Learning by Matching

Yi-Chun Chen
Department of Economics
National University of Singapore

Gaoji Hu
Nanyang Business School
Nanyang Technological University

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Background

- Two-sided markets:
  - Marriage market
  - Job market
  - College admission market
  - School choice
  - ...
Complete Information Assumption

**Assumption:** Information is complete (CI), i.e.,

Every agent’s characteristics and preferences are common knowledge.
Outline

1. Incorporate firm-specific info by means of partitional information structure
2. Path to stability
3. Proof
1. One-to-one job market:

2. Incomplete information:

3. Path to stability:
   Knuth (1976), Roth and Vande Vate (1990), Kojima and Ünver (2008), Klaus and Klijn (2007), Chen et al. (2010, 2016), Fujishige and Yang (2016)...

Related Literature
The Model
Agents

- **Agents**
  - $I \ni i$: a finite set of workers.
  - $J \ni j$: a finite set of firms.

- **Types**
  - $w : I \rightarrow W$, where $W$ is finite.
  - $f : J \rightarrow F$, where $F$ is finite. *f is public information.*
  - $\Omega \subset W^{\|I\|}$: a set of possible type assignment functions.
Values and Payoffs

- **Values for match \((w, f)\)**
  - worker premuneration value: \(\nu_{wf} \in \mathbb{R}\).
  - firm premuneration value: \(\phi_{wf} \in \mathbb{R}\).
  - surplus of the match: \(\nu_{wf} + \phi_{wf}\).

- **Payoffs**
  - \(\nu_{w(i),f(j)} + p\) for the worker.
  - \(\phi_{w(i),f(j)} - p\) for the firm.
Allocation

- **matching**: $\mu : I \rightarrow J \cup \emptyset$, one-to-one on $\mu^{-1}(J)$.

- **payment scheme**: $p$ associated with a matching function $\mu$.
  - $p_{i,\mu(i)} \in \mathbb{R}$ for each $i \in I$.
  - $p_{\mu^{-1}(j),j} \in \mathbb{R}$ for each $j \in J$.
  - $p_{\emptyset} = p_{i\emptyset} = 0$.

- $\mathcal{A} \ni (\mu, p)$: the set of all allocations.
  - $(\mu, p)$ is observable for all agents.
Information

- Assumptions about $w$:
  - $w \in \Omega \subset W^{|I|}$. 
Information

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  - \( w \in \Omega \subset W^{\left| I \right|} \).

- \( \Pi_j \): Information Partition of a firm \( j \in J \).
  - \( \Pi_j \) is a partition of \( \Omega \).
  - \( w' \in \Pi_j(w) \):
    Firm \( j \) thinks \( w' \) is possible when \( w \) is true.
Information

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  - $w \in \Omega \subset W^{|I|}$.

- $\Pi_j$: Information Partition of a firm $j \in J$.
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    Firm $j$ thinks $w'$ is possible when $w$ is true.

- $\Pi := \{\Pi_j\}_{j \in J}$.

- Complete info: every partition cell is a singleton.
State of the Market

A state of the matching market, \((\mu, p, w, \Pi)\), specifies

- an allocation \((\mu, p)\);

- a type assignment function \(w\); and

- a partition profile \(\Pi\).
Stability
Requirement 1 of Stability: Individual Rationality

Definition 1
A state \((\mu, p, w, \Pi)\) is said to be individually rational if

\[
\nu_{w(i),f(\mu(i))} + p_{i,\mu(i)} \geq 0 \quad \text{for all } i \in I \text{ and }
\]

\[
\phi_{w(\mu^{-1}(j)),f(j)} - p_{\mu^{-1}(j),j} \geq 0 \quad \text{for all } j \in J.
\]
Requirement 2 of Stability: No Blocking

- Following LMPS, 'a firm cares about the worst case of worker if she does not know his true type.'

