Order on Types based on Monotone Comparative Statics

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Comparative Statics

Comparative statics is one of the most important methodologies in Economics.

Parameter $\Rightarrow$ Optimal Solution or Equilibrium

e.g) Wealth $\Rightarrow$ Consumption

Cost $\Rightarrow$ Production in Cournot

Risk aversion $\Rightarrow$ Portfolio

Classical approach often appeals to implicit function theorem.
Monotone Comparative Statics

Monotone comparative statics (MCS) is an approach that utilizes the order structure of the game in which whenever a parameter increases, the set of equilibria also increases.

- No concavity or differentiability is needed.
- Single-crossing / supermodularity (or strategic complementarity) play a big role.

Our goal is to enlarge the applicability of MCS.
Example: Carlson and van Damme (1993)

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Invest</td>
<td>Invest</td>
<td>$\theta, \theta$</td>
</tr>
<tr>
<td>Not Invest</td>
<td>Not Invest</td>
<td>$0, \theta - 1$</td>
</tr>
</tbody>
</table>

where $\theta \in \mathbb{R}$ is the investment’s profitability.

- $\theta > 1 \Rightarrow$ each player has a dominant strategy to invest.

- $\theta \in [0, 1] \Rightarrow$ two pure strategy NE

- $\theta < 0 \Rightarrow$ each player has a dominant strategy not to invest.

Suppose \( \theta \) is common knowledge. As \( \theta \) increases, the set of equilibrium investment levels increase. Hence, MCS holds.

Milgrom and Shannon (1994) derive a necessary and sufficient condition for MCS to hold. It is the single-crossing condition.

But, what if there is incomplete information about \( \theta \)?
Example: Global Games

We use the same investment game but now assume the following:

• Each player $i$ observes $s_i \in \mathbb{R}$ as a noisy signal about $\theta$.

• There is a common prior on $(\theta, s_1, s_2)$.

• $i$’s posterior over $(\theta, s_{-i})$ upon observing $s_i$ is derived via Bayesian updating.

This is the setup often used in global games.
MCS under Incomplete Information about $\theta$


Suppose the common prior on $(\theta, s)$ exhibits affiliation. Then, as $s_i$ increases, $i$’s equilibrium investment level increases.

To enlarge the applicability of MCS, we dispense with the assumptions Athey made.

- One-dimensional signal structure
- Common prior
- Bayesian updating
What This Paper Does

We introduce an order on types: $t_i'$ is higher than $t_i$ in the sense of common certainty of optimism (CCO) if $t_i'$ is more optimistic that the news is good than $t_i$; $t_i'$ is more optimistic that all are optimistic that the news is good than $t_i$, and so on ad infinitum.

- **Sufficiency**: If $t_i'$ is higher than $t_i$ in the CCO order, $t_i'$ takes a higher action than $t_i$ in any supermodular game.

- **Necessity**: There is a supermodular game in which $t_i'$ is “not” higher than $t_i$ in the CCO order $\Rightarrow t_i'$ does “not” take a higher action than $t_i$. *This is our main theoretical contribution.*
Given a set $X$ and a partial order $\geq$: $\forall x, y \in X$, $x \lor y = \inf\{z \in X| z \geq x, z \geq y\}$ (join) and $x \land y = \sup\{z \in X| z \leq x, z \leq y\}$ (meet).

For $Y \subseteq X$, let $\bigvee Y \in X$ denote the least upper bound ("join") of $Y$, and $\bigwedge Y \in X$ denote the greatest lower bound ("meet") of $Y$.

A lattice is a set $X$ together with a partial order $\geq$ on $X$ such that the set is closed under meet and join operations.

A lattice $(X, \geq)$ is complete if every subset of $X$ has a meet and a join.
Complete Info Supermodular Games

\[ g = \langle I, \prod_{i \in I} A_i, \Theta, (u_i)_{i \in I} \rangle \] denotes a supermodular game where

(i) \( I = \{1, \ldots, I\} \): Set of Players;

(ii) \( A_i \): \( i \)'s action space; complete metric lattice;

(iii) \( \Theta \): a Polish parameter space; complete lattice;

(iv) \( u_i : A \times \Theta \to \mathbb{R} \): \( i \)'s payoff function.
(v) $u_i(\cdot)$ is **supermodular** on $A_i$: \forall \theta, a_{-i}, a_i, a'_i,$

$$u_i(a_i \lor a'_i, a_{-i}; \theta) + u_i(a_i \land a'_i, a_{-i}; \theta) \geq u_i(a_i, a_{-i}; \theta) + u_i(a'_i, a_{-i}; \theta).$$

and

(vi) $u_i(\cdot)$ has **increasing differences** in both $(a_i, a_{-i})$ and $(a_i, \theta)$:

