Mislaid Pieces in Finitely Additive Population Games

Maxwell B. Stinchcombe

Department of Economics, University of Texas at Austin

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1. Large Population Games

2. Finitely Additive Probabilities

3. PFAs in Population Games

4. PFAs in Economic Models
Basics

\[ \Gamma(\mu) = ((T, T, \mu), \mathbb{U}, G). \]
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- \((T, \mathcal{T}, \mu)\), a non-atomic probability space.
- \((A, d), \Delta(A), \mathbb{U}\) the closed unit ball in \(C(A \times \Delta(A))\).
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- $(T, \mathcal{T}, \mu)$, a non-atomic probability space.
- $(A, d)$, $\Delta(A)$, $\mathbb{U}$ the closed unit ball in $C(A \times \Delta(A))$.
- Or $\mathbb{U} \subset C(A \times M)$, $M = \{q \in \Delta(T \times A) : q(E \times A) = \mu(E)\}$. 
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- Or \(\mathbb{U} \subset C(A \times M), M = \{q \in \Delta(T \times A) : q(E \times A) = \mu(E)\}.\)
- \(\mathcal{G} : T \rightarrow \mathbb{U}, P = \mathcal{G}(\mu) \in \Delta(\mathbb{U}).\)
If $a : T \to \Delta(A)$ is the population strategy, the distribution is $\nu_a(E) = \int a(t)(E) \, d\mu(t)$, and agent $t$ receives utility $G(t)(a(t), \nu_a)$. 
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A strategy $a(\cdot)$ is an $\epsilon$-equilibrium if

$$\mu(\{t : G(t)(a(t), \nu_a) \geq \max_{b \in A} G(t)(b, \nu_a) - \epsilon\}) \geq 1 - \epsilon, \quad (1)$$

and is an equilibrium if it is a 0-equilibrium.
A probability is **finitely additive** if \( \mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) \) for \( E_1 \cap E_2 = \emptyset \).
Definitions

A probability is **finitely additive** if $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ for $E_1 \cap E_2 = \emptyset$.

A probability $\mu$ is **countably additive** iff

$$[E_n \downarrow \emptyset] \Rightarrow [\mu(E_n) \downarrow 0].$$
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A probability \( \mu \) is **countably additive** iff

\[
[E_n \downarrow \emptyset] \Rightarrow [\mu(E_n) \downarrow 0].
\]

Countable additivity is **not** “just a technical assumption.”
Definitions

Dfn: the **deficiency** of a finitely additive $\mu$ is

$$\sup\{\delta \geq 0 : \exists E_n \downarrow \emptyset \text{ and } \mu(E_n) \geq \delta\}.$$
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If the deficiency is 1, then $\mu$ is **purely finitely additive**. A probability is pfa iff there exists a strictly positive $g$ with $\int g \, d\mu = 0$. 

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Mislaid Pieces in Finitely Additive Population Games
Weak* Compactness

Banach space theory: \( \mu_\alpha \to_{w^*} \mu \) iff \( \int g \, d\mu_\alpha \to \int g \, d\mu \) for all bounded measurable \( g \).
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Alaoglu’s Theorem: the set of finitely additive probabilities is weak*-compact.
An Implication

Kingman (1967). There is a purely finitely additive $\mu$ on the set of polynomials with the same finite dimensional distributions as a Poisson process.
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- $\mathcal{P}$ is the set of polynomials on $[0, \infty)$.
- For $0 =: t_0 \leq t_1 < \cdots < t_n$ and $f \in \mathcal{P}$,
  $\text{proj}_{t_1,\ldots,t_n}(f) := (f(t_1), \ldots, f(t_n))$. 
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  $\text{proj}_{t_1,\ldots,t_n}(f) := (f(t_1), \ldots, f(t_n))$.
- $\mathcal{P}^\circ := \{\text{proj}_{t_1,\ldots,t_n}^{-1}(B^n) : B^n \subset \mathbb{R}^n \text{ measurable}\}$, $\mathcal{P} := \sigma(\mathcal{P}^\circ)$. 
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- $\mathcal{P}^\circ := \{\text{proj}_{t_1, \ldots, t_n}^{-1}(B^n) : B^n \subset \mathbb{R}^n$ measurable\}, $\mathcal{P} := \sigma(\mathcal{P}^\circ)$.
- FIDI’s — define $\mu' : \mathcal{P}^\circ \to [0, 1]$ by
  $\mathcal{L}(\{\text{proj}_{t_m}(\mu') - \text{proj}_{t_{m-1}}(\mu') : m = 1, \ldots n\})$ to be
  independent Poissons with parameters $(\lambda \cdot (t_m - t_{m-1}))$. 
An Implication

For any finite set \( 0 =: t_0 \leq t_1 < \cdots < t_n \), there is a non-empty, weak*-closed/compact set of probabilities \( \mu' \) on \( \mathbb{P} \) with these FIDIs.
An Implication

For any finite set $0=: t_0 \leq t_1 < \cdots < t_n$, there is a non-empty, weak*-closed/compact set of probabilities $\mu'$ on $\mathbb{P}$ with these FIDIs.

