Principal-agent models in economics

• Corporate finance
  – Capital structure and financing constraints
  – Executive compensation

• Macroeconomics / public finance
  – Taxation / insurance / incentives

• Personnel economics
  – Labor contracts

• Industrial organizations
Example: Optimal taxation
Dynamic Mirrlees model

- Werning, Farhi, Golosov, Tsyvinski
- Agent’s skill $\theta$ evolves over time (randomly, private info)
- Income $f(\text{effort, skill})$
- Gov’t wants to provide insurance, worries about incentives
- Revelation principle: agent reports $\theta$
- Incentives: agent’s deviation payoff matters (agent faces some taxes / transfers from reporting $\theta$, what is the payoff of another type who pretends to be $\theta$)?
Dynamic price discrimination (Battaglini 05)

• Buyer’s valuation $\theta$ private information
• Buyer reveals current valuation (through purchase decision)
• Seller makes future offers based on what the buyer has revealed
• Again, incentives to pretend to be another type matter
Long-lived private information

• **private information** $\theta$ can be long-lived
  – Public finance – skill is persistent (Farhi-Werning 2013)
  – Effort can shape principal’s belief about skill/profitability (Holmstrom career concerns, He-Wei-Yu-Gao 2016, DeMarzo-Sannikov 2017)

• **private information** can result from past hidden actions
  – Effort has lasting effect on profitability (Sannikov 2014)
  – Short-termism (boost today’s performance at future expense)

• These problems look different, but have much in common
Information rents

• Myerson 81: agent’s rents from being higher type $\theta$

• Fernandes and Phelan (2000):
  design dynamic contracts based on agent’s entire deviation value function $U(\theta)$
  tractability (?)

• First-order approach: account for only local incentives, $U'(\theta)$, hope for the best
  – Werning 01, Williams 06, Pavan-Segal-Toikka 14
General intuition

Distortions (give proper payoff for the intended type, but lower incentives to deviate / payoffs of other types)

If we did not have to worry about deviating types’ payoffs, we could do better

What are some examples of how to do it? And how to do it optimally?
Optimal asset management contracts with hidden savings

joint with Sebastian Di Tella (Stanford GSB)
Intro

• Optimal contract for financial intermediaries
  – Manage risky capital: classic portfolio problem
  – CRRA preferences over consumption
  – Hidden action: secretly divert funds
  – Hidden savings (self insure, undo incentive scheme)
What is interesting about this setting

• Connected to classic portfolio choice theory
• Scale invariance helps understand otherwise complex dynamics
• Precautionary motive worsens incentives
• Agent’s private info is about savings – agent’s private value of savings matter
• Future distortions improve incentives incentives incentives
  ex-ante / mitigate precautionary motive
• First-order conditions vs. full value function
Technology

- Principal gives agent capital $k_t$, agent generates return

$$dR_t = (\alpha + r - a_t)dt + \sigma \, dZ_t$$

- $a_t \geq 0$ is “stealing”. Hidden savings

$$dh_t = (rh_t + c_t - \hat{c}_t + \phi k_t a_t) dt, \quad \phi \leq 1$$

- $h_t \geq 0$
  - in the optimal contract without savings, agent wishes he could save
  - principal can try to push the agent to the borrowing limit

- Agent’s utility

$$E \left[ \int_0^\infty e^{-\rho t} \frac{\hat{c}_t^{1-\gamma}}{1-\gamma} \, dt \right]$$
Space of contracts, principal’s problem

- Contract \{c, k\} contingent on history of returns; full commitment
- Optimal to enforce \( a = 0 \), without loss of generality consider contracts that give the agent incentives not to save
- Minimize cost of compensation

\[
\min_{\{c,k\}} \quad E \left[ \int_0^\infty e^{-rt} (c_t - k_t \alpha) \, dt \right]
\]

s.t.

\[
U^{a=0, h=0} \geq W_0, \quad a_t = 0, \quad h_t = 0 \quad \text{are optimal for agent}
\]

- If competitive principals, \( W_0 \) set so principal breaks even
  - cost of compensation = agent’s initial savings
Some other questions

• What if the agent can choose $h < 0$ and/or $a < 0$ (i.e. borrow or boost returns)?
• What if cash diversion technology is nonlinear?
• What if the agent can also save secretly in his risky technology?
Autarky