$\forall a_i, a'_i \in A_i, \ a_{-i}, a'_i \in A_{-i},$ and $\theta, \theta' \in \Theta,$ whenever $(a_{-i}, \theta) \geq (a'_{-i}, \theta'),$ it follows that

$$u_i((a; \theta) \lor (a'; \theta')) + u_i((a; \theta) \land (a'; \theta')) \geq u_i(a; \theta) + u_i(a'; \theta').$$
**Incomplete Information Supermodular Games**

\[(T_i, \mathcal{T}_i, \pi_i)_{i \in I} \text{ is a type space where}
\]

- \(T_i\): \(i\)'s set of types;

- \(\mathcal{T}_i\): a sigma-algebra over \(T_i\); and

- \(\pi_i : T_i \rightarrow \Delta(\Theta \times T_{-i})\): \(i\)'s \(\mathcal{T}_i\)-measurable belief map.

\(G = (g, (T_i), (\mathcal{T}_i), (\pi_i))_{i \in I}\) now describes an **incomplete-information** supermodular game.
Belief Hierarchies induced by type $t_i$

$h^1(t_i) \in Z^1_i = \Delta(\Theta)$: the set of player $i$’s first-order beliefs;

$h^2(t_i) \in Z^1_i = \Delta(\Theta \times Z^1_{-i})$: the set of $i$’s second-order beliefs;

\[ \vdots \]

$h^k(t_i) \in Z^k_i = \Delta(\Theta \times Z^1_{-i} \times \cdots \times Z^{k-1}_{-i})$: $i$’s $k$th-order beliefs

where $k \geq 2$.

\[ \vdots \]

Finally, $Z^\infty_i = \prod_{k=1}^{\infty} Z^k_i$: the set of $i$’s all coherent infinite belief hierarchies.
First-Order Stochastic Dominance (FOSD)

Let \( X \) be a Polish space endowed with a closed partial order \( \succeq \).

A closed subset \( Y \subseteq X \) is an upper event of \( X \) if, \( \forall y, z \in X \), 
\[ y \in Y \text{ and } z \succeq y \implies z \in Y. \]

Let \( U(X) \) denote the set of all upper events of \( X \).

**Definition:** Let \( \beta, \beta' \in \Delta(X) \). \( \beta' \) (first-order) stochastically dominates \( \beta \) (denoted \( \beta' \succeq_{SD} \beta \)) if \( \beta'(Y) \geq \beta(Y) \) for any \( Y \in U(X) \).
Common Certainty of Optimism (CCO)

Suppose that (i) \( t_i' \) is more optimistic about \( \Theta \) than \( t_i \); (ii) \( t_i' \) is more optimistic about the optimism of other players about \( \Theta \); (iii) \( t_i' \) is more optimistic about the optimism about the optimism of other players about \( \Theta \) than \( t_i \); and so on ad infinitum.

In such a case, we say that \( t_i' \) is at least high as \( t_i \) in the order of common certainty of optimism and we denote it by \( t_i' \succeq_{\text{CCO}} t_i \).

Formally:

**Definition:** \( t_i' \succeq_{\text{CCO}} t_i \) if \( h^k(t_i') \succeq_{\text{SD}} h^k(t_i) \) for each \( k \in \mathbb{N} \).
Bayesian Nash Equilibrium (BNE)

Fix $G = (g, (T_i), (\mathcal{T}_i), (\pi_i))_{i \in I}$. $\sigma_i : T_i \rightarrow A_i$ denotes $i$’s $\mathcal{T}_i$-measurable pure strategy.

**Definition:** A strategy profile $\sigma^*$ is a (pure-strategy) Bayesian Nash equilibrium if, for each $i \in I$, $t_i \in T_i$, and $a_i \in A_i$,

$$\int_{\Theta \times T_{-i}} \left\{ u_i(\sigma^*_i(t_i), \sigma^*_{-i}(t_{-i}), \theta) - u_i(a_i, \sigma^*_i(t_{-i}), \theta) \right\} d\pi_i(t_i)[\theta, t_{-i}] \geq 0.$$  

$\Sigma^*$: the set of “all” BNE of $G = (g, (T_i), (\mathcal{T}_i), (\pi_i))_{i \in I}$.