Compactness implies non-emptiness of the intersection over all finite $0=: t_0 \leq t_1 < \cdots < t_n$. Any $\mu$ in the intersection is purely finitely additive.
Infinitely Steep Polynomials

Fix a Poisson realization $h : [0, \infty) \to \{0, 1, \ldots\}$ with jumps at $\tau_1 < \cdots < \tau_k < \cdots$. 

The finitely additive $\mu$ is "trying to" put mass 1 on polynomials having slopes at least $1/\epsilon$ for every $\epsilon > 0$. 

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Fix a Poisson realization $h : [0, \infty) \to \{0, 1, \ldots\}$ with jumps at $\tau_1 < \cdots < \tau_k < \cdots$.

Fix arbitrary $\epsilon > 0$ and interval $[0, 1/\epsilon]$. There exists $K$ such that $\tau_K \leq (1/\epsilon) < \tau_{K+1}$. There exists an $f \in \mathbb{P}$ with slope at least $1/\epsilon$ such that for $1 \leq k \leq K$,

$$[k \leq h(t) < (k + 1)] \Rightarrow [k \leq f(t) < (k + 1)]$$

$$[d(t, \tau_k) \geq \epsilon, \ 0 \leq t \leq 1/\epsilon] \Rightarrow [\vert h(t) - f(t)\vert < \epsilon].$$

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\end{align*}
\]

The finitely additive \( \mu \) is “trying to” put mass 1 on polynomials having slopes at least \( 1/\epsilon \) for every \( \epsilon > 0 \).
Let \( *\mathbb{P} \) be the nonstandard version of the polynomials. By overspill, there exists a strictly positive \( \epsilon \approx 0 \) such that for every Poisson realization \( h \), there is an \( f \in *\mathbb{P} \) such that for \( 1 \leq k \leq K \),

\[
[k \leq h(t) < (k + 1)] \Rightarrow [k \leq f(t) < (k + 1)]
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\[
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Let $\ast \mathbb{P}$ be the nonstandard version of the polynomials. By overspill, there exists a strictly positive $\epsilon \simeq 0$ such that for every Poisson realization $h$, there is an $f \in \ast \mathbb{P}$ such that for $1 \leq k \leq K$,

\[
[k \leq h(t) < (k + 1)] \Rightarrow [k \leq f(t) < (k + 1)]
\]

\[
[d(t, \tau_k) \geq \epsilon, \quad 0 \leq t \leq 1/\epsilon] \Rightarrow [|h(t) - f(t)| < \epsilon].
\]

$\ast \mu$ or $L(\ast \mu)$ is a probability on $\ast \mathbb{P}$ having the FIDIs of a Poisson process.
Let $\eta$ be a pfa probability on $\mathbb{N}$ with $\eta(E) = 0$ or $\eta(E) = 1$ for all $E \subset \mathbb{N}$. 
A Lightning Fast Introduction to NSA

Let \( \eta \) be a pfa probability on \( \mathbb{N} \) with \( \eta(E) = 0 \) or \( \eta(E) = 1 \) for all \( E \subset \mathbb{N} \).

For arbitrary non-empty set \( X \) and \( (x_m), (y_m) \in X^\mathbb{N} \), define \( (x_m) \sim (y_m) \) if \( \eta(\{m \in \mathbb{N} : x_m = y_m\}) = 1 \), let \( \langle x_m \rangle \) denote the equivalence class of \( (x_m) \), and define \( *X = X^\mathbb{N}/ \sim \) as the set of equivalence classes.
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For arbitrary non-empty set $X$ and $(x_m), (y_m) \in X^\mathbb{N}$, define $(x_m) \sim (y_m)$ if $\eta(\{m \in \mathbb{N} : x_m = y_m\}) = 1$, let $\langle x_m \rangle$ denote the equivalence class of $(x_m)$, and define $\ast X = X^\mathbb{N}/\sim$ as the set of equivalence classes.