Agent allocates wealth between risk-free asset, rate $r$, and risky asset with return

$$dR_t = (\alpha + r) dt + \sigma \, dZ_t,$$

Sharpe ratio $\pi = \frac{\alpha}{\sigma}$

What is the optimal portfolio weight on the risky asset?
Autarky

Agent allocates wealth between risk-free asset, rate r, and risky asset with return

\[ dR_t = (\alpha + r)dt + \sigma \, dZ_t, \quad \text{Sharpe ratio } \pi = \frac{\alpha}{\sigma} \]

What is the optimal portfolio weight on the risky asset?
Wealth volatility at optimal portfolio is \( \sigma^n = \pi/\gamma \), wealth follows

\[ \frac{dn_t}{n_t} = \lambda \, dR_t + (1 - \lambda) r \, dt - \frac{c_t}{n_t} \, dt, \quad \lambda \sigma = \frac{\pi}{\gamma} = \frac{\alpha}{\gamma \sigma} \]

Optimal consumption rate

\[ \frac{c_t}{n_t} = \frac{\rho + (\gamma - 1)r}{\gamma} + \frac{\gamma - 1}{2} (\sigma^n)^2 \]
Autarky

Agent allocates wealth between risk-free asset, rate $r$, and risky asset with return

$$dR_t = (\alpha + r) dt + \sigma \, dZ_t,$$
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$$\lambda \sigma = \frac{\pi}{\gamma} = \frac{\alpha}{\gamma \sigma}$$

Optimal consumption rate

$$\frac{c_t}{n_t} = \frac{\rho + (\gamma - 1)r}{\gamma} + \frac{\gamma - 1}{2}(\sigma^n)^2$$

Autarky outcome is suboptimal if $\phi < 1$ (principal can give insurance) … and even if $\phi = 1$
Simple insurance contracts (take $\phi = 1$)

Principal specifies $\lambda$ and swaps fraction $1 - \chi$ of realized (stochastic) growth in $n_t$ for risk-free growth $g$

$$\frac{dn_t}{n_t} = \chi \left( \lambda \, dR_t + (1 - \lambda) r \, dt - \frac{c_t}{n_t} \, dt \right) + (1 - \chi) g \, dt$$

Under what conditions (1) the principal breaks even, agent does not (2) divert funds or (3) save?

Use formulas from the previous slide,
- the agent’s wealth is $n_t/\chi + h_t$
- weight on hidden savings must be 0,
- consumption rate must be $c/(n/\chi + h)$
- given those, principal must break even
Simple insurance contracts (take $\phi = 1$)

Principal specifies $\lambda$ and swaps fraction $1 - \chi$ of realized (stochastic) growth in $n_t$ for risk-free growth $g$

$$\frac{dn_t}{n_t} = \chi \left( \lambda dR_t + (1 - \lambda) r dt - \frac{c_t}{n_t} dt \right) + (1 - \chi) g dt$$

Under what conditions (1) the principal breaks even, agent does not (2) divert funds or (3) save?

Agent’s perspective:

$$\frac{1}{\gamma} \frac{\chi \lambda \alpha + (1 - \chi)(g - r)}{\chi \lambda \sigma / \sigma^n} = \chi \lambda \sigma / \sigma^n$$

$$\frac{c_t}{n_t / \chi} = \rho + (\gamma - 1)r + \frac{\gamma - 1}{2} (\sigma^n)^2$$

$$g = \lambda \alpha + r - \frac{c}{n}$$
Is simple insurance better than autarky?

Mechanism designer’s perspective:

\[
\frac{dn_t}{n_t} = \left( \lambda \alpha + r - \frac{c_t}{n_t} \right) dt + \chi \lambda \sigma \ dt,
\]

as if the agent has higher Sharpe ratio \(\alpha/(\chi \sigma)\) (1st-order benefit) but with \(\lambda, c/n\) distorted from moral hazard (2nd-order cost)

Yes, \(\chi < 1\) (at least slightly) better than \(\chi = 1\).

