left|\cdot\right|^{\zeta_2/2} \times \left|\cdot\right|^{\zeta_3/2} \times \left<\left|\cdot\right|^{3/2}, \left|\cdot\right|^{3/2}, \left|\cdot\right|^{1/2}\right> \times 1_{SO_1(F)}$$

we have $\text{Jac}_{\left|\cdot\right|^{-\zeta_1/2}}(\Psi_{>\psi}, \eta) = \text{Jac}_{\left|\cdot\right|^{\zeta_2/2}, \left|\cdot\right|^{\zeta_3/2}, \left|\cdot\right|^{-\zeta_1/2}}(\pi(\psi, \varepsilon)) \neq 0$. Note that

$$\text{Jac}_{\left|\cdot\right|^{-\zeta_1/2}}(\pi(\psi, \varepsilon)) = \pi(\psi', \varepsilon')$$

with $(\psi')_d = (\psi_{>\psi})_d - S_2$ and $\varepsilon' = \varepsilon|A_{\psi'}$. Using [X17a, Lemma 5.6], Lemma 3.5 and the Aubert involution if necessary, we see that

$$\text{Jac}_{\left|\cdot\right|^{-\zeta_1/2}}(\pi(\psi, \varepsilon), \eta) \cong \text{Jac}_{\left|\cdot\right|^{-\zeta_1/2}, \left|\cdot\right|^{\zeta_2/2}, \left|\cdot\right|^{\zeta_3/2}, \left|\cdot\right|^{-\zeta_1/2}}(\pi(\psi, \varepsilon))$$

$$\left\{\begin{array}{ll} \text{St}_2 \times 1_{SO_1(F)} & \text{if } \zeta = +1, \\ \text{det}_2 \times 1_{SO_1(F)} & \text{if } \zeta = -1. \end{array}\right.$$
• If \( \eta = (-,+,+) \), since \( \pi(\psi,\varepsilon) \) is supercuspidal, we have \( \text{Jac}_{\psi} \left( \frac{1}{2} \pi(\psi,\varepsilon) \right) = 0 \) so that \( \pi_{\psi} \left( \psi, (\eta, \eta) \right) = 0 \).

• If \( \eta = (-,-,+), \) since \( \pi(\psi,\varepsilon) \) is the same representation as in the case (1), we have \( \pi_{\psi} \left( \psi, (\eta, \eta) \right) = 0 \).

Therefore, we have

\[
\Pi_{\psi} = \{ \pi_{\psi} \left( \psi, (\eta, \eta) \right) \}.
\]

In particular, \( \# \Pi_{\psi} = 2 \) while \( \#(\Sigma_{\psi} / \sim_{\psi}) = 4 \). Moreover,

\[
\bigoplus_{\pi \in \Pi_{\psi}} \pi
\]

is the maximal semisimple submodule of \( \langle -\zeta/2 \rangle \times \langle \zeta/2, -\zeta/2 \rangle \rtimes \text{SO}_{1}(F) \). Note that \( \Pi_{\psi} \subset \Pi_{\psi} \) but

\[
\# \Pi_{\psi} = \begin{cases} 2 & \text{with } \phi_{\psi} = \left( 0, \frac{1}{2} \right) \oplus S_{2} \oplus S_{2} \oplus |-\frac{1}{2} | \quad \text{if } \zeta = +1, \\ 1 & \text{with } \phi_{\psi} = \left( 0, \frac{1}{2} \right) \oplus | \frac{1}{2} | \oplus | \frac{1}{2} | \oplus | \frac{1}{2} | \quad \text{if } \zeta = -1. \end{cases}
\]

5.4. General case. Finally, let \( \psi \in \Psi(G_{n}) \) be a general \( A \)-parameter for \( G_{n} \). Then we can decompose

\[
\psi = \psi_{1} \oplus \psi_{0} \oplus \psi'_{0}.
\]

where \( \psi_{0} \) is an \( A \)-parameter for \( G_{n_{0}} \) of good parity, and \( \psi_{1} \) is a sum of irreducible representations of \( W_{F} \times \text{SL}_{2}(\mathbb{C}) \times \text{SL}_{2}(\mathbb{C}) \) which are not the same type as \( \psi \). Fix an admissible order \( >_{\psi_{0}} \) for \( \psi_{0} \). For \( (l_{1}, \eta) \in \Sigma_{\psi_{0}} \), we set

\[
\pi_{\psi} \left( \psi, (l_{1}, \eta) \right) = \tau_{\psi_{0}} \times \pi_{\psi_{0}} \left( \psi_{0}, (l_{1}, \eta) \right).
\]

Theorem 5.5. The representation \( \pi_{\psi} \left( \psi, (l_{1}, \eta) \right) \) is either zero or irreducible, and is independent of the choice of \( \psi_{1} \). The \( A \)-packet \( \Pi_{\psi} \) is given by

\[
\Pi_{\psi} = \{ \pi_{\psi} \left( \psi, (l_{1}, \eta) \right) \mid (l_{1}, \eta) \in \Sigma_{\psi_{0}} / \sim_{\psi_{0}} \} \setminus \{0\}.
\]

To summarize, we obtain Moeglin’s multiplicity-free result for \( A \)-packets.

Theorem 5.6 (Moeglin [Moe11], Xu [X17b, Theorem 8.12]). For \( \psi \in \Psi(G_{n}) \), the \( A \)-packet \( \Pi_{\psi} \) is multiplicity-free, i.e., \( \Pi_{\psi} \) is a subset of \( \text{Irr}_{\text{unit}}(G_{n}(F)) \).

5.5. Complementary results. There are useful results of Moeglin.

Proposition 5.7 ([Moe09b, 4.2 Corollaire]). For \( \psi, \psi' \in \Psi(G_{n}) \), if \( \Pi_{\psi} \cap \Pi_{\psi'} = \emptyset \), then \( \psi' \cong \psi' \).

We call \( \psi \in \Psi(G_{n}) \) unramified if \( \psi|_{I_{F} \times \text{SL}_{2}(\mathbb{C}) \times \{1_{2}\}} \) is trivial, where \( I_{F} \) is the inertia subgroup of \( W_{F} \).

Proposition 5.8 ([Moe09b, 4.4 Proposition]). If \( \psi \in \Psi(G_{n}) \) is unramified, then \( \Pi_{\psi} \) has a unique unramified representation of \( G_{n}(F) \).
6. A non-vanishing criterion

In the previous sections, we have constructed \( \Pi_\psi \) explicitly for any \( \psi \in \Psi(G_\alpha) \). However, when \( \psi \) is of good parity, the representation \( \pi_{>0}(\psi, l, \eta) \) can be zero for \( (l, \eta) \in \Sigma_\psi \). In this section, we give a procedure to determine whether \( \pi_{>0}(\psi, l, \eta) \) is zero or not. This is the work in [X].

By the construction in \( \S 5.4 \), we may assume in this section that \( \psi \) is of good parity.

6.1. Definitions and the algorithm. As in the previous section, we write

\[
\psi = \bigoplus \bigoplus \rho \otimes S_{a_i} \otimes S_{b_i}.
\]

We set

\[
d_i = \min\{a_i, b_i\}, \quad \zeta_i = \begin{cases} 1 & \text{if } a_i > b_i, \\ -1 & \text{if } a_i < b_i. \end{cases}
\]

When \( a_i = b_i \), we choose \( \zeta_i \in \{\pm 1\} \) arbitrarily.

**Definition 6.1.** Suppose that \( \rho \otimes S_a \otimes S_b \subset \psi \), and \( r \in \mathbb{Z} \) with \( r > 0 \). Let \( J \) be a subset of \( I_\rho \), and \( J^c \) be its complement in \( I_\rho \).

1. We say that \( \rho \otimes S_a \otimes S_b \) is in **level** \( r \) “far away” from \( J \) if

\[
\frac{|a - b|}{2} > 2^{|r|} \left( \sum_{j \in J} \left( \frac{a_j + b_j}{2} - 1 \right) + |J| \sum_{i \in I_\rho} d_i \right).
\]

In this case, we write

\[
\rho \otimes S_a \otimes S_b \gg_r \psi_J = \bigoplus_{j \in J} \rho \otimes S_{a_j} \otimes S_{b_j}.
\]

2. We say that \( J \) is “**separated**” from \( J^c \) if the following conditions are satisfied.

   a. For any \( j \in J \) and \( j' \in J^c \), either

   \[
   \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1 \quad \text{or} \quad \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1.
   \]

   This condition is equivalent that

   \[
   \text{Hom}_{\text{SL}_2(\mathbb{C})}(S_{a_j} \otimes S_{b_j}, S_{a_{j'}} \otimes S_{b_{j'}}) = 0.
   \]

   b. For any admissible order \( >_J \) on \( J \), there exists a parameter \( \psi_{J_\rho} = \bigoplus_{j \in J_\rho} \rho \otimes S_{a_j} \otimes S_{b_j} \) with DDR which dominates \( \psi_J = \bigoplus_{j \in J} \rho \otimes S_{a_j} \otimes S_{b_j} \) such that for any \( j \in J \) and \( j' \in J^c \),

   \[
   \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1 \implies \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1.
   \]

   c. For any admissible order \( >_{J^c} \) on \( J^c \), there exists a parameter \( \psi_{J_\rho} = \bigoplus_{j \in J_\rho} \rho \otimes S_{a_j} \otimes S_{b_j} \) with DDR which dominates \( \psi_{J^c} = \bigoplus_{j \in J_\rho} \rho \otimes S_{a_j} \otimes S_{b_j} \) such that for any \( j \in J \) and \( j' \in J^c \),

   \[
   \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1 \implies \frac{|a_j - b_j|}{2} > \frac{a_j + b_j}{2} - 1.
   \]
(3) The index set \( I_\rho \) is in "good shape" if we can index \( I_\rho = \{1, \ldots, N_\rho\} \) such that \( a_i + b_i \geq a_{i-1} + b_{i-1} \) and \( |a_i - b_i| \geq |a_{i-1} - b_{i-1}| \) for any \( i \), and we can divide \( I_\rho = \bigcup_j I^{(j)}_\rho \) such that

(a) \( I^{(j)}_\rho = \{i, i-1\} \) with \( \zeta_i = \zeta_{i-1} \), or \( I^{(j)}_\rho = \{i\} \);

(b) \( I^{(j)}_\rho \) is "separated" from \( I_\rho \setminus I^{(j)}_\rho \).

Then there is a natural order \( > \) on \( I_\rho \) given by \( i > i' \).

(4) We say that \( \psi \) is in the \textbf{generalized basic case} if \( I_\rho \) is in "good shape" for any \( \rho \).

When \( \psi \) is in the generalized basic case with natural order \( > \), there is a criterion for the non-vanishing of \( \pi_{\psi}(\psi, \ell, \eta) \) (Proposition 6.2). We reduce the general case to the generalized basic case changing an admissible order \( > \) (Proposition 6.5), and using three reduction operators "Pull" (Propositions 6.6, 6.7), "Expand" (Proposition 6.8), and "Change sign" (Proposition 6.9).

More precisely, we use the following algorithm.

\textbf{Algorithm:} Let

\[ \psi = \bigoplus_{\rho} \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i} \in \Psi_{SP}(G_\pi), \]

and \( (\ell, \eta) \in \Sigma_\psi / \sim_\psi \). Choose an admissible order \( > \) on \( I_\rho \) for each \( \rho \), and we index \( I_\rho = \{1, 2, \ldots, N_\rho\} \) such that \( i > i' \).

\textbf{Step 1:} Is \( \psi \) in the generalized basic case?
- If yes, use Proposition 6.2 (after using Proposition 6.5 unless \( > \) is a natural order).
- If no, choose \( \rho \) and \( 1 \leq m \leq N_\rho \) such that \( I_\rho \) is not in "good shape", and for \( i > m \),

\[ \rho \otimes S_{a_i} \otimes S_{b_i} \gg \bigoplus_{j=1}^m \rho \otimes S_{a_j} \otimes S_{b_j}, \]

and such that \( \{m+1, \ldots, N_\rho\} \) are in "good shape". Go to Step 2.