It may be the case that $\Sigma^*$ is empty.
Lattice Structure of the set of BNE

We call \( \sigma \in \Sigma^* \) the **least** equilibrium if, for each \( \sigma^* \in \Sigma^* \), \( i \), and \( t_i \), we have \( \sigma^*_i(t_i) \succeq_{A_i} \sigma_i(t_i) \),

and similarly, call \( \bar{\sigma} \in \Sigma^* \) the **greatest** equilibrium if, for each \( \sigma^* \in \Sigma^* \), \( i \in I \), and \( t_i \), we have \( \bar{\sigma}_i(t_i) \succeq_{A_i} \sigma^*_i(t_i) \).

In addition, \( \Sigma^* \) has the following lattice structure: for any \( \sigma^* \in \Sigma^* \), \( i \in I \), and \( t_i \), we have that \( \bar{\sigma}_i(t_i) \succeq_{A_i} \sigma^*_i(t_i) \succeq_{A_i} \sigma_i(t_i) \).

Due to this structure, we only focus on the **least** equilibrium in the rest of the analysis.
The Least Interim Correlated Rationalizability (ICR)

Let $A_i^0[t_i] = A_i$ and $a_i^0[t_i] = \bigwedge A_i^0[t_i]$.

$$A_i^1(t_i) = \arg \max_{a_i \in A_i^0(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, a_{-i}^0(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],$$

and $a_i^1(t_i) = \bigwedge A_i^1(t_i)$.

We assume that $a_i^1(\cdot)$ is a measurable mapping and $A_i^1(t_i)$ is a complete sublattice.

$$\Rightarrow a_i^1(t_i) \in A_i^1(t_i).$$

By supermodularity, any $a_i$ such that $a_i \not\succeq_{A_i} a_i^1(t_i)$ is a never-best response against $a_{-i}^0(\cdot)$. 

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The Least Interim Correlated Rationalizability (ICR) Cont.

By induction, for each \( k \geq 1 \),

\[
A_i^{k+1}(t_i) = \arg \max_{a_i \in A_i^k(t_i)} \int_{\Theta \times T_{-i}} u_i(a_i, a_{-i}^k(t_{-i}); \theta) d\pi_i(t_i)[\theta, t_{-i}],
\]

and \( a_i^{k+1}(t_i) = \bigwedge A_i^{k+1}(t_i) \).

Again, we assume \( a_i^{k+1}(\cdot) \) is a measurable mapping and \( A_i^{k+1}(t_i) \) is a complete sublattice.

\[
\Rightarrow a_i^{k+1}(t_i) \in A_i^{k+1}(t_i).
\]

By supermodularity, any \( a_i \) such that \( a_i \not\leq A_i a_i^{k+1}(t_i) \) is a never-best response against \( a_i^{k+1} \).
The Least Interim Correlated Rationalizability (ICR) Cont.

Finally, define

$$a_i^\infty(t_i) = \bigvee\{a_i^1(t_i), a_i^2(t_i), \ldots\}.$$ 

$A_i$ is a complete lattice $\Rightarrow a_i^\infty(t_i) \in A_i$.

if $a_i^\infty(t_i)$ is a best response to $a_{-i}^\infty(\cdot) \Rightarrow \sigma$ defined by $\sigma_i(t_i) = a_i^\infty(t_i)$ constitutes an equilibrium.

By construction, $\sigma$ must be the least equilibrium of the game.
Characterization of the Least Equilibrium

Therefore,

**Proposition:** Assume that, for each $i, t_i$, and $k \geq 1$, (i) $A^k_i(t_i)$ is a complete sublattice, (ii) $a^k_i(\cdot) = \bigwedge A^k_i(\cdot)$ is a measurable mapping, and (iii) $a^\infty_i(t_i)$ is a best response to $a^-\infty_i(\cdot)$. Then, $\sigma$ defined by $\sigma_i(t_i) = a^\infty_i(t_i)$ for each $i$ and $t_i$ constitutes the least equilibrium.

Van Zandt and Vives (2007) propose more primitive assumptions for the existence of the least equilibrium: (i) $A_i$ is a compact metric lattice; (ii) $u_i(\cdot)$ is bounded, continuous in $a_i$ and measurable in $\theta$; and (iii) $\pi_i(\cdot)$ is measurable.
Sufficiency of Common Certainty of Optimism for MCS

**Theorem:** Let $G = (g, (T_i), (\mathcal{I}_i), (\pi_i))_{i \in I}$ be an incomplete information supermodular game that satisfies: for each $i \in I$, $t_i \in T_i$, and $k \geq 1$, (i) $A_i^k(t_i)$ is a complete sublattice; (ii) $\alpha_i^k(\cdot) = \bigwedge A_i^k(\cdot)$ is a measurable mapping; and (iii) $a_i^\infty(t_i)$ is a best response to $a_i^\infty$.