Remark: simple insurance contract is fully incentive compatible even if the agent can choose \(a < 0\) and \(h < 0\) (i.e. raise returns or use hidden borrowing)
Distortions...
Distortions…

Hard to talk about here because it’s a static contract…

Incentives and distortions are mixed in together
Towards the optimal contract:
recursive structure subject to first-order incentive constraints
Continuation value and local IC

Continuation value

\[
dU_t = \left( \rho U_t - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \left( (dR_t - (\alpha + r) \ dt) \right)
\]

promise keeping

Incentive compatibility: “skin in the game”

\[
\Delta_t \geq c_t^{-\gamma} \phi k_t
\]

This is costly because the agent is risk averse

Remark: \(c_t\) affects incentives, but depends on precautionary motive

when agent can save: agent may divert resources and save
Euler equation

Since the agent can have hidden savings, $e^{-(\rho-r) t} c_t^{-\gamma}$ must be a supermartingale under $a = 0$

$$\frac{dc_t}{c_t} = \left(\frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_t^c)^2\right) dt + \sigma_t^c dZ_t + dL_t$$

- Precautionary motive for savings
- This is necessary, but what about double deviations?
  - Steal and consume later
  - We need to verify global IC
Exploiting scale invariance

- Use state variables
  \[ x_t = ((1 - \gamma)U_t)^{\frac{1}{1-\gamma}} > 0 \]
  \[ \hat{c}_t = \frac{c_t}{x_t} \]
  Captures the agent’s precautionary motive

- The agent’s consumption reflects precautionary savings motive
- In stationary insurance contract
  \[ \hat{c} = \left( \frac{\rho + (\gamma - 1)r}{\gamma} + \frac{\gamma - 1}{2}(\sigma^n)^2 \right)^{\frac{1}{1-\gamma}} \]
  ... decreasing in \( \sigma^n \)

- Upper bound (over all contracts)
  \[ \hat{c}_h = \left( \frac{\rho + r(\gamma - 1)}{\gamma} \right)^{\frac{1}{1-\gamma}} \]
Incentives and distortions

Power of incentives

\[ \Delta_t = \frac{dU_t}{dR_t} \]

Incentive not to “steal”

\[ \Delta_t \geq c_t^{-\gamma} \phi k_t \]

Depends on \( c_t \), which depends on the precautionary motive
Features of the optimal contract

Incentive not to “steal”

\[ \Delta_t \geq c_t^{-\gamma} \phi k_t \quad \text{or} \quad \sigma_t^x \geq \hat{c}_t^{-\gamma} \frac{\phi k_t}{x_t} \sigma \]

Initially, agent gets more capital relative to autarky, \( \sigma_0^x > \frac{1}{\gamma} \frac{\alpha}{\phi \sigma} \) Sharpe ratio

Incentives maintained through future distortions (\( \sigma_t^x < \sigma_0^x \))

… especially after bad outcomes, portfolio weight on capital goes down
Formal recursive problem

After some algebra, from the laws of motion of $U$ and $c$, we obtain

$$
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma_t^x)^2 \right) dt + \sigma_t^x dZ_t
$$

$$
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t \hat{c} + \frac{1 + \gamma}{2} (\sigma_t^r)^2 - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right) dt + \sigma_t \hat{c} dZ_t + dL_t
$$

The IC constraint is binding

$$
\sigma_t^x \geq \hat{c}_t^{r} \frac{k_t}{\phi \sigma} \quad \Rightarrow \quad \frac{\alpha k_t}{\text{fund profit rate}} = \sigma_t^x \hat{c}_t^r x_t \frac{\alpha}{\phi \sigma}
$$

Promised safety (higher $\hat{c}_t$) relaxes IC
Principal’s problem:

Flow cost

\[ c_t - \alpha k_t = \hat{c}_t x_t - \sigma_t^{x} \hat{c}_t^\gamma x_t \frac{\alpha}{\phi \sigma} \]

State variables

\[ \frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma_t^{x})^2 \right) dt + \sigma_t^{x} dZ_t \]

\[ \frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{(\sigma_t^{x})^2}{2} + \gamma \sigma_t^{x} \sigma_t^\gamma + \frac{1 + \gamma}{2} (\sigma_t^\gamma)^2 - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} \right) dt + \sigma_t^\gamma dZ_t \]

Controls \( \sigma_t^{x}, \sigma_t^\gamma \)

- Plus free choice: promise of safety \( \hat{c}_0 \) sets “budget” \([\hat{c}_0, \hat{c}_h]\) for future risk exposure
- Trade-off: higher initial \( \hat{c}_0 \) improves incentives initially, but requires future distortions
Three effects of risk exposure

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} \left( \sigma_t^x \right)^2 \right) dt + \sigma_t^x dZ_t
\]

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{\left( \sigma_t^x \right)^2}{2} + \gamma \sigma_t^x \sigma_t^x + \frac{1 + \gamma}{2} \left( \sigma_t^x \right)^2 - \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} \right) dt + \sigma_t^x dZ_t + dL_t
\]