\textbf{Step 2:} Choose \( 1 \leq i_0 \leq m \) such that

\[ \max_{1 \leq i \leq m} (a_i + b_i) = a_{i_0} + b_{i_0}. \]

Consider

\[ S = \{i \leq m \mid S_{a_i} \otimes S_{b_i} \subseteq S_{a_{i_0}} \otimes S_{b_{i_0}}, \zeta_i = \zeta_{i_0}\}, \]

\[ \overline{S} = \{i \leq m \mid S_{a_i} \otimes S_{b_i} \subseteq S_{a_{i_0}} \otimes S_{b_{i_0}}, \zeta_i = \zeta_{i_0}\}. \]

Then there are three possibilities.
- If \( S \neq \emptyset \), go to Step 3a.
- If \( S = \emptyset \) but \( \overline{S} \supseteq \{i_0\} \), go to Step 3b.
- If \( \overline{S} = \{i_0\} \), go to Step 3c.

\textbf{Step 3a:} Take \( i'_0 \in S \) such that

\[ \max_{i \in S} (a_i + b_i) = a_{i'_0} + b_{i'_0}. \]

Using Proposition 6.5, we rearrange the order \( > \) for \( i \leq m \) so that

\[ \rho \otimes S_{a_m} \otimes S_{b_m} = \rho \otimes S_{a_{i_0}} \otimes S_{b_{i_0}}. \]
\[ \rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}} = \rho \boxtimes S_{a_1'} \boxtimes S_{b_1'} . \]

Then we can “Pull” the pair \( \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \) and \( \rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}} \) using Proposition 6.6. Consequently, the non-vanishing of \( \pi_{\psi}(\psi, l, n) \) is equivalent to those of 3 representations \( \{ \pi_{\psi}(\psi^*, l^*, n^*) \} \). If we replace \( \psi \) with \( \psi^* \), one can replace \( m \) with \( m - 1 \) or \( m - 2 \). After such replacements, go back to Step 2 if \( m \geq 1 \). If \( m = 0 \), then \( I_\rho \) is in “good shape”, and go back to Step 1.

**Step 3b:** Take \( i'_0 \in \overline{S} \) such that \( i'_0 \neq i_0 \). Using Proposition 6.5, we rearrange the order \( \psi \) for \( i \leq m \) as in Step 3a. Then we can “Pull” the pair \( \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \) and \( \rho \boxtimes S_{a_{m-1}} \boxtimes S_{b_{m-1}} \) using Proposition 6.7. Consequently, the non-vanishing of \( \pi_{\psi}(\psi, l, n) \) is equivalent to those of 2 representations \( \{ \pi_{\psi^*}(\psi^*, l^*, n^*) \} \). If we replace \( \psi \) with \( \psi^* \), one can replace \( m \) with \( m - 1 \) or \( m - 2 \). After such replacements, go back to Step 2 if \( m \geq 1 \). If \( m = 0 \), then \( I_\rho \) is in “good shape”, and go back to Step 1.

**Step 3c:** Using Proposition 6.5, we rearrange the order \( \psi \) for \( i \leq m \) so that

\[ \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} = \rho \boxtimes S_{a_{i_0}} \boxtimes S_{b_{i_0}} . \]

Then we can “Expand” \( \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \) using Proposition 6.8. Consequently, the non-vanishing of \( \pi_{\psi}(\psi, l, n) \) is equivalent to that of a representation \( \pi_{\psi^*}(\psi^*, l^*, n^*) \). Go to Step 4.

**Step 4:** For the parameter \( \psi^* \) obtained in Step 3c, consider the set \( S^* \) as in Step 2 with \( \rho \boxtimes S_{a_{i_0}} \boxtimes S_{b_{i_0}} := \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \), which is the “Expansion” of \( \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \).

- If \( S^* \neq \emptyset \), go back to Step 3a after replacing \( \psi \) with \( \psi^* \).
- If \( S^* = \emptyset \), it is necessary that \( |a_m - b_m| \leq 1 \), and \( \zeta_i \neq \zeta_n \) for \( i < m \). Go to Step 5.

**Step 5:** Using Proposition 6.5, we rearrange the order \( \psi \) for \( i \leq m \) so that

\[
\begin{pmatrix}
1 \\
2 \\
m
\end{pmatrix}
\approx
\begin{pmatrix}
m \\
1 \\
m - 1
\end{pmatrix}.
\]

Then we can “Change sign” of \( \rho \boxtimes S_{a_{m}} \boxtimes S_{b_{m}} \) using Proposition 6.9. Consequently, the non-vanishing of \( \pi_{\psi}(\psi^*, l^*, n^*) \) is equivalent to that of a representation \( \pi_{\psi^*}(\psi^{**}, l^{**}, n^{**}) \). If we replace \( \psi^* \) with \( \psi^{**} \), the set \( S^* \) becomes non-empty. After such replacements, go back to Step 1.

**Step 6:** By the above steps, the non-vanishing of \( \pi_{\psi}(\psi, l, n) \) is equivalent to those of several representations \( \{ \pi_{\psi}(\psi^*, l^*, n^*) \} \), where each \( \psi^* \) is in the generalized basic case. Use Proposition 6.2, we obtain the conditions on \( (l, n) \) for \( \pi_{\psi^*}(\psi, l, n) \neq 0 \).

In the rest of this section, we state several propositions appearing in this algorithm.

### 6.2. Generalized basic case

Recall that \( \psi \) is in the generalized basic case if for each \( \rho \), the index set \( I_\rho \) is in “good shape”, i.e., \( I_\rho = \{ 1, \ldots, N_\rho \} \) with \( a_i + b_i \geq a_{i-1} + b_{i-1} \) and \( |a_i - b_i| \geq |a_{i-1} - b_{i-1}| \) for any \( i \), and we can divide \( I_\rho = \bigcup_j I^{(j)}_\rho \) such that

- (a) \( I^{(j)}_\rho = \{ i, i - 1 \} \) with \( \zeta_i = \zeta_{i-1} \), or \( I^{(j)}_\rho = \{ i \} \);
- (b) \( I^{(j)}_\rho \) is “separated” from \( I_\rho \setminus I^{(j)}_\rho \).

In this case, we use the natural order \( \psi \) on \( I_\rho \) such that \( i > \psi i - 1 \). In the generalized basic case, there is a non-vanishing criterion for \( \pi_{\psi}(\psi, l, n) \).
Proposition 6.2 ([X, Proposition 4.3]). When \( \psi \) is in the generalized basic case, \( \pi_{\psi}(\psi, l, \eta) \neq 0 \) if and only if for any \( \rho \) and any \( I^j_{\rho} \), if \( I^j_{\rho} = \{i, i - 1\} \) has two elements, then
\[
\begin{align*}
\eta_i = (-1)^{d_{i-1}}\eta_{i-1} & \implies l_i + l_{i-1} > \frac{a_i - b_i}{2} - \frac{|a_i - b_i|}{2} - 1, \\
\eta_i \neq (-1)^{d_{i-1}}\eta_{i-1} & \implies -\frac{|a_i - b_i| - |a_{i-1} - b_{i-1}|}{2} \leq l_i - l_{i-1} \leq \frac{(a_i + b_i) - (a_{i-1} + b_{i-1})}{2}.
\end{align*}
\]

Remark 6.3. Let \( \psi \) be in the generalized basic case, and \( I_\rho = \bigcup_j I^j_{\rho} \) be the division as in Definition 6.1 (3).

1. If \( \# I^j_{\rho} = 1 \) for any \( \rho \) and any \( j \), then by Definition 6.1 (2)-(a), \( \psi \) has a DDR.
2. We say that \( \psi \) is in the basic case if there exists a unique pair \( (\rho, j) \) such that \( \# I^j_{\rho} = 2 \). M"{a}glin gave a non-vanishing criterion for \( \pi_{\psi}(\psi, l, \eta) \) when \( \psi \) is in the basic case such that \( I^j_{\rho} = \{i, i - 1\} \) with \( S_{a_i} \otimes S_{b_i} \cong S_{a_{i-1}} \otimes S_{b_{i-1}} \). Proposition 6.2 is a generalization of M"{a}glin’s result.

Using Proposition 6.2, we obtain the following necessary conditions on non-vanishing of \( \pi_{\psi}(\psi, l, \eta) \) in general.

Lemma 6.4 ([X, Lemmas 4.6, 4.7]). Fix an admissible order \( >_\psi \) on \( I_\rho \), and let \( k > \psi k - 1 \) be two adjacent elements in \( I_\rho \) such that \( \zeta_k = \zeta_{k-1} \). Suppose that \( \pi_{\psi}(\psi, l, \eta) \neq 0 \).

1. If \( S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}} \), then
\[
\begin{align*}
\eta_k = (-1)^{d_{k-1}}\eta_{k-1} & \implies l_k + l_{k-1} > d_{k-1} - 1, \\
\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1} & \implies 0 \leq l_k - l_{k-1} \leq d_k - d_{k-1}.
\end{align*}
\]
2. If \( S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}} \), then
\[
\begin{align*}
\eta_k = (-1)^{d_{k-1}}\eta_{k-1} & \implies l_k + l_{k-1} > d_{k-1} - 1, \\
\eta_k \neq (-1)^{d_{k-1}}\eta_{k-1} & \implies 0 \leq l_k - l_{k-1} \leq d_k - d_{k-1} - d_k.
\end{align*}
\]

We denote by \( \Sigma^{(+)}_\psi \) (resp. \( \Sigma^{(-)}_\psi \)) the subset of \( \Sigma_\psi \) satisfying the conditions in Lemma 6.4 (1) (resp. (2)) for any \( \rho \) and any adjacent pair \( \{k, k-1\} \) in \( I_\rho \) with \( \zeta_k = \zeta_{k-1} \). Note that if \( (l, \eta) \sim_\psi (l', \eta') \) and \( (l, \eta) \in \Sigma^{(\pm)}_\psi \), then \( (l', \eta') \in \Sigma^{(\pm)}_\psi \).

6.3. Change of admissible orders. Let \( \psi \in \Psi_{gp}(G_n) \). We choose an admissible order \( >_\psi \) on \( I_\rho \), and we index \( I_\rho = \{1, \ldots, N_\rho\} \) such that \( i >_\psi i - 1 \). Now for fixed \( 1 < k \leq N_\rho \), we denote by \( >'_\psi \) the order on \( I_\rho \) obtained so that \( i >'_\psi j \) if and only if \( i >_\psi j \) and \( (i, j) \neq (k, k-1) \), or \( (i, j) = (k-1, k) \). Assume that \( >'_\psi \) is also an admissible order.

When \( \zeta_k = \zeta_{k-1} \) and \( S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}} \), we define \( (l', \eta') = S^+(l, \eta) \) for \( (l, \eta) \in \Sigma^{(+)}_\psi \) as follows:

- If \( i \neq k, k - 1 \), then \( l'_i = l_i \) and \( \eta'_i = \eta_i \).
• If $\eta_k = (-1)^{d_k-1} \eta_{k-1}$, then
  $$
  \begin{align*}
  l'_{k-1} &= l_{k-1}, \\
  l'_k &= l_k + 2l_{k-1} - d_{k-1}, \\
  \eta'_{k-1} &= -(-1)^{d_k} \eta_{k-1}, \\
  \eta'_k &= (-1)^{d_k-1} \eta_k.
  \end{align*}
  $$

• If $\eta_k \neq (-1)^{d_k-1} \eta_{k-1}$ and $l_k - 2l_{k-1} < (d_k + 1)/2 - d_{k-1}$, then
  $$
  \begin{align*}
  l'_{k-1} &= l_{k-1}, \\
  l'_k &= l_k - 2l_{k-1} + d_{k-1}, \\
  \eta'_{k-1} &= -(-1)^{d_k} \eta_{k-1}, \\
  \eta'_k &= (-1)^{d_k-1} \eta_k.
  \end{align*}
  $$

• If $\eta_k \neq (-1)^{d_k-1} \eta_{k-1}$ and $l_k - 2l_{k-1} \geq (d_k + 1)/2 - d_{k-1}$, then
  $$
  \begin{align*}
  l'_{k-1} &= l_{k-1}, \\
  l'_k &= -l_k + 2l_{k-1} + d_k - d_{k-1}, \\
  \eta'_{k-1} &= -(-1)^{d_k} \eta_{k-1}, \\
  \eta'_k &= -(-1)^{d_k-1} \eta_k.
  \end{align*}
  $$

When $\zeta_k = \zeta_{k-1}$ and $S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}}$, we define $(l', \eta') = S^{-}(l, \eta)$ for $(l, \eta) \in \Sigma^{(\psi)}$ as follows:

• If $i \neq k, k - 1$, then $l'_i = l_i$ and $\eta'_i = \eta_i$.

