Ambiguity and the Centipede Game: Strategic Uncertainty in Multi-Stage Games

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(with Jürgen Eichberger and David Kelsey)

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(a) Stack of 1 dollar coins

(b) 2nd stack (equally high)
... or snatch?
The Centipede Game

- Game tree with two (initial) stacks of one hundred 1-dollar coins
Q. How much consistency should we require between beliefs of players about opponents’ behavior and those opponents’ actual behavior to justify calling a situation an (at least temporary) equilibrium?

Q. How should beliefs be updated and the attendant dynamic inconsistencies handled?
Relationship to Literature

● Ambiguity in Games
  – normal Form games
    * games of incomplete information: Azrieli & Teper (2011), Kajii & Ui (2005), Grant, Meneghel & Tourky (2016)
  – extensive form games with incomplete information
    * Hanany, Klibanoff & Mukerji (2016) ambiguity concerns type of opponent while the strategy is unambiguous.

Consistent Planning

- At $t = 0$, DM chooses one of three bets:
  
  $\bar{b}$: accept guaranteed payment of $x$;
  
  $b_G$: a bet which pays $q$ if a signal’s realization is $G$ in period 1;
  
  $b_W$: a bet which pays 1 if an event $W$ obtains in period 2.

- At $t = 1$, if she did not choose $\bar{b}$ and the signal’s realization is $G$ then she has option to switch from $b_W$ to $b_G$ (or vice versa).
Consistent Planning

- State-contingent pay-offs of the three bets are:

<table>
<thead>
<tr>
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- Suppose \( b_W \succ \bar{b} \succ b_G \) but \( b_G \succ^G b_W \).

- If DM anticipates she will prefer \( b_G \) should she be told the signal’s realization is \( G \), then may decide to choose \( \bar{b} \) at \( t = 0 \).

- We refer to this as consistent planning (Siniscalchi [2011]).
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The Neo-additive Model of Ambiguity

Preferences admit a representation of the form:

\[ \int u(a) \, d\nu = (1 - \delta) \mathbb{E}_\pi u(a) + \delta [\alpha m(a) + (1 - \alpha) M(a)] \]

- \( \nu \) is a neo-additive capacity,
  - \( \pi \) is a probability measure over the state space;
  - \( \delta \) is a measure of perceived ambiguity;
  - \( \alpha \) measures ambiguity-attitude, \( \alpha = 1 \) (respectively, \( \alpha = 0 \))
    corresponding to pure pessimism (respectively, optimism).
- \( M(a) \) denotes the maximum utility of act \( a \),
- \( m(a) \) denotes the minimum utility of act \( a \),
- \( \mathbb{E}_\pi u(a) \) denotes the expected utility of act \( a \).
Updating Ambiguous Beliefs

Suppose that event $E \subseteq S$ is observed.

Let $\nu^E$ denote the updated capacity then:

$$a \succ^E b \iff \int u(a(s)) d\nu^E(s) \succ \int u(b(s)) d\nu^E(s).$$

**Generalized Bayesian Updating (GBU)**

$$\nu^E(A) = \frac{\nu(A \cap E)}{\nu(A \cap E) + 1 - \nu(E^c \cup A)} = \frac{\nu(A \cap E)}{\nu(A \cap E) + \bar{\nu}(A^c \cap E)},$$

where $\bar{\nu}(E) = 1 - \nu(E^c)$. 

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Generalized Bayesian Updating (GBU)

For a neo-additive capacity \((\pi, \delta, \alpha)\),
its GBU is the neo-additive capacity \((\pi^E, \delta^E, \alpha^E)\),

where \( \pi^E(A) = \frac{\pi(A \cap E)}{\pi(E)} \), \( \delta^E = \frac{\delta}{\delta + (1-\delta)\pi(E)} \), and \( \alpha^E = \alpha \).
Multi-Stage Games

Definition

A multi-stage game $\Gamma$ is a triple $\langle \{1, 2\}, H, (u_1, u_2) \rangle$, where $H = H \cup Z$ is the union of all non-terminal and terminal histories, and $u_i$ is player $i$’s pay-off function.

Definition

A (pure) strategy of a player $i = 1, 2$ is a function $s_i$ which assigns to each history $h \in H$ an action $a_i \in A_i^h$.

- $S_i$ (resp., $S_{-i}$) denotes $i$’s (resp., her opponent’s) strategy set.
- $S = S_i \times S_{-i}$ denotes the set of strategy profiles.
- For each non-terminal history $h \in H$:
  - $S(h) \subset S$ denotes the set of strategy profiles that lead to the play of history $h$ with respective marginals $S_i(h)$ and $S_{-i}(h)$.
  - $S_i^h, S_{-i}^h$ & $S^h$ are the corresponding continuation sets.
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Admissible “beliefs”

- To study impact of *ambiguity*, we put restrictions on a player’s
  - perception of ambiguity about her opponent's choice of strategy;
  - attitude towards any such perceived ambiguity.