Then, $t_i' \succeq_{CCO} t_i \Rightarrow \sigma_i(t_i') \succeq_{A_i} \sigma_i(t_i)$. 

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Necessity of CCO: Optimism-Elicitation Game

This is our main result.

**Theorem:** There is a supermodular game with the property that, for any player \(i \in I\) and two types \(t_i, t'_i\), we have that \(t'_i \succeq_{CCO} t_i\) if and only if \(\sigma_i(t'_i) \succeq_{A_i} \sigma_i(t_i)\), where \(\sigma\) is the least equilibrium of this supermodular game.
Flavor of the Proof: a Single Agent Case

**Step 1:** Any upper set on $\Theta$ can be approximated by a countable set.

Each $U_n$ denotes an upper set such that the closure of $\bigcup_{n=1}^{\infty} U_n$ is equivalent to the set of all upper sets.

**Step 2:** The agent’s strategy is $\beta : U_n \mapsto [0, 1]$ and $\beta$ is monotone: $U_n \subseteq U_m \Rightarrow \beta(U_n) \leq \beta(U_m)$.

$\beta$ is defined as a capacity rather than a probability measure so that $B = \{ \beta : U_n \mapsto [0, 1] | \beta \text{ is monotone} \}$ constitutes a complete lattice. If we choose a topology on $B$ right, we can make $B$ a compact metric space.
**Step 3:** The agent’s payoff function using a strategy $\beta$ in state $\theta$ is

$$u(\beta, \theta) = \sum_{n=1}^{\infty} \left[ \beta(U_n)1_{U_n}(\theta) - \frac{\beta(U_n)^2}{2} \right] \mu(U_n),$$

where $1_{U_n}$ denotes the indicator function and $\mu$ is a full support distribution over all $\{U_n\}$.

**Step 4:** It is always optimal to choose the truthful probability assessment of $U_n$. 
Flavor of How to Extend to the Multiple Players Case

Set $X^1 = \emptyset; \ X^2 = (\Delta(X^1))^{I-1}; \ \text{and} \ X^k = (\Delta(X^1 \times \cdots \times X^{k-1}))^{I-1}$ for each $k \geq 3$, where $I$ stands for the number of players.

Finally, define $X^\infty = \prod_{k=1}^{\infty} X^k$.

**Step I:** Any upper set over $X^k$ can be approximated by a countable set.

Each $U_n^{(k)}$ denotes an upper set on $X^k$ such that the closure of $\bigcup_{n=1}^{\infty} U_n^{(k)}$ is equivalent to the set of all upper sets on $X^k$. 
**Step II:** Each agent’s strategy $\beta = (\beta^k)_{k=1}^\infty$ is such that $\beta^k : U_n^{(k)} \mapsto [0, 1]$.

$\beta^k$ is monotone: $U_n^{(k)} \subseteq U_m^{(k)} \Rightarrow \beta^k(U_n^{(k)}) \leq \beta^k(U_m^{(k)})$

**Step III:** Each agent’s payoff function using strategy $\beta$ in state $x \in X^\infty$ is

$$u(\beta, x) = \sum_{k=1}^\infty \delta^{k-1} \left[ \sum_{n=1}^\infty \left[ \beta^k(U_n)1^k_{U_n^{(k)}}(x^k) - \frac{(\beta^k(U_n^{(k)}))^2}{2} \right] \mu^k(U_n) \right],$$

where $0 < \delta < 1$; $x^k$ is the restriction of $x$ to $X^k$; $1^k_{U_n^{(k)}}$ denotes the indicator function on $X^k$; and $\mu^k$ is a full support distribution over all $\{U_n^{(k)}\}$. 

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Step IV: The unique rationalizable strategy profile leads to each agent's choosing the truthful probability assessment of $U_n^{(k)}$.

So, $\sigma_i(t_i) = \bar{\sigma}_i(t_i)$. 
Summary

- This paper introduces an order on types by which MCS is valid in all supermodular games with incomplete information.

- We fully characterize this order in terms of common certainty of optimism: $t'_i$ is higher than $t_i$ if $t'_i$ is more optimistic that the news is good for all than $t_i$; $t'_i$ is more optimistic that all are more optimistic that the news is good for all than $t_i$, and so on ad infinitum.

- Our work-in-progress investigates all possible orders on types induced by stochastic dominance and shows that our CCO order is the maximal one.