• If $\eta_k = (-1)^{d_k-1} \eta_{k-1}$, then
  $$
  \begin{align*}
  l'_k &= l_k, \\
  l'_{k-1} &= l_{k-1} + 2l_k - d_k, \\
  \eta'_k &= -(-1)^{d_k-1} \eta_k, \\
  \eta'_{k-1} &= (-1)^{d_k-1} \eta_{k-1}.
  \end{align*}
  $$

• If $\eta_k \neq (-1)^{d_k-1} \eta_{k-1}$ and $l_{k-1} - 2l_k < (d_{k-1} + 1)/2 - d_k$, then
  $$
  \begin{align*}
  l'_k &= l_k, \\
  l'_{k-1} &= l_{k-1} - 2l_k + d_k, \\
  \eta'_k &= -(-1)^{d_k-1} \eta_k, \\
  \eta'_{k-1} &= (-1)^{d_k-1} \eta_{k-1}.
  \end{align*}
  $$

• If $\eta_k \neq (-1)^{d_k-1} \eta_{k-1}$ and $l_{k-1} - 2l_k \geq (d_{k-1} + 1)/2 - d_k$, then
  $$
  \begin{align*}
  l'_k &= l_k, \\
  l'_{k-1} &= -l_{k-1} + 2l_k + d_{k-1} - d_k, \\
  \eta'_k &= -(-1)^{d_k-1} \eta_k, \\
  \eta'_{k-1} &= -(-1)^{d_k-1} \eta_{k-1}.
  \end{align*}
  $$
Then one can check that \( S^\pm: \Sigma^{(\pm)} / \sim_\psi \to \Sigma^{(\pm)} / \sim_\psi \) and that they are inverse to each other. Hence \( S^+ \) and \( S^- \) are bijective.

When \( \zeta_k \neq \zeta_{k-1} \), not assuming any extra conditions on \( a_k, b_k, a_{k-1}, b_{k-1} \), we define \((I', \eta') = U(l, \eta)\) for \((l, \eta) \in \Sigma_\psi\) by \( l'_i = l_i \) and \( \eta'_i = \eta_i \) for \( i \neq k, k - 1 \), and by

\[
\begin{align*}
\begin{cases}
  l'_{k-1} &= l_{k-1}, \\
  l'_k &= l_k,
\end{cases}
\end{align*}
\]

Then one can check that \( U(l, \eta) \in \Sigma_\psi \) and that \( U \circ U = \text{id} \). Hence \( U: \Sigma_\psi / \sim_\psi \to \Sigma_\psi / \sim_\psi \) is bijective.

**Proposition 6.5 ([X, Propositions 5.1, 5.3]).**

1. Suppose that \( \zeta_k = \zeta_{k-1} \) and \( S_{a_k} \otimes S_{b_k} \supseteq S_{a_{k-1}} \otimes S_{b_{k-1}} \). Then \( \pi_{>\psi}(\psi, l, \eta) = 0 \) unless \((l, \eta) \in \Sigma^{(+)}_\psi / \sim_\psi\), in which case,

\[
\pi_{>\psi}(\psi, l, \eta) \cong \pi_{>\psi}(\psi, l', \eta')
\]

with \((l', \eta') = S^+(l, \eta) \in \Sigma^{(-)}_\psi / \sim_\psi\).

2. Suppose that \( \zeta_k = \zeta_{k-1} \) and \( S_{a_k} \otimes S_{b_k} \subseteq S_{a_{k-1}} \otimes S_{b_{k-1}} \). Then \( \pi_{>\psi}(\psi, l, \eta) = 0 \) unless \((l, \eta) \in \Sigma^{(-)}_\psi / \sim_\psi\), in which case,

\[
\pi_{>\psi}(\psi, l, \eta) \cong \pi_{>\psi}(\psi, l', \eta')
\]

with \((l', \eta') = S^-(l, \eta) \in \Sigma^{(+)}_\psi / \sim_\psi\).

3. Suppose that \( \zeta_k \neq \zeta_{k-1} \). Then

\[
\pi_{>\psi}(\psi, l, \eta) \cong \pi_{>\psi}(\psi, l', \eta')
\]

for any \((l, \eta) \in \Sigma_\psi / \sim_\psi\) with \((l', \eta') = U(l, \eta) \in \Sigma_\psi / \sim_\psi\).

**6.4. Reduction operator 1: “Pull”**. In this subsection, we introduce a reduction operator “Pull”. Choose an admissible order \( \succ \). We index \( I_\rho = \{1, \ldots, N_\rho\} \) such that \( i \succ \psi i - 1 \). First, we suppose that there exists \( m \) such that

- for \( i > m \),

\[
\rho \otimes S_{a_i} \otimes S_{b_i} \gg 1 \bigoplus_{j=1}^{m} \rho \otimes S_{a_j} \otimes S_{b_j};
\]

- \( S_{a_m} \otimes S_{b_m} \supseteq S_{a_{m-1}} \otimes S_{b_{m-1}} \) and \( \zeta_m = \zeta_{m-1} \).

We denote by \( \succ' \) the order on \( I_{\rho} \) obtained so that \( i \succ j \succ' j \) if and only if \( i \succ j \) and \( (i, j) \neq (m, m - 1) \), or \( (i, j) = (m - 1, m) \). Set \((a^m_m, b^m_m), (a^m_{m-1}, b^m_{m-1}), (a^m_i, b^m_j), \) and \((a^m_i, b^m_j)\) so that

- \( \max\{a^m_m, b^m_m\} = \max\{a_m, b_m\} + 2T_m \), \( \min\{a^m_m, b^m_m\} = \min\{a_m, b_m\} \), and so that \( \zeta^*_m = \zeta_m \), where \( T_m \) is an arbitrary positive integer such that

\[
\rho \otimes S_{a_i} \otimes S_{b_i} \gg 1 \rho \otimes S_{a^*_m} \otimes S_{b^*_m}
\]

for \( i > m \);
\begin{itemize}
  \item \( \max \{ a_{m-1}^*, b_{m-1}^* \} = \max \{ a_{m-1}, b_{m-1} \} + 2T_{m-1} \), \( \min \{ a_{m-1}^*, b_{m-1}^* \} = \min \{ a_{m-1}, b_{m-1} \} \), and so that \( \zeta_{m-1}^* = \zeta_{m-1} \), where \( T_{m-1} \) is the integer given by
  \[
  T_{m-1} = T_m + \frac{|a_m - b_m| - |a_{m-1} - b_{m-1}|}{2};
  \]
  \item \( \max \{ a_m^*, b_m^* \} = \max \{ a_m, b_m \} + 2T, \min \{ a_m^*, b_m^* \} = \min \{ a_m, b_m \} \), and so that \( \zeta_m^* = \zeta_m \), where \( T \) is an arbitrary positive integer such that
  \[
  T < \frac{|a_i - b_i| - (a_m + b_m)}{2} + 1
  \]
  for \( i > m; \)
  \item \( \max \{ a_{m-1}^*, b_{m-1}^* \} = \max \{ a_{m-1}, b_{m-1} \} + 2T, \min \{ a_{m-1}^*, b_{m-1}^* \} = \min \{ a_{m-1}, b_{m-1} \} \), and so that \( \zeta_{m-1}^* = \zeta_{m-1} \), where \( T \) is an arbitrary positive integer such that
  \[
  T < \frac{|a_i - b_i| - (a_{m-1} + b_{m-1})}{2} + 1
  \]
  for \( i > m. \)
\end{itemize}

Define \( \psi^*, \psi^b, \) and \( \psi^\natural \) by
\[
\psi^* = \psi - \left( \bigoplus_{i=m-1}^m \rho \otimes S_{a_i} \otimes S_{b_i} \right) - \left( \bigoplus_{i=m-1}^m \rho \otimes S_{a_i^*} \otimes S_{b_i^*} \right),
\]
\[
\psi^b = \psi - \rho \otimes S_{a_m} \otimes S_{b_m} + \rho \otimes S_{a_{m-1}^\natural} \otimes S_{b_{m-1}^\natural},
\]
\[
\psi^\natural = \psi - \rho \otimes S_{a_{m-1}^\natural} \otimes S_{b_{m-1}^\natural} + \rho \otimes S_{a_{m-1}^\natural} \otimes S_{b_{m-1}^\natural}.
\]

We may identify \( \Sigma_\psi \) with \( \Sigma_\psi^* \) canonically for \( * \in \{ \natural, b, \natural \}. \)

**Proposition 6.6** ([X, Proposition 6.1]). Let \( (l, \eta) \in \Sigma_\psi \) and set \( (l', \eta') = S^+(l, \eta) \). Then \( \pi_{>\psi}(\psi^*, l, \eta) \neq 0 \) if and only if all of
\[
\pi_{>\psi}(\psi^*, l, \eta), \quad \pi_{>\psi}(\psi^b, l, \eta), \quad \pi_{>\psi}(\psi^\natural, l', \eta')
\]
are nonzero.

Next, we suppose that there exists \( m \) such that
\begin{itemize}
  \item for \( i > m, \)
  \[
  \rho \otimes S_{a_i} \otimes S_{b_i} \gg 1 \bigoplus_{j=1}^m \rho \otimes S_{a_j} \otimes S_{b_j};
  \]
  \item \( S_{a_m} \otimes S_{b_m} = S_{a_{m-1}} \otimes S_{b_{m-1}} \) and \( \zeta_m = \zeta_{m-1}; \)
  \item there is no \( i < n \) such that \( S_{a_i} \otimes S_{b_i} \subsetneq S_{a_m} \otimes S_{b_m} \) and \( \zeta_i = \zeta_m. \)
\end{itemize}

Set \( (a_m^*, b_m^*), (a_{m-1}^*, b_{m-1}^*), \) and \( (a_m^b, b_m^b) \) so that
\begin{itemize}
  \item \( \max \{ a_m^*, b_m^* \} = \max \{ a_m, b_m \} + 2T, \min \{ a_m^*, b_m^* \} = \min \{ a_m, b_m \} \), and so that \( \zeta_m^* = \zeta_m \), where \( T_m \) is an arbitrary positive integer such that
  \[
  \rho \otimes S_{a_m} \otimes S_{b_m} \gg 1 \rho \otimes S_{a_m^*} \otimes S_{b_m^*},
  \]
  for \( i > m. \)
Proposition 6.8 ([X, Proposition 6.4]). We may identify

\[ \max\{a_{m-1}, b_{m-1}^*\} = \max\{a_{m-1}, b_{m-1}\} + 2T_{m-1}, \min\{a_{m-1}, b_{m-1}^*\} = \min\{a_{m-1}, b_{m-1}\}, \]

and so that \( \zeta_{m-1} = \zeta_{m-1}' \), where \( T_{m-1} = T_m \).

\[ \max\{a_m, b_m^*\} = \max\{a_m, b_m\} + 2T, \min\{a_m, b_m^*\} = \min\{a_m, b_m\}, \]

and so that \( \zeta_m = \zeta_m' \), where \( T \) is an arbitrary positive integer such that

\[ T < \frac{|a_i - b_i| - (a_m + b_m)}{2} + 1 \]

for \( i > m \).

Define \( \psi^a \) and \( \psi^b \) by

\[ \psi^a = \psi - \left( \bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \right) + \left( \bigoplus_{i=m-1}^{m} \rho \boxtimes S_{a_i}^* \boxtimes S_{b_i}^* \right), \]

\[ \psi^b = \psi - \rho \boxtimes S_{a_m} \boxtimes S_{b_m} + \rho \boxtimes S_{a_m}^* \boxtimes S_{b_m}^* . \]

We may identify \( \Sigma_\psi \) with \( \Sigma_{\psi^*} \) canonically for \( \ast \in \{a, b\} \).

