The very odd lattice of cumulative distributions

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Outline of talk

- The lattice of monotone functions
- Decomposable sets
- The odd property
- The open question
- The Tourky-Meneghel theorem
- Outline of proof of Tourky-Meneghel
- Application: a fixed point theorem
- Application to games.
Let $C$ be a compact subset of $[0, 1]$ that contains $0, 1$. Let $X(C)$ be the set of cumulative distributions on with support that is a subset of $[0, 1]$ and values in $C$. That is, the set of upper semi-continuous functions $f$ from $[0, 1]$ to $C$ satisfying $f(0) = 0$ and $f(1) = 1$. Viewing it as a subset of $L_1$ we see the following:

1. As a subset of $L_1$ it is norm compact.
2. It is a lattice under the usual ordering.
3. If $f$ first order stochastically dominates $g$, $g \geq f$, then the order interval $\{g \geq h \geq f\}$ is path connected.
4. It is a continuous lattice.
Decomposable sets

Let \( f, g : [0, 1] \rightarrow \mathbb{R} \) be measurable functions and \( E \subseteq [0, 1] \) be a measurable set. We write \( f_E g \) for the function

\[
f_E g(a) = \begin{cases} 
    f(a) & \text{if } a \in E, \\
    g(a) & \text{otherwise}.
\end{cases}
\]

A subset \( D \) of \( L_1 \) is decomposable if \( f, g \in D \) implies \( f_E g \in D \) for all measurable \( E \in [0, 1] \).
If $D$ is a closed decomposable set, and $D \cap X(C)$ is non empty, then $D \cap X(C)$ has the following properties:

1. Norm compact.
2. It is a lattice under the usual ordering.
3. If $f$ first order stochastically dominates $g$, $g \succeq f$, then the order interval $\{g \succeq h \succeq f\}$ is path connected.
4. It is a continuous lattice.
If $D$ is a closed decomposable set, and $D \cap X(C)$ is non empty, then the following holds true:

1. $X(C)$ is a norm compact absolute retract.
2. $D \cap X(C)$ is a norm compact absolute retract.

Once we observe that $D \cap X(C)$ has the lattice properties, there is nothing new in proving this. It has been understood in lattice theory for a very long time.
Characterise the sets $Y$ in $L_1$ such that

1. $Y$ is a norm-compact absolute retract.
2. If $D$ is any set that is closed and decomposable and $D \cap Y$ is nonempty, then $D \cap Y$ is an absolute retract.

I’ll call such sets “OAR” Absolute Retracts. (OAR stands for OAR Absolute Retracts. :-) )
Why are we interested

1. It is interesting.

2. If \( Y \) is a “OAR” Absolute Retract and \( F: Y \to L_1 \) is a closed, decomposable valued correspondence such that \( F(x) \cap Y \) is nonempty for all \( x \in Y \), then \( F \) has a fixed point in \( Y \).

3. Decomposable valued mappings arise in many applications.
The Tourky-Meneghel theorem

**Theorem:** If $X$ is a set that is compact in $L_\infty$, then there is a compact set $Y \subseteq L_1$, with $X \subseteq Y$, such that $Y$ is an “OAR” Absolute Retract.
Q: Suppose that $D$ is a closed decomposable subset of $L_1$ and $F : D \to D$ is a non-empty valued decomposable-valued mapping. If there is a (pointwise) compact set $X \subseteq D$ such that $F(f) \cap X \neq \emptyset$ for all $f \in D$, then does there exist $f^* \in D$ satisfying $f^* \in F(f^*)$?

A: Prior literature—yes when $D$ is compact, but that means it is a singleton (obvious).
The Meneghel-Tourky theorem answers this in the affirmative.
Meneghel-Tourky: If $X$ is a compact set in $L_\infty$, then there is a compact set $Y \subseteq L_1$, $X \subseteq Y$, satisfying:

- If $F : Y \rightarrow L_1$ is a closed, decomposable-valued correspondence satisfying $F(f) \cap Y \neq \emptyset$, then $F$ has a fixed point in $Y$.
- If $F : L_1 \rightarrow L_1$ is a closed, decomposable-valued correspondence satisfying $F(f) \cap X \neq \emptyset$, then $F$ has a fixed point in $Y$. 
Let $C$ be a compact subset of $[0, 1]$, $X$ be the set of all constant functions $f : [0, 1] \to C$.

1. $X$ is compact in $L_\infty$.
2. $Y$ of Meneghel-Tourky is the set of monotone functions $g : [0, 1] \to C$.
3. $Y$ is “AOR” Absolute Retract, and $X$ need not be an Absolute Retract.
Outline of proof

Let $X$ be a compact metric space. Let $C \subset [0, 1]$ be the Cantor ternary set. By the Hahn-Mazurkiewicz theorem there exists a continuous function such that $f : C \to X$ is onto.

So set of constant functions $f : [0, 1] \to C$ maps to $X$. The set of monotone functions $g : [0, 1] \to C$ maps back to the desired $Y \supset X$. 