\(\sigma_t^x = \hat{c}_t^{-\gamma} \frac{k_t}{x_t} \phi \sigma\) \iff \(\alpha k_t = \sigma_t^x \hat{c}_t^\gamma x_t \frac{\alpha}{\phi \sigma}\)

(1) Risk exposure raises output

(2) Compensation for risk

(3) “Promised safety” constraint: make contract safer in the future to maintain promised precautionary motive
Mitigating precautionary motive

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma (\sigma_t^x)^2}{2} \right) dt + \sigma_t^x dZ_t
\]

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{(\sigma_t^x)^2}{2} + \gamma \sigma_t^x \sigma_t \hat{c}_t + \frac{1}{2} \frac{1 + \gamma (\sigma_t^\hat{c})^2}{1 - \gamma} - \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} \right) dt + \sigma_t^\hat{c} dZ_t + dL_t
\]

promise of safety

higher \( \sigma_t^x \) uses the "risk exposure" budget

but \( \sigma_t^\hat{c} < 0 \) mitigates the effect of risk exposure (agent insured after bad outcomes)

also \( \sigma_t^\hat{c} < 0 \) adds to risk exposure budget when \( x_t \) goes up
The HJB equation

Cost function \( v(x, \hat{c}) = \hat{v}(\hat{c})x \)

\[
\begin{align*}
    r\hat{v}(\hat{c}) &= \min_{\sigma^x, \sigma^\hat{c}} \hat{c} - \sigma^x \hat{c}^\gamma x_t \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right) + \\
    \hat{v}'(\hat{c}) &= \left( \frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} + (1+\gamma)\sigma^\hat{c} \left( \sigma^x + \frac{\sigma^\hat{c}}{2} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} \right) \right) + \hat{v}''(\hat{c}) \frac{(\sigma^\hat{c})^2}{2}
\end{align*}
\]

Special contracts: (1) optimal
(2) deterministic (set \( \sigma_t^\hat{c} = 0 \) choose \( \sigma_t^x \) optimally)
(3) stationary (set \( \sigma_t^\hat{c} = 0 \), \( \sigma_t^x \) to keep \( \hat{c}_t \) fixed)
    - autarky benchmark, set \( \sigma_t^x = \alpha / (\gamma \phi \sigma) \)
(4) without hidden savings, make \( \hat{c} \) a control

\[
    r\hat{v} = \min_{\sigma^x, \hat{c}} \hat{c} - \sigma^x \hat{c}^\gamma x_t \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right)
\]
Cost to attain unit utility ($x = 1$)

$\gamma = 1/3,$

$r = \rho = 5\%,$

$\alpha = 1.7\%,$

$\phi = 0.5$

$\sigma = 40\%$

---

The graph shows a cost function with the following features:

- Autarky
- Stationary contracts
- Optimal contract with savings
- Optimal contract, no hidden savings

The parameters used in the graph are:

- $r = \rho = 5\%$
- $\alpha = 1.7\%$
- $\phi = 0.5$
- $\sigma = 40\%$
Optimal contract properties

• Cost function minimized at $\hat{c}_l \in (0, \hat{c}_h)$, strictly increasing on $[\hat{c}_l, \hat{c}_h]$ and flat on $[0, \hat{c}_l]$ with $v'(\hat{c}_l) = 0$

• Initially, $\mu_0^\hat{c} > 0$ and $\sigma_0^\hat{c} = 0$, $\sigma_0^x$ maximizes

$$\sigma^x \hat{c}^y x_t \frac{\alpha}{\phi \sigma} - \hat{v} \gamma (\sigma^x)^2$$

(*)

• After time 0, on $(\hat{c}_l, \hat{c}_h)$, $\sigma_t^\hat{c} < 0$, $\sigma_t^x > 0$ is lower
Risk reduction and precautionary motive over time
Risk reduction after bad outcomes
Verifying global IC

• Our constraints prevent
  – stealing and immediately consuming
  – saving compensation to consume later

• But what if the agent steals and saves it for later?