Proposition 6.7 ([X, Proposition 6.3]). Let \( (\lambda, \eta) \in \Sigma_\psi \). Then \( \pi_{> \psi}(\psi, (\lambda, \eta)) \neq 0 \) if and only if all of

\[ \pi_{> \psi}(\psi^a, (\lambda, \eta)), \quad \pi_{> \psi}(\psi^b, (\lambda, \eta)) \]

are nonzero.

6.5. Reduction operator 2: “Expand”. In this subsection, we introduce a reduction operator “Expand”. Choose an admissible order \( >_\psi \). We index \( I_\rho = \{1, \ldots, N_\rho\} \) such that \( \lambda \sim >_\psi \lambda - 1 \). Suppose that there exists \( m \) such that

- for \( i > m \),
  \[ \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \gg 2 \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j}; \]

- \( a_m + b_m \geq a_i + b_i \) for \( i < m \);

- there is no \( i < n \) satisfying

Let

\[ t_m = \min\left\{ \left\lfloor \frac{|a_{m-1} - b_{m-1}|}{2} \right\rfloor \right\} \cup \left\{ \frac{|a_{m-1} - b_{m-1}| - |a_i - b_i|}{2} \mid i < m, \xi_i = \zeta_m \right\} . \]

Here, \([x]\) denotes the greatest integer which is not larger than \( x \). Set \( (a_m^*, b_m^*) \) so that

\[ \max\{a_m^*, b_m^*\} = \max\{a_m, b_m\}, \min\{a_m^*, b_m^*\} = \min\{a_m, b_m\} + 2t_m, \]

and so that \( \zeta_m^* = \zeta_m \).

Define \( \psi^* \) by

\[ \psi^* = \psi - \rho \boxtimes S_{a_m} \boxtimes S_{b_m} + \rho \boxtimes S_{a_m}^* \boxtimes S_{b_m}^* . \]

Proposition 6.8 ([X, Proposition 6.4]). For any \( (\lambda, \eta) \in \Sigma_\psi \), we set \((\lambda^*, \eta^*)\) to be \( \lambda^*_i = \lambda_i \) for \( i \neq m \), \( \lambda^*_m = \lambda_m + t_m \), and \( \eta^*_i = \eta_i \) for any \( i \). Then

\[ \pi_{> \psi}(\psi, (\lambda, \eta)) \neq 0 \iff \pi_{> \psi}(\psi^*, (\lambda^*, \eta^*)) \neq 0. \]
6.6. Reduction operator 3: “Change sign”. In this subsection, we introduce a reduction operator “Change sign”. Choose an admissible order $\triangleright $. We index $I_\rho = \{1, \ldots, N_\rho\}$ such that $i \triangleright _\rho i - 1$. Suppose that there exists $m$ such that

- for $i > m$,
  \[ \rho \boxtimes S_{a_i} \boxtimes S_{b_i} \triangleright m \bigoplus_{j=1}^{m} \rho \boxtimes S_{a_j} \boxtimes S_{b_j}; \]
- $a_1 + b_1 \geq a_i + b_i$ and $\zeta_i \neq \zeta_1$ for $1 < i \leq m$;
- $|a_1 - b_1| \leq 1$.

Set $(a_1^*, b_1^*) = (b_1, a_1) + (|a_1 - b_1|, |a_1 - b_1|)$ and $\zeta_1^* = -\zeta_1$. Define $\psi^*$ by

\[ \psi^* = \psi - \rho \boxtimes S_{a_1} \boxtimes S_{b_1} + \rho \boxtimes S_{a_1^*} \boxtimes S_{b_1^*}. \]

**Proposition 6.9** ([X, Propositions 6.5, 6.6]). Let $(l, \eta) \in \Sigma_\psi$. When $l_1 = d_1/2$, we assume that $\eta_1 = -1$. We set $(l_1^*, \eta_1^*)$ to be $l_i = l_i$, $\eta_i^* = \eta_i$ for $i \neq 1$ and

\[ l_1^* = \begin{cases} l_1 + 1 & \text{if } |a_1 - b_1| = 1, \quad \eta_1 = +1, \\ l_1 & \text{otherwise}, \end{cases} \quad \eta_1^* = \begin{cases} \eta_1 & \text{if } |a_1 - b_1| = 0, \\ -\eta_1 & \text{if } |a_1 - b_1| = 1. \end{cases} \]

Then

\[ \pi_{\triangleright_\psi}(\psi, l, \eta) \neq 0 \iff \pi_{\triangleright_\psi}(\psi^*, l_1^*, \eta_1^*) \neq 0. \]

6.7. Example. We give an example for adapting the algorithm. We again consider $\psi \in \Psi_{gp}(SO_7)$ such that $\psi_d = S_2^{\ominus 3}$. As in Example 6.4, $\Sigma_\psi = \{(0, \eta) \mid \eta \in \{\pm 1\}^3, \eta_1 \eta_2 \eta_3 = 1\}$.

When $\zeta_1 = \zeta_2 = \zeta_3$, by Lemma 6.4, $\pi_{\triangleright_\psi}(\psi, 0, \eta) = 0$ unless $\eta = (+, +, +)$. Since $\Pi_\psi \neq \emptyset$, we have $\pi_{\triangleright_\psi}(\psi, 0, (+, +, +)) \neq 0$.

When $\zeta_1 = \zeta_2 = \zeta \neq \zeta_3$, by Lemma 6.4, $\pi_{\triangleright_\psi}(\psi, 0, \eta) = 0$ unless $\eta_1 = \eta_2$. Assume this condition. Note that $\eta_3 = +1$. We apply Algorithm in §6.1.

**Step 1 for $\psi$:** The parameter $\psi$ is not in the generalized basic case. Setting $m = 3$, go to Step 2.

**Step 2 for $\psi$:** Take $i_0 = 3$. Then $S = \emptyset$ and $\bar{S} = \{3\}$. Go to Step 3c.

**Step 3c for $\psi$:** Proposition 6.8 gives no information. Go to Step 4.

**Step 4 for $\psi$:** Since $S = \emptyset$, go to Step 5.

**Step 5 for $\psi$:** Let $\triangleright_\psi$ be the new order such that $2 \triangleright_\psi 1 > \triangleright_\psi 3$. Then by Proposition 6.5 (3), we have $\pi_{\triangleright_\psi}(\psi, 0, \eta) \equiv \pi_{\triangleright_\psi}(\psi, 0, (-\eta_1, -\eta_2, \eta_3))$. By Proposition 6.9, $\pi_{\triangleright_\psi}(\psi, 0, (-\eta_1, -\eta_2, \eta_3)) \neq 0$ if and only if $\pi_{\triangleright_\psi}(\psi^*, l_1^*, \eta_1^*) \neq 0$, where $\psi^* = \psi - S_{a_1} \boxtimes S_{b_3} \boxtimes S_{a_3+1} \boxtimes S_{b_3+1}$, and $\eta_1^* = (-\eta_1, -\eta_2, -\eta_3) = -\eta$. Go back to Step 1.

**Step 1 for $\psi^*$:** The parameter $\psi^*$ is not in the generalized basic case. Setting $m = 3$, go to Step 2.

**Step 2 for $\psi^*$:** Take $i_0 = 3$. Since $\zeta_1^* = \zeta_2^* = \zeta_3^*$, we have $S = \{1, 2\}$. Go to Step 3a.

**Step 3a for $\psi^*$:** Take $i_0 = 2$. By Proposition 6.9 (2), $\pi_{\triangleright_\psi}(\psi^*, l_1^*, \eta_1^*) \neq 0$ if and only if $\pi_{\triangleright_\psi}(\psi^*, \eta^*, l_1^*) \neq 0$, where $\eta^* = (+, +, +)$ if $\eta = (+, +, +)$, and $\eta^* = (-, -, +)$ if $\eta = (-, -, +)$. By Proposition 6.6, $\pi_{\triangleright_\psi}(\psi^*, l_1^*, \eta_1^*) \neq 0$ if and only if all of $\pi_{\triangleright_\psi}(\psi^*, \eta_1^*, l_1^*)$, $\pi_{\triangleright_\psi}(\psi^*, l_1^*, \eta_1^*)$, $\pi_{\triangleright_\psi}(\psi^*, \eta_1^*, l_1^*)$ are nonzero, where

- $\{a_3^3, b_3^2\} = \{5, 2\}$, $\{a_2^3, b_2^2\} = \{4, 1\}$, $\{a_1^3, b_1^2\} = \{2, 1\}$, and $\zeta_1^* = \zeta_2^* = \zeta_3^* = \zeta$;
• \( \{a_3^3, b_3^3\} = \{7, 2\}, \{a_2^3, b_2^3\} = \{2, 1\}, \{a_1^3, b_1^3\} = \{2, 1\}, \) and \( \zeta_3^3 = \zeta_2^3 = \zeta_1^3 = \zeta; \)
• \( \{a_3^5, b_3^5\} = \{3, 2\}, \{a_2^5, b_2^5\} = \{8, 1\}, \{a_1^5, b_1^5\} = \{2, 1\}, \) and \( \zeta_3^5 = \zeta_2^5 = \zeta_1^5 = \zeta; \)
• \( 2 >_{\psi^3} 3 >_{\psi^2} 1, \ell^2 = 0, \) and \( \bar{\eta}^2 = (+, -, +) \) if \( \bar{\eta} = (-, +, +), \) and \( \bar{\eta}^3 = (-, +, -) \) if \( \eta = (-, -, +). \) Go back to Step 1.

**Step 1 for** \( \psi^3, \psi^2, \psi^1; \) All of \( \psi^3, \psi^2, \) and \( \psi^1 \) are in the generalized basic case. By Proposition 6.2, we see that all of \( \pi >_{\psi} (\psi^3, \ell^*, \eta^{**}), \pi >_{\psi} (\psi^2, \ell^*, \eta^{**}), \) and \( \pi >_{\psi} (\psi^1, \ell^*, \eta^{**}) \) are nonzero. Go to Step 6.

**Step 6 for** \( \psi; \) As a consequence, we see that \( \pi >_{\psi} (\psi, 0, \eta) \neq 0 \) for \( \eta = (+, +, +) \) and \( \bar{\eta} = (-, -, +). \) This is compatible with Example 5.4. (Note that in Example 5.4 (2), we assume that \( \zeta_2 = \zeta_3 = \zeta \neq \zeta_1, \) but now we assume that \( \zeta_1 = \zeta_2 = \zeta \neq \zeta_3. \)
Part 2. The Archimedean case

The theory of A-packets are also established in the archimedean case. In Part 2, we review the archimedean case for $G_n = \text{SO}_{2n+1}$ or $\text{Sp}_{2n}$.

7. A-parameters

In this section, we recall the A-parameters in the archimedean case.

7.1. Weil groups and their representations. The Weil groups of $C$ and $R$ are given by $W_C = C^\times$, $W_R = C^\times \sqcup C^\times j$, respectively, where $j^2 = -1, \ jzj^{-1} = \bar{z}$ for $z \in C^\times$.

Then there exists a canonical exact sequence

$$1 \longrightarrow W_C \longrightarrow W_R \longrightarrow \text{Gal}(C/R) \longrightarrow 1.$$

The norm maps $| \cdot | : W_C \rightarrow R^\times$ and $| \cdot | : W_R \rightarrow R^\times$ are given by $|z| = z\bar{z}$ for $z \in W_C \subset W_R$, and $|j| = 1$. Note that $| \cdot |$ on $W_C = C^\times$ is not the absolute value but the modulus character of $C^\times$ on $C$.

For each integer $k$, we define a unitary character $\chi_k$ of $W_C = C^\times$ by

$$\chi_k(z) = z^{-k}(z\bar{z})^{\frac{k}{2}}$$

for $z \in C^\times$. We sometimes write $\chi_k(z)$ as $\left(z/\bar{z}\right)^{\frac{k}{2}}$, but one has to keep $\chi_k(-1) = (-1)^k$ in mind. Any character of $W_C$ is of the form $| \cdot |^\alpha \chi_k$ for some $\alpha \in C$ and $k \in Z$.