Definition 2

A state \((\mu, p, w, \Pi)\) is said to be **blocked** if there exists a worker-firm pair \((i, j)\) and a payment \(p \in \mathbb{R}\) such that

\[
\nu_{w(i), f(j)} + p > \nu_{w(i), f(\mu(i))} + p_{i, \mu(i)} \quad \text{and} \\
\phi_{w'(i), f(j)} - p > \phi_{w'(\mu^{-1}(j)), f(j)} - p_{\mu^{-1}(j), j}
\]

for all \(w' \in \Pi_j(w)\)
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for all $w' \in \Pi_j(w)$ satisfying

$$\nu_{w'(i),f(j)} + p > \nu_{w'(i),f(\mu(i))} + p_{i,\mu(i)}.$$
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\nu_{w'(i),f(j)} + p > \nu_{w'(i),f(\mu(i))} + p_{i,\mu(i)}.
\]

Consistency: A firm can observe the type of her own employee, if any.

\[
\forall w' \in \Pi_j(w), w'(\mu^{-1}(j)) = w(\mu^{-1}(j)).
\]
Example 1

- One worker $\alpha$ with possible types $w = -1$ (true) and $w' = 1$.
  Two firms $a$ and $b$. Firms’ type: $f_a = 1$ and $f_b = -1$.
  Values: $\nu_{wf} = \phi_{wf} = \nu_{wf}$.

- Allocation: No firm is matched with the worker.

- $\Pi_a = \{\{w\}, \{w'\}\}$ and $\Pi_b = \{\{w, w'\}\}$.
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- $(\alpha, a)$ is a blocking pair at $w'$ but not at $w$, i.e., $N_a = \{\{w\}, \{w'\}\}$.
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$\Pi_b = \{\{w, w'\}\}$.

▶ $(\alpha, a)$ is a blocking pair at $w'$ but not at $w$, i.e., $N_a = \{\{w\}, \{w'\}\}$.

▶ ’The state is not blocked by firm $a’ \implies$ firm $b$ can learn $N_a$, i.e.,

$$\Pi_b \lor N_a = \{\{w\}, \{w'\}\}.$$
Requirement 3 of Stability: Informational Stability

The fact of IR and no blocking provides no information to agents.

1. Partition Representation
2. Information Aggregation
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1. Partition Representation

2. Information Aggregation

1. Given a state \((\mu, p, w, \Pi)\), let \(N^{(\mu, p, \Pi)}\) be a partition of \(\Omega\):

\[ N^{(\mu, p, \Pi)}(w') = N^{(\mu, p, \Pi)}(w'') \text{ if and only if either neither } (\mu, p, w', \Pi) \text{ nor } (\mu, p, w'', \Pi) \text{ is blocked or both of them are blocked.} \]
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\]

2. Aggregating two pieces of information → Join of two partitions.

- Inferences:
  \[
  [H_{\mu, p}(\Pi)]_j := N^{(\mu, p, \Pi)} \lor \Pi_j, \forall j \in J, \text{ i.e.,}
  \]
  \[
  [H_{\mu, p}(\Pi)]_j(w') := \Pi_j(w') \cap N^{(\mu, p, \Pi)}(w'), \forall w' \in \Omega, \forall j \in J.
  \]
Stability

Definition 3

A state \((\mu, p, w, \Pi)\) is said to be stable if

1. it is individually rational,
2. it is not blocked by any pair, and
3. \(\Pi\) is a fixed point of \(H_{\mu, p}\), i.e. \(H_{\mu, p}(\Pi) = \Pi\).
Learning and Blocking

Consider \((\mu, p, w^*, \Pi)\) where \(\Pi\) and \((\mu, p)\) are common knowledge.

- The state is not blocked:

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\Pi \rightarrow H_{\mu, p}(\Pi).
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Learning and Blocking

Consider \((\mu, p, w^*, \Pi)\) where \(\Pi\) and \((\mu, p)\) are common knowledge.

- The state is not blocked:
  \[
  \Pi \rightarrow H_{\mu,p}(\Pi).
  \]

- The state is blocked by \((i, j; p)\).
  Extra information described by \(B(\mu, p, \Pi; i, j; p)\):
  \[
  B(\mu, p, \Pi; i, j; p)(w') = B(\mu, p, \Pi; i, j; p)(w'') \text{ if and only if either } (i, j; p) \text{ blocks both } (\mu, p, w', \Pi) \text{ and } (\mu, p, w'', \Pi) \text{ or neither.}
  \]
  \[
  \forall j', \Pi_{j'} \rightarrow \Pi_{j'} \lor B(\mu, p, \Pi; i, j; p)
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  \forall j', \quad \Pi_{j'} \rightarrow \Pi_{j'} \lor B(\mu, p, \Pi; i, j; p)
  \]