• Stealing raises the marginal utility of future consumption

\[ E_t^a \left[ e^{(r-\rho)(s-t)} c_{s-\gamma} \right] \geq c_t^{-\gamma} \]

so this could be really attractive

• But there’s hope: the agent’s marginal utility decreases when he consumes extra diverted amount
Validity of the first-order approach

• **Theorem**: Consider any contract that satisfies the first-order constraints and has the property

\[ \forall t, \sigma_t^c < 0. \]

• Then the agent’s deviation utility is bounded from above by

\[ x_t(h_t) \leq x_t + h_t \hat{c}_t^\gamma. \]

• In particular, if \( h_t = 0 \), the agent can’t get \( > x_t \) by deviating.
Intuition

• More risk $\implies$ hidden savings are worth more

• If risk went up after bad outcomes, stealing and saving may be attractive: stealing makes bad outcomes more likely

• But $\sigma_t^\hat{c} < 0$ guarantees that risk and precautionary motive go down after bad outcomes

• Remark: sufficient condition $\sigma_t^\hat{c} < 0$ identifies a whole class of IC contracts (including optimal and others e.g. stationary)
Utility bound and distortions

- Principal pushes the bound

\[ x_t + h_t \hat{c}_t^{-\gamma} \]

down by distortions that reduce the agent’s risk and raise \( \hat{c}_t \).
Extensions

- Market with return

\[ d\tilde{R}_t = r\, dt + \tilde{\sigma} \left( \tilde{\pi} \, dt + d\tilde{Z}_t^P \right) = r\, dt + \tilde{\sigma} \, d\tilde{Z}_t^O \]

Sharpe ratio

- Agent’s strategy market-neutral wlog

\[ dR_t = (\alpha + r - a_t)\, dt + \sigma \, dZ_t \]

orthogonal to market

- Agent may save in the market, maybe even his strategy

\[ dh_t = (rh_t + c_t - \hat{c}_t + \phi k_t a_t)\, dt \\
+ \tilde{x}_t (d\tilde{R}_t - r\, dt) + x_t ((\alpha + r)\, dt + \sigma \, dZ_t - r\, dt) \]
The HJB equation

\[ r \hat{v}(\hat{c}) = \min_{\sigma^x, \sigma^c, \sigma^\hat{c}, \sigma^\hat{\hat{c}}} \hat{c} - \sigma^x \hat{c}^\gamma x, \pi + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^x)^2 + \frac{\gamma}{2} (\sigma^\hat{x})^2 - \tilde{\pi} \sigma^x \right) + \hat{v}' \hat{c} \left( \frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} + (1+\gamma)\sigma^\hat{c} \left( \sigma^x + \sigma^\hat{c} \right) + \frac{(\sigma^\hat{x})^2}{2} + (1+\gamma)\tilde{\sigma}^\hat{c} \left( \tilde{\sigma}^x + \tilde{\sigma}^\hat{c} \right) - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \tilde{\pi} \tilde{\sigma}^\hat{c} \right) \]

\[ + \hat{v}'' \hat{c}^2 \frac{(\sigma^\hat{c})^2 + (\tilde{\sigma}^\hat{c})^2}{2} \]

If the agent can secretly save in the market and/or his own strategy, we have additional constraints on controls

\[ \tilde{\sigma}^c = \tilde{\sigma}^x + \tilde{\sigma}^\hat{c} = \frac{\tilde{\pi}}{\gamma} \] if the agent can secretly trade the market (long and short)

\[ \sigma^c = \sigma^x + \sigma^\hat{c} \geq \phi \frac{\pi}{\gamma} \] if the agent can secretly save in his strategy (long only)

Each of these constraints lowers \( \hat{c}^h \), raises the cost function \( \hat{v}(\hat{c}) \)
\( \gamma = 3, \)  
\( r = \rho = 5\%, \)  
\( \pi = 0.2\%, \)  
\( \phi = 0.5 \)  
\( \pi^\sim = 0.05 \)
\begin{align*}
\gamma &= 3, \\
\rho &= 5\%, \\
\pi &= 0.2\%, \\
\phi &= 0.5 \\
\pi^\sim &= 0.05
\end{align*}
Cost function (all 4 cases)

\[ \gamma = 3, \quad r = \rho = 5\%, \quad \pi = 0.2\%, \quad \phi = 0.5 \quad \pi \sim = 0.05 \]
Conclusions

• Tractable contractual environment for finance and macro
  – moral hazard + hidden savings
  – CRRA preferences

• Hidden savings and incentives
  – agent’s precautionary motive depends on future risk exposure
  – precautionary motive leads to lower consumption, higher temptation
to divert returns, and tighter incentive constraint

• Distortions in the optimal contract
  – optimal contract manages the precautionary motive by lowering risk
 exposure over time and after bad outcomes

• Analytical verification of the first-order approach
  – double deviations are unattractive if precautionary motive weakens
  after bad outcomes
Thank you!