There are exactly two quadratic characters of $W_R$. One is the trivial character $1$, and the other is the sign character $\text{sgn} : W_R \rightarrow \{ \pm 1 \}$ given by $\text{sgn}(j) = -1$ and $\text{sgn}(z) = 1$ for $z \in C^\times$. Any character of $W_R$ is of the form $| \cdot |^\alpha \text{sgn}^\delta$ for some $\alpha \in C$ and $\delta \in \{0, 1\}$. The character $| \cdot |\text{sgn}$ implies an isomorphism

$$W_R^{ab} \cong R^\times, \ j \mapsto -1, \ z \mapsto z\bar{z}.$$

Via this isomorphism, we identify $W_R^{ab}$ with $R^\times$. In particular, any character of $W_R$ is regarded as a character of $R^\times$.

For each integer $k$, we define a 2-dimensional representation $\rho_k : W_R \rightarrow \text{GL}_2(C)$ by

$$\rho_k(j) = \begin{pmatrix} 0 & (-1)^k \\ 1 & 0 \end{pmatrix}, \ \rho_k(z) = \begin{pmatrix} \chi_k(z) & 0 \\ 0 & \chi_k(z) \end{pmatrix}$$

for $z \in C^\times$.

It is the induced representation from the character $\chi_k$ of $W_C$. Note that $\rho_k \cong \rho_{-k}$, $\rho_0 \cong 1 \oplus \text{sgn}$, and that $\rho_k$ is irreducible when $k \neq 0$. Moreover, $\rho_k$ is orthogonal (resp. symplectic) if $k$ is
even (resp. \( k \) is odd). Any irreducible representation of \( W_\mathbb{R} \) is a character or a 2-dimensional representation of the form
\[
| : |^\alpha \rho_k
\]
for some \( \alpha \in \mathbb{C} \) and \( k \in \mathbb{Z} \) with \( k > 0 \).

By abuse of notation, for \( k \in \mathbb{Z} \), we denote by \( \rho_k \) the irreducible (limit of) discrete series representation of \( GL_2(F) \) with minimal \( O(2) \)-type \( \pm(|k| + 1) \).

### 7.2. The case of \( GL_N \).

Let \( F = \mathbb{R} \) or \( F = \mathbb{C} \). A homomorphism
\[
\psi : W_F \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C})
\]
is a **representation** of \( W_F \times SL_2(\mathbb{C}) \) if
- \( \psi|W_\mathbb{C} \) is continuous;
- \( \psi|SL_2(\mathbb{C}) \) is algebraic.

An **A-parameter** for \( GL_N(F) \) is a representation \( \psi : W_F \times SL_2(\mathbb{C}) \to GL_N(\mathbb{C}) \) such that \( \psi(W_F) \) is bounded.

For an A-parameter \( \psi \) for \( GL_N(F) \), one can associate an irreducible unitary representation \( \tau_\psi \) of \( GL_N(F) \) as follows: When \( \psi \) is irreducible, \( \psi \) is of the form \( \psi = \chi \boxtimes S_d \) for some character \( \chi \) of \( W_F \), or \( \psi = \rho_k \boxtimes S_d \) for some \( k > 0 \) (with \( F = \mathbb{R} \)). When \( \psi = \chi \boxtimes S_d \) so that \( N = d \), we set \( \tau_\psi = \chi \circ \det_d \). When \( F = \mathbb{R} \) and \( \psi = \rho_k \boxtimes S_d \) so that \( N = 2d \), we set \( \tau_\psi \) to be the unique irreducible subrepresentation of
\[
\rho_k \cdot |^\frac{d-1}{2} \times \rho_k| \cdot |^\frac{d-3}{2} \times \cdots \times \rho_k| \cdot |^\frac{d-1}{2} .
\]
In general, \( \psi \) can be decomposed into a direct sum
\[
\psi = \psi_1 \oplus \cdots \oplus \psi_r,
\]
where \( \psi_1, \ldots, \psi_r \) are irreducible representations of \( W_F \times SL_2(\mathbb{C}) \). Then we set
\[
\tau_\psi = \tau_{\psi_1} \times \cdots \times \tau_{\psi_r},
\]
which is irreducible.

### 7.3. The case of \( SO_{2n+1} \) and \( Sp_{2n} \).

We denote by \( SO_{2n+1} \) the special orthogonal group with respect to a quadratic space over \( F \) of dimension \( 2n + 1 \). We do not assume here that \( SO_{2n+1} \) is (quasi-)split over \( F \). Namely, when \( F = \mathbb{R} \), the group \( SO_{2n+1}(\mathbb{R}) \) of \( \mathbb{R} \)-points is isomorphic to \( SO(p, q) \) for some \( (p, q) \) with \( p + q = 2n + 1 \).

An **A-parameter** for \( SO_{2n+1} \) is a symplectic representation
\[
\psi : W_F \times SL_2(\mathbb{C}) \to Sp_{2n}(\mathbb{C})
\]
such that \( \psi(W_F) \) is bounded. Similarly, an **A-parameter** for \( Sp_{2n} \) is an orthogonal representation
\[
\psi : W_F \times SL_2(\mathbb{C}) \to SO_{2n+1}(\mathbb{C})
\]
such that \( \psi(W_F) \) is bounded. For \( G = SO_{2n+1} \) or \( G = Sp_{2n} \), we set \( \Psi(G) \) to be the set of equivalence classes of A-parameters for \( G \). We say that \( \psi \in \Psi(G) \) is **tempered** if \( \psi|\{\{1\} \times SL_2(\mathbb{C}) \) is trivial. We denote by \( \Phi_{\text{temp}}(G) \) the subset of \( \Psi(G) \) consisting of tempered A-parameters.

One can define the **component group** \( A_\psi \) of \( \psi \), and the **central element** \( z_\psi \in A_\psi \) as in the \( p \)-adic case. Then \( S_\psi = \pi_0(Z_G(\text{Im}(\psi)))/Z(G)^{W_F} \) is canonically isomorphic to \( A_\psi/\langle z_\psi \rangle \).
As explained in Theorem 0.1 (at least when $G$ is split), for $\psi \in \Psi(G)$, there is an $A$-packet $\Pi_\psi$, which is a finite multiset over $\text{Irr}_{\text{unit}}(G(F))$, together with a map

$$\Pi_\psi \to \hat{A}_\psi, \ \pi \mapsto \langle \cdot, \pi \rangle_\psi$$

enjoying certain (twisted and standard) endoscopic character identities such that $\langle z_\psi, \pi \rangle_\psi = 1$ for any $\pi \in \Pi_\psi$. The purpose of Part 2 is to review a construction of $A$-packets when $F = \mathbb{R}$ or $F = \mathbb{C}$.

8. Complex case

In this section, we consider $F = \mathbb{C}$. Let $G = \text{SO}_{2n+1}$ or $\text{Sp}_{2n}$ over $\mathbb{C}$. We identify $G$ with $G(\mathbb{C})$. Fix a Borel subgroup $B$ of $G$, and a maximal compact subgroup $K$ of $G$. Define

- $T = B \cap K$, a maximal torus in $K$;
- $H = Z_G(T)$, a Cartan subgroup of $G$;
- $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{b} = \text{Lie}(B)$, $\mathfrak{t}_0 = \text{Lie}(K)$, $\mathfrak{t}_0 = \text{Lie}(T)$, $\mathfrak{h} = \text{Lie}(H)$, the Lie algebras;
- $W = W(\mathfrak{g}, \mathfrak{h})$ the Weyl group;
- $a_0 = \sqrt{-1}t_0$, and $A = \exp(a_0)$.

Then $H = TA$, and $(B, H)$ is a Borel pair of $G$. Set $\hat{G} = \text{Sp}_{2n}(\mathbb{C})$ when $G = \text{SO}_{2n+1}$, and $\hat{G} = \text{SO}_{2n+1}(\mathbb{C})$ when $G = \text{Sp}_{2n}$. Fix a Borel pair $(\hat{B}, \hat{H})$ of $\hat{G}$. The Lie algebra of $\hat{G}$ is denoted by $\hat{\mathfrak{g}}$.

8.1. Local Langlands correspondence over $\mathbb{C}$. Recall that an $L$-parameter is a continuous homomorphism

$$\phi : W_\mathbb{C} = \mathbb{C}^\times \to \hat{G}$$

such that the image consists of semisimple elements. Let $\Phi(G)$ be the set of conjugacy classes of $L$-parameters. Taking a conjugation if necessary, we may assume that the image of $\phi$ is contained in $\hat{H}$. Since $\phi$ is semisimple, we can decompose it into a direct sum

$$\phi = | \cdot |^{s_1} \chi_{k_1} \oplus \cdots \oplus | \cdot |^{s_N} \chi_{k_N}$$

with $N = 2n$ or $N = 2n + 1$. We define $\lambda_\phi, \mu_\phi \in X_*(\hat{H}) \otimes \mathbb{C} = X^*(H) \otimes \mathbb{C} = \mathfrak{t}^* \cong \mathbb{C}^N$ by

$$\lambda_\phi = \left( s_1 + \frac{k_1}{2}, \ldots, s_N + \frac{k_N}{2} \right), \quad \mu_\phi = \left( s_1 - \frac{k_1}{2}, \ldots, s_N - \frac{k_N}{2} \right).$$

This satisfies that $\lambda_\phi - \mu_\phi = (k_1, \ldots, k_N) \in X^*(H)$. The map $\phi \mapsto (\lambda_\phi, \mu_\phi)$ gives a canonical bijection

$$\Phi(G) \to \{ (\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^* \mid \lambda - \mu \in X^*(H) \}/\Delta W.$$

For $\lambda, \mu \in \mathfrak{h}^*$ such that $\lambda - \mu \in X^*(H)$, take $\phi : \mathbb{C}^\times \to \hat{H}$ such that $\lambda = \lambda_\phi$ and $\mu = \mu_\phi$. Then $\phi$ can be regarded as a character of $H$. We set $X(\lambda, \mu)$ to be the $K$-finite part of the normalized induction $\text{Ind}_G^H(\phi)$. We call $X(\lambda, \mu)$ the principal series representation with parameter $(\lambda, \mu)$. Define $\overline{X}(\lambda, \mu)$ by the unique irreducible subquotient of $X(\lambda, \mu)$ containing the $K$-representation of extremal weight $\lambda - \mu$. We call $\overline{X}(\lambda, \mu)$ the Langlands subquotient of $X(\lambda, \mu)$. The $W \times W$-orbit of $(\lambda, \mu)$ is called the infinitesimal character of $\overline{X}(\lambda, \mu)$.
**Proposition 8.1** (Zhelobenko). The map \((\lambda, \mu) \mapsto \overline{X}(\lambda, \mu)\) gives a bijection
\[\{ (\lambda, \mu) \in \mathfrak{h}^* \times \mathfrak{h}^* \mid \lambda - \mu \in X^*(H) \}/W \to \text{Irr}(G),\]
where \(\text{Irr}(G)\) is the set of equivalence classes of irreducible \((\mathfrak{g}, K)\)-modules.

**Corollary 8.2** (Local Langlands correspondence over \(\mathbb{C}\)). There exists a canonical bijection
\[\Phi(G) \to \text{Irr}(G), \ \phi \mapsto \overline{X}(\lambda_\phi, \mu_\phi).\]

8.2. **Reduction.** Next, we consider the \(A\)-parameters \(\psi \in \Psi(G)\). To construct the \(A\)-packets \(\Pi_\psi\), we consider a subset \(\Psi_{gp}(G)\) of \(\Psi(G)\). We define that an \(A\)-parameter \(\psi\) belongs to \(\Psi_{gp}(G)\) if \(\psi\) is a sum of irreducible self-dual representations of the same type as \(\psi\). In this case, we say that \(\psi\) is **of good parity**.

Since \(W_\mathbb{C} = \mathbb{C}^\times\) has no irreducible self-dual representations other than \(1\), any parameter \(\psi \in \Psi_{gp}(G)\) is of the form
\[\psi = \bigoplus_{i=1}^t S_{d_i},\]
where \(d_1 \geq \cdots \geq d_t\) are positive even (resp. odd) integers such that \(d_1 + \cdots + d_t = 2n\) (resp. \(d_1 + \cdots + d_t = 2n + 1\)) when \(G = \text{SO}_{2n+1}\) (resp. when \(G = \text{Sp}_{2n}\)).