State updating: \((\mu', p', w^*, \Pi') \xrightarrow{(i, j; p)} (\mu, p, w^*, \Pi)\), if

- \((i, j; p)\) is satisfied in the new state, and

- for all \(j' \neq j\), \(\Pi'_{j'} = \Pi_{j'} \lor B(\mu, p, \Pi; i, j; p)\).
A learning-blocking path is a sequence of states \( \{(\mu^l, p^l, w^*, \Pi^l)\}_{l=0}^L \) s.t. for any two adjacent states \((\mu^l, p^l, w^*, \Pi^l)\) and \((\mu^{l+1}, p^{l+1}, w^*, \Pi^{l+1})\),

- if \((\mu^l, p^l, w^*, \Pi^l)\) is not blocked, then \((\mu^{l+1}, p^{l+1}) = (\mu^l, p^l)\) and \(\Pi^{l+1} = H_{\mu^l, p^l}(\Pi^l)\);

- if \((\mu^l, p^l, w^*, \Pi^l)\) is blocked, then \((\mu^{l+1}, p^{l+1}, w^*, \Pi^{l+1}) \xleftarrow{(i,j;p)} (\mu^l, p^l, w^*, \Pi^l)\), where \((i,j;p)\) is a blocking combination for \((\mu^l, p^l, w^*, \Pi^l)\).
Main Result

Theorem 1
Suppose payments permitted in the job market are all integers. Then for an arbitrary initial state, there exists a finite Learning-Blocking Path starting with it that leads to a stable state.
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**Theorem 2**
Suppose payments permitted in the job market are all integers. Then the random learning-blocking path starting from an arbitrary state converges with probability one to a stable state.

**Theorem 3**
\((\mu, p, w)\) is an incomplete-info. stable outcome in the sense of LMPS if and only if there exists a partition profile \(\Pi\) such that \((\mu, p, w, \Pi)\) is stable.
Proof of Theorem 1

Initial state: \((\mu, p, t^*, \Pi)\), assumed to be IR.

\[
(\mu, p, t^*, \Pi) : \begin{cases} 
\text{Blocked} \\
\text{Not blocked}
\end{cases}, \quad (\mu, p, t^*, H_{\mu, p}(\Pi)) : \begin{cases} 
\text{Blocked} \\
\text{Not blocked}
\end{cases} \ldots
\]

Finite time: blocked OR stable.
Proof of Theorem 1

Initial state: \((\mu, p, t^*, \Pi)\) is blocked, where \((i^1, j^1)\) is a blocking pair.

A new state: \((\mu', p', t^*, \Pi')\).
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\[ \alpha = i \]
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$\alpha = i$

\[
\begin{array}{c c c c}
i^1 & i & i^2 & i^3 \\
\mid & \mid & \mid & / \\
 j^1 & j^2 & j^3 & \\
\end{array}
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When tracking stops: the set contains no blocking pair
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When tracking stops: the set contains no blocking pair

OR there is one more direct observation.
Efficiency (CI Stability) of Stable States?

A partial answer:

(LMPS) Under Monotonicity and Supermodularity, every incomplete-information stable outcome is efficient.
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Example 2

- One worker $\beta$ with possible types $w_\beta = 1$ (true) and $w'_\beta = -1$.
- One firm $b$ with type: $f_b = 1$.
- Values: $\nu_{wf} = |wf|$ and $\phi_{wf} = wf$.  

\[ \text{Status quo: no match and } \Pi_b = \{ w_\beta, w'_\beta \} \]
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Values: $\nu_{wf} = |wf|$ and $\phi_{wf} = wf$.

▶ Status quo: no match and $\Pi_b = \{w_\beta, w'_\beta\}$.

▶ The status quo is
  ▶ incomplete-information stable but
  ▶ not efficient (not complete-information stable).
Conclusion

1. Stability with one-sided incomplete information.
   
i Describes firms’ information by firm specific and flexible partitions.
   
ii Makes (II) stability a natural extension of (CI) stability. Isolates the role played by information (requirement 3).
   

2. Path to stability.
   
i Describes information updating along a blocking path.
   
ii Shows the convergence of Learning-Blocking Paths.
   
iii Robustness of convergence w.r.t. learning pattern.

3. Connection with LMPS’s stability notions.
   
i Generates the same set of stable allocations as LMPS.
   
ii Different conceptual starting points: one state V.S. a set of outcomes.