- We do this by assigning each player $i$ a set of *admissible* capacities:
  e.g. for a neo-expected payoff maximizer we have

$$C_i = \{ \nu_i = \nu(\cdot \mid \alpha_i, \delta_i, \pi_i) : \pi_i \in \Delta(S_{-i}) \}$$
Conditional neo-expected payoffs

If Player $i$’s initial “belief” about how her opponent is choosing his strategy corresponds to the capacity $\nu(\cdot | \alpha_i, \delta_i, \pi_i)$;

Then, given the game has reached stage $t$ via history $h$, her GBU capacity $\nu^h_i$ leads her to evaluate the neo-expected payoff of her continuation strategy $s^h_i$ according to:

$$V^h_i \left( s^h_i | \nu_i \right) = \left( 1 - \delta^h_i \right) \mathbb{E}_{\pi_i^h} u_i \left( s^h_i, \cdot \right)$$

$$+ \delta^h_i \left[ \alpha_i \min_{s^h_{-i} \in S^h_{-i}} u^h_i \left( s^h_i, s^h_{-i} \right) + (1 - \alpha_i) \max_{s^h_{-i} \in S^h_{-i}} u^h_i \left( s^h_i, s^h_{-i} \right) \right] ,$$

where $\delta^h_i = \frac{\delta_i}{\delta_i + (1 - \delta_i) \pi_i(S_{-i}(h))}$,

and $\pi_i^h$ is the Bayesian update of $\pi_i$ (whenever $\delta^h_i < 1$).
Support of a Capacity

Definition

If $\nu_i$ is a capacity on $S_{-i}$, define

$$\text{supp } \nu_i = \{s_{-i} \in S_{-i} : \forall A \subsetneq S_{-i}, s_{-i} \notin A; \nu_i(A \cup s_{-i}) > \nu_i(A)\}.$$

- $\text{supp } \nu_i$ comprises those strategies of $i$’s opponent which always get positive weight in the Choquet integral, no matter which of $i$’s strategies is being evaluated.

- For a neo-additive capacity $\nu_i = \nu(\cdot | \alpha_i, \delta_i, \pi_i)$
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Consistent-Planning Equilibrium Under Ambiguity

- Players are neo-expected payoff maximizers
- They update beliefs using Generalized Bayesian Updating.
- In any continuation after any history of play, they choose a best action according to their (updated) beliefs.
- They anticipate how information they may receive in the future will change their preferences (consistent planning)
A Consistent Planning Equilibrium Under Ambiguity (CP-EUA) is a profile of capacities $\langle \nu_1, \nu_2 \rangle$ such that for each player $i = 1, 2$,

$$s_i \in \text{supp } \nu_{-i}$$

$$\Rightarrow V_i^h \left( s_i^h | \nu_i \right) \geq V_i^h \left( (a_i, s_i^h (-t)) | \nu_i \right)$$

for every $a_i \in A_i^h$, every $h \in H^{t-1}$ and every $t = 1, \ldots, T$
Existence of CP-EUA

Proposition

Let $\Gamma$ be a multi-stage game with 2 neo-expected payoff maximizing players. Then $\Gamma$ has at least one CP-EUA for any given parameters $\alpha_1, \alpha_2, \delta_1, \delta_2$, where $0 \leq \alpha_i \leq 1$, $0 < \delta_i \leq 1$, for $i = 1, 2$. 
Returning to the Centipede Game

Suppose $\delta_1 = \delta_2 = \delta$ and $\alpha_1 = \alpha_2 = \alpha$

Recall $\delta$ measures ambiguity, $\alpha$ reflects ambiguity-attitude.
1. **Cooperation**
   If sufficient ambiguity and players are sufficiently ambiguity-loving (that is, provided $\delta(1 - \alpha) \geq \frac{1}{3}$) then equilibrium involves playing $r$ (i.e, “continue”) until the final node.
   
   - At final node player 2 chooses $d$ (down) since it is a dominant strategy for player 2 at that point.

2. **Non-cooperation**
   With high levels of ambiguity-aversion (that is, provided $\alpha \geq \frac{2}{3}$) only equilibrium is playing $d$ at every node.
   
   - Similar to Nash equilibrium.
Cooperation versus Non-cooperation

- No cooperation: \( \alpha > \frac{2}{3} \)
- Continue: \( \delta(1-\alpha) > \frac{1}{3} \)
Mixed Equilibria

**Proposition**

Let $\Gamma$ be a $M$ stage centipede game, where $M \geq 4$, then $\Gamma$ does not have a pure strategy equilibrium when $\alpha < \frac{2}{3}$ and $\delta(1 - \alpha) < \frac{1}{3}$.

- Kilka and Weber, (2001) experimentally estimate neo-additive preferences and find on average $\alpha = \delta = \frac{1}{2}$.
  - This is compatible with the present case.

- Under these assumptions there is a mixed strategy equilibrium.
  - $1$ believes likelihood of $2$ choosing $r$ at $M - 2$ is $p$.
  - $2$ believes likelihood of $1$ choosing $r$ at $M - 1$ is $q$. 
Mixed Equilibria

(i) 2’s belief about how 1 “randomizes” at node $M-1$ should make 2 at node $M-2$ indifferent between selecting either $d$ or $r$; i.e.,

$$M - 1 = (1 - \delta)((1 - q)(M - 2) + q(M + 1)) + \delta(\alpha(M - 2) + (1 - \alpha)(M + 1))$$
(ii) given the *GBU* of 1’s belief conditional on reaching node $M - 1$, 1 should be *indifferent* between selecting either $d$ or $r$; i.e.,

$$M = \left(1 - \delta^{M-1}\right) (M - 1) + \delta^{M-1} \left(\alpha (M - 1) + (1 - \alpha) (M + 2)\right)$$

where $\delta^{M-1} = \frac{\delta}{\delta + (1 - \delta)p}$. 

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Extensions/Work in Progress

- Extend to a larger class of games e.g. multi-player games.
- Add a type space.
- Extend to related games.
  - Alternating offers bargaining.
  - Chain store paradox.
  - Repeated Games
  - Herding in financial markets.
  - Asset price bubbles.