In general, \(\psi \in \Psi(G)\) can be decomposed into a direct sum
\[\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee,\]
where \(\psi_0\) is an \(A\)-parameter of good parity for a classical group \(G_0\) of the same type as \(G\), and \(\psi_1\) is a sum of irreducible representations of \(W_\mathbb{C} \times \text{SL}_2(\mathbb{C})\) which are not the same type as \(\psi\). In this setting, the \(A\)-packet \(\Pi_\psi\) can be described by \(\Pi_{\psi_0}\).

**Theorem 8.3** ([MR17, Theorem 6.12]). Let \(\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee\) be as above. Then for any \(\pi_0 \in \Pi_{\psi_0}\), the induced representation
\[\tau_{\psi_1} \times \pi_0\]
is irreducible, and does not depend on the choice of \(\psi_1\). The \(A\)-packet \(\Pi_\psi\) is given by
\[\Pi_\psi = \{ \tau_{\psi_1} \times \pi_0 \mid \pi_0 \in \Pi_{\psi_0} \} .\]

8.3. **The case of good parity.** In this subsection, we assume that \(\psi \in \Psi(G)\) is of good parity. In this case, we may regard \(\psi\) as an algebraic representation
\[\psi : \text{SL}_2(\mathbb{C}) \to \widehat{G}.\]
By differential, we obtain a map \(d\psi : \mathfrak{sl}_2(\mathbb{C}) \to \widehat{\mathfrak{g}}\). Taking a conjugation if necessary, we may assume that
\[d\psi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{h} \cong \mathfrak{h}^* .\]
Its \(W\)-orbit is denoted by \(\lambda_\psi \in \mathfrak{h}^*/W\).

We set \(U_\psi\) to be the \(\widehat{G}\)-orbit of
\[d\psi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \widehat{\mathfrak{g}},\]
which is a special nilpotent orbit. By the duality of Lusztig–Spaltenstein, we obtain a nilpotent orbit \(O_\psi \subset \mathfrak{g}\) from \(U_\psi\). Using the Killing form, we may identify \(\mathfrak{g} \cong \mathfrak{g}^*\), and we regard \(O_\psi\) as a nilpotent orbit in \(\mathfrak{g}^*\).
Associated to a representation \( \pi \) of \( G \), one can define a subset in the nilpotent cone in \( \mathfrak{g}^* \), denoted by \( \text{WF}(\pi) \) and called the \textit{wavefront set}. Since we focus on complex groups, when \( \pi \) is irreducible, \( \text{WF}(\pi) \) is the closure of a nilpotent orbit.

**Definition 8.4** (Barbasch–Vogan [BV85]). For \( \psi \in \Psi_{\mathfrak{gp}}(G) \), we define a packet \( \Pi^\text{BV}_\psi \) by the set of irreducible Harish-Chandra modules \( \pi \) of \( G \) with infinitesimal character \((\lambda_\psi, \lambda_\psi)\), and with wavefront set \( \text{WF}(\pi) = \overline{\mathcal{O}_\psi} \).

Let \( A_\psi \) be the component group of \( \psi \). There is a quotient \( \widetilde{A}_\psi \) of \( A_\psi \), called \textbf{Lusztig’s quotient}, such that the packet \( \Pi^\text{BV}_\psi \) is parametrized by the character group \( \widetilde{A}_\psi \), i.e., there is a bijection

\[
\widetilde{A}_\psi \ni \eta \mapsto \pi_\eta \in \Pi^\text{BV}_\psi.
\]

**Theorem 8.5** ([MR17, Theorem 10.1]). Let \( \psi \in \Psi_{\mathfrak{gp}}(G) \). Then we have \( \Pi_\psi = \Pi^\text{BV}_\psi \). Moreover, the map \( \Pi_\psi \ni \pi \mapsto (\cdot, \pi)_\psi \in \widetilde{A}_\psi \) is given by \( \pi_\eta \mapsto \eta \). In particular, this map is injective, and the image of this map is \( \widetilde{A}_\psi \).

Using a deep result of Barbasch [B89] for his classification of the unitary dual of \( G \), one can describe the \( A \)-packet \( \Pi^\text{BV}_\psi \) for \( \psi \in \Psi_{\mathfrak{gp}}(G) \) more precisely. In the rest of this section, we explain this description.

### 8.4. Barbasch–Vogan packets: Type \( B_n \).

In this subsection, we set \( G = \text{SO}_{2n+1} \). Then \( \psi \in \Psi(G) \) is of the form \( \psi = \oplus_{i=1}^{t'} S_{d_i} \), where \( d_i \) is even for any \( i \), and \( \sum_{i=1}^{t'} d_i = 2n \). When \( t' \) is odd, we set \( t = t' \). When \( t' \) is even, we set \( d_t = 0 \) and \( t = t' + 1 \). We may assume that \( d_1 \geq \cdots \geq d_t \).

We define a subalgebra \( \mathfrak{m} \) of \( \mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C}) \) by

\[
\mathfrak{m} = \mathfrak{gl} \left( \frac{d_2 + d_3}{2}, \mathbb{C} \right) \times \cdots \times \mathfrak{gl} \left( \frac{d_{t-1} + d_t}{2}, \mathbb{C} \right) \times \mathfrak{so}_{d_t+1}(\mathbb{C}).
\]

Set \( k = (t - 1)/2 \). For each \( j \in \{1, \ldots, k\} \), let \( F^j \) be the irreducible finite dimensional holomorphic representation of \( \mathfrak{gl}((d_2 + d_{j+1})/2, \mathbb{C}) \) with infinitesimal character

\[
\lambda^j_1 = \left( \frac{d_2 - d_3}{2}, \frac{d_2 - 3}{2}, \ldots, -\frac{d_{j+1} - 3}{2}, -\frac{d_{j+1} - 1}{2} \right).
\]

Namely, \( F^j \) is the 1-dimensional representation given by

\[
F^j: \mathfrak{gl} \left( \frac{d_2 + d_{j+1}}{2}, \mathbb{C} \right) \ni X \mapsto \frac{d_2 - d_{j+1}}{4} \text{tr}(X) \in \mathbb{C}.
\]

When \( d_{2j} > d_{2j+1} \), we define \( \bar{F}^j \) by the irreducible finite dimensional holomorphic representation of \( \mathfrak{gl}((d_2 + d_{j+1})/2, \mathbb{C}) \) with infinitesimal character

\[
\lambda^{j \downarrow}_{-1} = \left( \frac{d_2 - 1}{2}, \ldots, \frac{d_{j+1} + 3}{2}, \frac{d_{j+1} - 1}{2}, \ldots, -\frac{d_{j+1} - 1}{2}, -\frac{d_{j+1} + 1}{2} \right).
\]

Namely, \( \lambda^{j \downarrow}_{-1} \) is obtained from \( \lambda^j_1 \) by the sign change of \((d_{2j+1} + 1)/2\).

For each \( j \in \{1, \ldots, k\} \), we define a representation \( F^j \) of \( \mathfrak{gl}((d_2 + d_{j+1})/2, \mathbb{C}) \) by

\[
F^j(X) = F^j(X) \otimes F^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_2 + d_{j+1})/2, \mathbb{C}).
\]
In addition, when $d_{2j} > d_{2j+1}$, we define a representation $F_{j-1}^j$ of $\mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C})$ by

$$F_{j-1}^j(X) = F^j(X) \otimes \bar{F}^j(X) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C}).$$

Note that for $\epsilon_j \in \{\pm 1\}$ such that $\epsilon_j = 1$ if $d_{2j} = d_{2j+1}$, the representation $F_{j}^j$ of the Lie algebra $\mathfrak{gl}((d_{2j} + d_{2j+1})/2, \mathbb{C})$ can be lifted to an irreducible representation of the Lie group $GL((d_{2j} + d_{2j+1})/2, \mathbb{C})$, which is denoted by the same notation $F_{j}^j$. For example,

$$F_1^j : GL((d_{2j} + d_{2j+1})/2, \mathbb{C}) \ni g \mapsto \left(\det(g)\overline{\det(g)}\right)^{\frac{d_{2j} - d_{2j+1}}{4}} \in \mathbb{C}^\times.$$

Let $P = MN$ be a parabolic subgroup of $G$ with $\text{Lie}(M) = \mathfrak{m}$. Then for $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{\pm 1\}^k$ such that $\epsilon_j = 1$ if $d_{2j} = d_{2j+1}$, we define $\pi_\epsilon$ by the irreducible subquotient of the parabolic induction

$$F_{\epsilon_1}^1 \times \cdots \times F_{\epsilon_k}^k \times 1_{SO_{d_j+1}(\mathbb{C})}$$

containing its minimal $K$-type. If we set

$$\lambda = \left(\lambda_1^\epsilon, \ldots, \lambda_k^\epsilon, \frac{d_1 - 1}{2}, \ldots, \frac{1}{2}\right),$$

$$\mu = \left(\lambda_1^\epsilon, \ldots, \lambda_k^\epsilon, \frac{d_1 - 1}{2}, \ldots, \frac{1}{2}\right),$$

then

$$\pi_\epsilon \cong \overline{X}(\lambda, \mu).$$

The Barbasch–Vogan packet $\Pi^\text{BV}_\psi$ is given by

$$\Pi^\text{BV}_\psi = \left\{ \pi_\epsilon \mid \epsilon \in \{\pm 1\}^k, \epsilon_j = 1 \quad \text{if} \quad d_{2j} = d_{2j+1} \right\}.$$

In particular, $\#\Pi^\text{BV}_\psi = 2^m$ with $m = \#\left\{ j \in \{1, \ldots, k\} \mid d_{2j} \neq d_{2j+1} \right\}$.

**8.5. Barbasch–Vogan packets: Type $C_n$.** In this subsection, we set $G = \text{Sp}_{2n}$. Then $\psi \in \Psi(G)$ is of the form $\psi = \oplus_{i=1}^t S_{d_i}$, where $d_i$ is odd for any $i$, and $\sum_{i=1}^t d_i = 2n + 1$. In particular, we note that $t$ is odd. We may assume that $d_1 \geq \cdots \geq d_t$.

We define a subalgebra $\mathfrak{m}$ of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ by

$$\mathfrak{m} = \mathfrak{gl}\left(\frac{d_1 + d_2}{2}, \mathbb{C}\right) \times \cdots \times \mathfrak{gl}\left(\frac{d_{t-2} + d_{t-1}}{2}, \mathbb{C}\right) \times \mathfrak{sp}_{d_{t-1}}(\mathbb{C}).$$

Set $k = (t - 1)/2$. For each $j \in \{1, \ldots, k\}$, let $F^j$ be the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C})$ with infinitesimal character

$$\lambda^j = \left(\frac{d_{2j-1} - 1}{2}, \frac{d_{2j-1} - 3}{2}, \ldots, -\frac{d_{2j - 3}}{2}, -\frac{d_{2j - 1}}{2}\right).$$

Namely, $F^j$ is the 1-dimensional representation given by

$$F^j : \mathfrak{gl}\left(\frac{d_{2j-1} + d_{2j}}{2}, \mathbb{C}\right) \ni X \mapsto \frac{d_{2j-1} - d_{2j}}{4}\text{tr}(X) \in \mathbb{C}.$$
When $d_{2j-1} > d_{2j}$, we define $\tilde{F}^j_\epsilon$ by the irreducible finite dimensional holomorphic representation of $\mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C})$ with infinitesimal character

$$\lambda^j_{-1} = \left(\frac{d_{2j-1} - 1}{2}, \ldots, \frac{d_{2j} + 3}{2}, \frac{d_{2j} - 1}{2}, \ldots, -\frac{d_{2j} - 1}{2}, -\frac{d_{2j} + 1}{2}\right).$$

Namely, $\lambda^j_{-1}$ is obtained from $\lambda^1_{-1}$ by the sign change of $(d_{2j} + 1)/2$.

For each $j \in \{1, \ldots, k\}$, we define a representation $\mathcal{F}^j_1$ of $\mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C})$ by

$$\mathcal{F}^j_1(X) = F^j(X) \otimes F^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C}).$$

In addition, when $d_{2j-1} > d_{2j}$, we define a representation $\mathcal{F}^j_{-1}$ of $\mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C})$ by

$$\mathcal{F}^j_{-1}(X) = F^j(X) \otimes \overline{F}^j(\overline{X}) \quad \text{for} \quad X \in \mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C}).$$

Note that for $\epsilon_j \in \{\pm 1\}$ such that $\epsilon_j = 1$ if $d_{2j-1} = d_{2j}$, the representation $\mathcal{F}^j_{\epsilon_j}$ of the Lie algebra $\mathfrak{gl}((d_{2j-1} + d_{2j})/2, \mathbb{C})$ can be lifted an irreducible representation of the Lie group $\text{GL}((d_{2j-1} + d_{2j})/2, \mathbb{C})$, which is denoted by the same notation $\mathcal{F}^j_{\epsilon_j}$. For example,

$$\mathcal{F}^j_1 : \text{GL}((d_{2j-1} + d_{2j})/2, \mathbb{C}) \ni g \mapsto \left(\text{det}(g) \overline{\text{det}(g)}\right)^{\frac{d_{2j-1} - d_{2j}}{4}} \in \mathbb{C}^\times.$$

Let $P = MN$ be a parabolic subgroup of $G$ with $\text{Lie}(M) = \mathfrak{m}$. Then for $\epsilon = (\epsilon_1, \ldots, \epsilon_k) \in \{\pm 1\}^k$ such that $\epsilon_j = 1$ if $d_{2j-1} = d_{2j}$, we define $\pi_\epsilon$ by the irreducible subquotient of the parabolic induction

$$\mathcal{F}^{j_1}_\epsilon \times \cdots \times \mathcal{F}^{j_k}_\epsilon \rtimes 1_{\text{Sp}_{d_{2j}}(\mathbb{C})}$$

containing its minimal $K$-type. If we set

$$\lambda = \left(\lambda^1_1, \ldots, \lambda^k_1, \frac{d_1 - 1}{2}, \ldots, 1\right),$$

$$\mu = \left(\lambda^1_{-1}, \ldots, \lambda^k_{-1}, \frac{d_1 - 1}{2}, \ldots, 1\right),$$

then

$$\pi_\epsilon \cong X(\lambda, \mu).$$

The Barbasch–Vogan packet $\Pi^\text{BV}_\psi$ is given by

$$\Pi^\text{BV}_\psi = \left\{ \pi_\epsilon \mid \epsilon \in \{\pm 1\}^k, \; \epsilon_j = 1 \text{ if } d_{2j-1} = d_{2j} \right\}.$$

In particular, $\# \Pi^\text{BV}_\psi = 2^m$ with $m = \# \{j \in \{1, \ldots, k\} \mid d_{2j-1} \neq d_{2j}\}$.

9. Real case

In this section, we consider $F = \mathbb{R}$. Let $G = \text{SO}_{2n+1}$ or $\text{Sp}_{2n}$ over $\mathbb{R}$. 
9.1. Filtration of $A$-parameters. For $\psi \in \Psi(G)$, we define a unitary representation $\psi_d$ of $W_C = \mathbb{C}^\times$ by
\[
\psi_d(z) = \psi \left( z, \begin{pmatrix} (z/\overline{z})^{\frac{1}{2}} & 0 \\ 0 & (z/\overline{z})^{-\frac{1}{2}} \end{pmatrix} \right) \quad \text{for } z \in \mathbb{C}^\times.
\]

Definition 9.1. We define a chain
\[
\Psi(G) \supset \Psi_{gp}(G) \supset \Psi_{vreg}(G) \supset \Psi_{unip}(G)
\]
as follows:

1. $\psi \in \Psi_{gp}(G)$ if $\psi$ is a sum of irreducible self-dual representations of the same type as $\psi$. In this case, we say that $\psi$ is of good parity.
2. $\psi \in \Psi_{vreg}(G)$ if $\psi$ is of good parity and $\psi$ is of the form
\[
\psi = \bigoplus_{i=1}^{r} \rho_{k_i} \otimes S_{d_i} \oplus \bigoplus_{j \in J} \text{sgn}^\delta \otimes S_{d_j}^0
\]
such that $k_i - k_{i+1} \geq d_i + d_{i+1}$ for $1 \leq i < r$ and $k_r \geq d_r + \max_{j \in J} d_j$. In this case, we say that $\psi$ is very regular.
3. $\psi \in \Psi_{unip}(G)$ if $\psi$ is of good parity and $\psi|_{W_C}$ is trivial, i.e., $\psi$ is very regular with $r = 0$. In this case, we say that $\psi$ is unipotent.

We also define $\Psi_{AJ}(G) \subset \Psi_{vreg}(G)$ so that:
4. $\psi \in \Psi_{AJ}(G)$ if $\psi$ is of good parity and $\psi_d$ is multiplicity-free, i.e., $\psi$ is very regular with $\# J \leq 1$. In this case, we say that $\psi$ is Adams–Johnson.

For $\psi \in \Psi_{AJ}(G)$, Adams–Johnson [AJ87] constructed a packet $\Pi_{AJ}$ using derived functor modules $A_q(\lambda)$ for $\lambda$ in the good range. It is called an Adams–Johnson packet. Later, Arancibia–Mœglin–Renard [AMR] showed that $\Pi_\psi = \Pi_{AJ}$, i.e., Arthur’s packets are Adams–Johnson packets when $\psi \in \Psi_{AJ}(G)$.

On the other hand, Mœglin [Mœ17] constructed a packet $\Pi_\psi$ for $\psi \in \Psi_{unip}(G)$ using theta liftings. It is called a unipotent packet. After these works, Mœglin–Renard [MRa] constructed a packet $\Pi_\psi$ for $\psi \in \Psi_{vreg}(G)$ from unipotent packets using cohomological inductions. They showed that this $A$-packet is multiplicity-free.

To extend the $A$-packet to $\psi \in \Psi_{gp}(G)$, Mœglin–Renard [MRb] used the translation principle. However, since this translation must be used while crossing the walls, the translation functor is difficult to understand, and the multiplicity-free result cannot easily be deduced from the case where $\psi$ is very regular.

Finally, for general $\psi$, the packet $\Pi_\psi$ is constructed by irreducible parabolic inductions ([MRa, Proposition 4.3, Théorème 4.4]). In particular, the multiplicity-free result for general $\psi$ is reduced to the case where $\psi$ is of good parity ([MRa, Corollaire 4.5]).

9.2. Adams–Johnson packets. Note that a representation $\psi$ of $W_{\mathbb{R}} \times \text{SL}_2(\mathbb{C})$ is in $\Psi_{AJ}(\text{SO}_{2n+1})$ if and only if
\[
\psi = \bigoplus_{i=1}^{r} \rho_{k_i} \otimes S_{d_i} \oplus \text{sgn}^\delta \otimes S_{d_0},
\]
where

- \( k_i > 0 \) and \( d_i > 0 \) for \( 1 \leq i \leq r \);
- \( k_i + d_i \equiv 0 \mod 2 \) for \( 1 \leq i \leq r \) and \( d_0 \equiv 0 \mod 2 \);
- \( 2 \sum_{i=1}^{r} d_i + d_0 = 2n \);
- \( \delta \in \{0, 1\} \);
- \( k_i - k_{i+1} \geq d_i + d_{i+1} \) for \( 1 \leq i < r \) and \( k_r \geq d_r + d_0 \),

and is in \( \Psi_{A,J}(Sp_{2n}) \) if and only if

\[
\psi = \left( \bigoplus_{i=1}^{r} \rho_{k_i} \boxtimes S_{d_i} \right) \oplus \text{sgn}^\delta \boxtimes S_{d_0},
\]

where

- \( k_i > 0 \) and \( d_i > 0 \) for \( 1 \leq i \leq r \);
- \( k_i + d_i \equiv 1 \mod 2 \) for \( 1 \leq i \leq r \) and \( d_0 \equiv 1 \mod 2 \);
- \( 2 \sum_{i=1}^{r} d_i + d_0 = 2n + 1 \);
- \( \delta \in \{0, 1\} \) such that \( \delta \equiv \sum_{i=1}^{r} d_i \mod 2 \);
- \( k_i - k_{i+1} \geq d_i + d_{i+1} \) for \( 1 \leq i < r \), and \( k_r \geq d_r + d_0 \).

In this subsection, we fix such \( \psi \).

We use the following coordinates for \( SO(p,q) \) and \( Sp_{2n}(\mathbb{R}) \):

\[
SO(p,q) = \left\{ g \in SL_{p+q}(\mathbb{R}) \left| \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix} g = \begin{bmatrix} 1_p & 0 \\ 0 & -1_q \end{bmatrix} \right. \right\},
\]

\[
Sp_{2n}(\mathbb{R}) = \left\{ g \in GL_{2n}(\mathbb{R}) \left| t g \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1_n \\ -1_n & 0 \end{bmatrix} \right. \right\}.
\]

We choose a maximal torus \( T \) of \( G \) defined over \( \mathbb{R} \) such that \( T(\mathbb{R}) \) is compact as follows. When \( G(\mathbb{R}) = SO(p,q) \), setting

\[
r_{\theta} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},
\]

the group \( T(\mathbb{R}) \) consists of the matrices of the form

\[
\begin{pmatrix} r_{\theta_1} & \cdots & 1 \\ \vdots & \ddots & \vdots \\ r_{\theta_{(p-1)/2}} \\ 1 \end{pmatrix}
\]

or

\[
\begin{pmatrix} r_{\theta_1} & \cdots & r_{\theta_{p/2}} \\ \vdots & \ddots & \vdots \\ r_{\theta_{(q-1)/2}} \end{pmatrix}
\]

and is in \( \Psi_{A,J}(Sp_{2n}) \) if and only if

\[
\psi = \left( \bigoplus_{i=1}^{r} \rho_{k_i} \boxtimes S_{d_i} \right) \oplus \text{sgn}^\delta \boxtimes S_{d_0},
\]

where

- \( k_i > 0 \) and \( d_i > 0 \) for \( 1 \leq i \leq r \);
- \( k_i + d_i \equiv 1 \mod 2 \) for \( 1 \leq i \leq r \) and \( d_0 \equiv 1 \mod 2 \);
- \( 2 \sum_{i=1}^{r} d_i + d_0 = 2n + 1 \);
- \( \delta \in \{0, 1\} \) such that \( \delta \equiv \sum_{i=1}^{r} d_i \mod 2 \);
- \( k_i - k_{i+1} \geq d_i + d_{i+1} \) for \( 1 \leq i < r \), and \( k_r \geq d_r + d_0 \).
for $\theta_i, \theta'_j \in \mathbb{R}$ according to $p \neq q \equiv 0 \mod 2$ or $p \neq q \equiv 1 \mod 2$. When $G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R})$, we set

$$T(\mathbb{R}) = \left\{ \begin{pmatrix} a_1 & \cdots & b_1 \\ \vdots & \ddots & \vdots \\ -b_1 & \cdots & a_n \\ -b_n & \cdots & a_1 \end{pmatrix} \in \mathbb{R}^{n \times n} \mid a_i, b_i \in \mathbb{R}, a_i^2 + b_i^2 = 1 \right\}.$$ 

Let $\theta$ be the Cartan involution given by

$$\theta: g \mapsto t_0 gt_0^{-1}$$

where $t_0 = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$ if $G(\mathbb{R}) = \text{SO}(p,q)$, and $t_0 = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ if $G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R})$. Note that $t_0 \in T(\mathbb{R})$ such that $t_0^2 \in Z(\text{G}(\mathbb{R}))$. Take a $\theta$-stable Borel subgroup $B$ of $G$ containing $T$.

Let $\Sigma_\psi$ be the set of matrices

$$\begin{pmatrix} p_1 & q_1 \\ \vdots & \vdots \\ p_r & q_r \end{pmatrix}$$

such that

- $p_i$ and $q_i$ are non-negative integers for $i = 1, \ldots, r$;
- $p_i + q_i = d_i$ for $i = 1, \ldots, r$;
- $2 \sum_{i=1}^r p_i \leq p$ and $2 \sum_{i=1}^r q_i \leq q$ if $G(\mathbb{R}) = \text{SO}(p,q)$ with $p + q = 2n + 1$.

Note that there exists a canonical bijection

$$\Sigma_\psi \cong \begin{cases} \mathcal{G}_{d_0/2} \times \left( \prod_{i=1}^r \mathcal{G}_{d_i} \right) \backslash \mathcal{G}_n / \mathcal{G}_{[p/2]} \times \mathcal{G}_{[q/2]} & \text{if } G(\mathbb{R}) = \text{SO}(p,q), \\
\prod_{i=1}^r \mathcal{P}_2(d_i) & \text{if } G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R}), \end{cases}$$

where $[x]$ denotes the greatest integer which is not larger than $x$, and $\mathcal{P}_2(d_i)$ is the set of pairs of integers $(p_i, q_i)$ with $p_i, q_i \geq 0$ such that $p_i + q_i = d_i$.

For $w = (p_i, q_i) \in \Sigma_\psi$, we take a $\theta$-stable parabolic subgroup $Q_w = L_w U_w$ of $G$ containing $B$ such that the Levi $L_w$ is defined over $\mathbb{R}$, and its $\mathbb{R}$-points $L_w(\mathbb{R})$ is given as follows: Define
\( \iota : \text{M}_{p,q}(\mathbb{C}) \to \text{M}_{2p,2q}(\mathbb{R}) \) by

\[
\iota \left( \begin{pmatrix} x_{1,1} & \cdots & x_{1,q} \\ \vdots & \ddots & \vdots \\ x_{p,1} & \cdots & x_{p,q} \end{pmatrix} \right) + \sqrt{-1} \left( \begin{pmatrix} y_{1,1} & \cdots & y_{1,q} \\ \vdots & \ddots & \vdots \\ y_{p,1} & \cdots & y_{p,q} \end{pmatrix} \right) = \left( \begin{array}{ccc} x_{1,1} & y_{1,1} & \cdots & x_{1,q} & y_{1,q} \\ -y_{1,1} & x_{1,1} & \cdots & -y_{1,q} & x_{1,q} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{p,1} & y_{p,1} & \cdots & x_{p,q} & y_{p,q} \\ -y_{p,1} & x_{p,1} & \cdots & -y_{p,q} & x_{p,q} \end{array} \right)
\]

for \( x_{i,j}, y_{i,j} \in \mathbb{R} \). We put

\[
\eta = \begin{pmatrix} 1 & -1 \\ \vdots & \ddots & \vdots \\ 1 & -1 \end{pmatrix} \in \text{O}(2q).
\]

Note that \( \eta \iota(d)\eta^{-1} = \iota(\bar{d}) \) for \( d \in \text{M}_q(\mathbb{C}) \). When \( G(\mathbb{R}) = \text{SO}(p, q) \), setting \( p_0 = p - 2 \sum_{i=1}^r p_i \) and \( q_0 = q - 2 \sum_{i=1}^r q_i \) so that \( p_0 + q_0 = d_0 + 1 \), the group \( L_w(\mathbb{R}) \) consists of the matrices of the form

\[
h = \begin{pmatrix} \iota(a_1) & \ldots & \iota(b_1)\eta^{-1} \\ \vdots & \iota(a_r) & \ldots & \iota(b_r)\eta^{-1} \\ \eta(c_1) & \ldots & \eta(d_1)\eta^{-1} \\ \vdots & \eta(c_r) & \ldots & \eta(d_r)\eta^{-1} \end{pmatrix}
\]

for \( a_i \in \text{M}_{p_i,p_i}(\mathbb{C}), b_i \in \text{M}_{p_i,q_i}(\mathbb{C}), c_i \in \text{M}_{q_i,p_i}(\mathbb{C}), d_i \in \text{M}_{q_i,q_i}(\mathbb{C}) \) such that \( \begin{pmatrix} a_i \\ c_i \end{pmatrix} \in \text{U}(p_i, q_i) \) for \( i = 1, \ldots, r \), and \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SO}(p_0, q_0) \). When \( G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R}) \), the group \( L_w(\mathbb{R}) \) consists of the matrices of the form

\[
h = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_1' & -b_1' \\ c_1' & -d_1' \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
where \( \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} + \sqrt{-1} \begin{pmatrix} a'_j & b'_j \\ c'_j & d'_j \end{pmatrix} \in U(p_i, q_i) \) for \( i = 1, \ldots, r \), and \( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{d_0 - 1}(\mathbb{R}) \). Note that \( L_w \) contains \( T \), and

\[
L_w(\mathbb{R}) \cong U(p_1, q_1) \times \cdots \times U(p_r, q_r) \times \begin{cases} \mathrm{SO}(p_0, q_0) & \text{if } G(\mathbb{R}) = \mathrm{SO}(p, q), \\ \mathrm{Sp}_{d_0 - 1}(\mathbb{C}) & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}). \end{cases}
\]

Set

\[
\lambda^j = \left( \frac{k_j + d_j - 1}{2}, \frac{k_j + d_j - 3}{2}, \ldots, \frac{k_j - d_j + 1}{2} \right) \in \left( \frac{1}{2} \mathbb{Z} \right)^{d_j}
\]

and

\[
\lambda^j = \left\{ \begin{array}{ll}
\left( \lambda^j, \frac{d_0 - 1}{2}, \frac{d_0 - 3}{2}, \ldots, \frac{1}{2} \right) & \text{if } G = \mathrm{SO}_{2n + 1}, \\
\left( \lambda^j, \frac{d_0 - 1}{2}, \frac{d_0 - 3}{2}, \ldots, 1 \right) & \text{if } G = \mathrm{Sp}_{2n}.
\end{array} \right.
\]

Let \( \rho \) be the half sum of positive roots of \( T \) with respect to \( B \), so that

\[
\rho = \left\{ \begin{array}{ll}
\left( n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2} \right) & \text{if } G = \mathrm{SO}_{2n + 1}, \\
(n, n - 1, \ldots, 1) & \text{if } G = \mathrm{Sp}_{2n}.
\end{array} \right.
\]

Fix \( w = (p_i, q_i)_i \in \Sigma_\psi \). Define a unitary character \( \chi_\lambda \psi : L_w(\mathbb{R}) \to \mathbb{C}^\times \) as follows: If \( h \in L_w(\mathbb{R}) \) is of the above form, we set

\[
\chi_\lambda \psi(h) = \left\{ \begin{array}{ll}
\text{spin}^\delta \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \prod_{j=1}^r \det \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \lambda^j - n + \frac{1}{2} + (d_1 + \cdots + d_j - 1) \right) & \text{if } G(\mathbb{R}) = \mathrm{SO}(p, q), \\
\prod_{j=1}^r \det \left( \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} + \sqrt{-1} \begin{pmatrix} a'_j & b'_j \\ c'_j & d'_j \end{pmatrix} \right) \lambda^j - n + (d_1 + \cdots + d_j - 1) & \text{if } G(\mathbb{R}) = \mathrm{Sp}_{2n}(\mathbb{R}),
\end{array} \right.
\]

where \( \text{spin} : \mathrm{SO}(p_0, q_0) \to \{ \pm 1 \} \) is the spinor character. When \( p_0q_0 = 0 \), we interpret spin to be the trivial character. The restriction of \( \chi_\lambda \psi \) to \( T(\mathbb{R}) \) is equal to \( \lambda^j - \rho \in X^+(T) \cong \mathbb{Z}^n \).

For each \( w \in \Sigma_\psi \), we define

\[
\pi_w = A_{Q_w}(\chi_\lambda \psi)
\]

to be the derived functor modules. Since the character \( \chi_\lambda \psi \) is in the good range, i.e., \( \text{Re} \langle \lambda \psi, \alpha \rangle > 0 \) for any simple root \( \alpha \) of \( T \) appearing \( \text{Lie}(U_w) \), it is nonzero and irreducible with infinitesimal character \( \lambda^j \psi \). Moreover, \( \pi_w \not\cong \pi_{w'} \) if \( w \neq w' \).

**Definition 9.2** (Adams–Johnson [AJ87]). For \( \psi \in \Psi_{AJ}(G_n) \), we define a packet \( \Pi_{\psi}^{AJ} \) by

\[
\Pi_{\psi}^{AJ} = \{ \pi_w \mid w \in \Sigma_\psi \}.
\]

Arancibia–Mœglin–Renard identified Adams–Johnson packets with Arthur’s one.

**Theorem 9.3** ([AMR]). For \( \psi \in \Psi_{AJ}(G) \), we have

\[
\Pi_\psi = \Pi_{\psi}^{AJ}.
\]
The map $\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \hat{A}_\psi$ is determined by

$$\langle z_\psi, \pi_w \rangle_\psi = \begin{cases} \langle -1 \rangle_\psi^{[\frac{p+q}{2}]} & \text{if } G(\mathbb{R}) = \text{SO}(p, q), \\ 1 & \text{if } G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R}) \end{cases}$$

and

$$\langle \alpha_i, \pi_w \rangle_\psi = (-1)^{\frac{p_i - q_i - \delta_i}{2}}$$

for $w = (p_i, q_i) \in \Sigma_\psi$ and $\alpha_i \in A_\psi$ associated to $\rho_{\psi_i} \boxtimes S_{d_i}$, where

$$\delta_i = \begin{cases} 0 & \text{if } d_i \equiv 0 \mod 2, \\ (-1)^{\sum_j d_j_i} & \text{if } d_i \equiv 1 \mod 2. \end{cases}$$

Remark 9.4. We remark that

$$\sum_{\substack{p+q=2n+1 \\text{p is odd}}} \# \left( \mathcal{G}_{d_i} \times \left( \prod_{i=1}^{r} \mathcal{S}_{d_i} \right) / \mathcal{G}_{d_i} \times \mathcal{S}_{\frac{r}{2}} \right) = \left( \frac{d_0}{2} + 1 \right) \prod_{i=1}^{r} (d_i + 1),$$

$$\prod_{i=1}^{r} \mathcal{P}_2(d_i) = \prod_{i=1}^{r} (d_i + 1).$$

One might regard these computations as an analogue to Lemma 4.2. Hence $A$-parameters with DDR may be regarded as a $p$-adic analogue of “Adams–Johnson” parameters.

As an example, we consider $\psi = \phi \in \Psi_{A_1}(G) \cap \Phi(G)$, i.e., the case where $d_1 = \cdots = d_r = 1$ and $d_0 \in \{0, 1\}$ so that $r = n$. In this case, for any $w = (p_i, q_i) \in \Sigma_\psi$, we have $(p_i, q_i) \in \{(1, 0), (0, 1)\}$. Moreover, the Levi $L_w$ is just the fixed maximal torus $T$. The irreducible representation $\pi_w$ is the discrete series representation of Harish-Chandra parameter

$$\left( \lambda_{i_1}, \ldots, \lambda_{i_{p_0}}, \lambda_{j_1}, \ldots, \lambda_{j_{q_0}} \right)$$

if $G(\mathbb{R}) = \text{SO}(p, q)$,

$$\left( (p_1 - q_1)\lambda_1, \ldots, (p_n - q_n)\lambda_n \right)$$

if $G(\mathbb{R}) = \text{Sp}_{2n}(\mathbb{R})$,

where we write $\lambda_\psi = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$, and we define $i_1 < \cdots < i_{p_0}$ and $j_1 < \cdots < j_{q_0}$ so that $\{i_1, \ldots, i_{p_0}\} = \{i \in \{1, \ldots, n\} \mid p_i = 1\}$ and $\{j_1, \ldots, j_{q_0}\} = \{j \in \{1, \ldots, n\} \mid q_j = 1\}$. Hence $p_0 = \lfloor p/2 \rfloor$ and $q_0 = \lfloor q/2 \rfloor$. The pairing $\langle \cdot, \pi_w \rangle_\phi$ is given by

$$\langle \alpha_i, \pi_w \rangle_\phi = (-1)^{i-1}(p_i - q_i)$$

for $i = 1, \ldots, n$. 
References


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