If $\epsilon = \langle \epsilon_m \rangle$ in $\ast \mathbb{R}$ and $\epsilon_m \downarrow 0$, then we say that $\epsilon$ is infinitesimal because, for all $r > 0$, $\eta(\{m : 0 < \epsilon_m < r\}) = 1$, so $0 < \epsilon < r$. 

Maxwell B. Stinchcombe

Mislaid Pieces in Finitely Additive Population Games
For $E_n \downarrow \emptyset$ and each $E_n \neq \emptyset$, we do not have $^\ast E_n \downarrow \emptyset$, a form of compactness.
For $E_n \downarrow \emptyset$ and each $E_n \neq \emptyset$, we do not have $*E_n \downarrow \emptyset$, a form of compactness.

For measurable $E$, $*\mu(*E) = \mu(E)$, so $E_n \downarrow \emptyset$ and $\mu(E_n) \equiv 1$ yield $*\mu(\cap_n *E_n) = \langle 1, 1, 1, \ldots \rangle$. 
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- For $E_n \downarrow \emptyset$ and each $E_n \neq \emptyset$, we do not have $^*E_n \downarrow \emptyset$, a form of compactness.

- For measurable $E$, $^*\mu(^*E) = \mu(E)$, so $E_n \downarrow \emptyset$ and $\mu(E_n) \equiv 1$ yield $^*\mu(\cap_n^*E_n) = \langle 1, 1, 1, \ldots \rangle$.

- For $E = \langle E_n \rangle$, $^*\mu(E) = \langle \mu(E_n) \rangle$, so domain of $^*\mu$ is large.
A quick look at $\ast \mathbb{P}$. 
Nonstandard Polynomials

A quick look at $^\ast\mathbb{P}$.

- Fix a Poisson realization $h : [0, \infty) \to \{0, 1, \ldots\}$ with jumps at $\tau_1 < \cdots$. 
Nonstandard Polynomials

A quick look at \( \ast \mathbb{P} \).

- Fix a Poisson realization \( h : [0, \infty) \to \{0, 1, \ldots\} \) with jumps at \( \tau_1 < \cdots \).

- For each \( m \) and \( K \) jumps of \( h \) in \( [0, m] \), let \( f_m \) be a polynomial with, for \( k = 1, \ldots, K \),

\[
[k \leq h(t) < (k + 1)] \Rightarrow [k \leq f_m(t) < (k + 1)]
\]

\[
[d(t, \tau_k) \geq \epsilon, \ 0 \leq t \leq m] \Rightarrow [|h(t) - f(t)| < 1/m].
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- Let \( f = \langle f_m \rangle \).

Claim: \( *\mu \) puts mass 1 on the infinitely steep polynomials.
Overview

Recall $\Gamma(\mu) = ((T, T, \mu), U, G)$. 

Two pfa examples from Khan, Kiao, Rath, Sun. The first has approximate equilibria but no equilibrium, the second has no approximate equilibria. In the first, the pfa $G(\mu)$ is "trying to" put mass 1 on a single utility function. In the second, $G(\mu)$ is "trying to" put mass 1 on infinitely steep continuous functions. 

Will then analyze the equilibria of the games $^\ast \Gamma(\mu) := ((^\ast T, \sigma(^\ast T), ^\ast \mu), st V(^\ast U), st V(^\ast G))$. 

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Will then analyze the equilibria of the games

$\Gamma^*(\mu) := ((*T, \sigma(*T), \circ\mu), \text{st}_V(*\mathbb{U}), \text{st}_V(*\mathcal{G}))$. 
Approximate Equilibria

$T = [1, \infty)$, $\mathcal{T}$ is the (usual) Borel $\sigma$-field, and $\mu$ is a non-atomic, pfa probability on $T$ with $\mu([t, \infty)) \equiv 1$. The common space of actions is $A = \{0, 1\}$, $U$ is the closed unit ball in $C(A \times [0, 1])$ where $[0, 1]$ representing $\nu(a = 1)$. 

Example 1: $G(t) = a \cdot (1 - \nu(t))$. If $\nu(a) > 0$ is equilibrium, then $a^* = 1$ is only a best response for $t$ in the null set $(0, 1/\nu(a) - \nu(a) > 0) \Rightarrow [\nu(a) = 0]$. If $\nu(a) = 0$ is equilibrium, then for all $t \in T$, $1/t > \nu(a)$, so everyone should (apparently) play the action 1, making $\nu(a) = 1$. For $\epsilon$-equilibria, any tiny set of people play $a = 1$. 

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Example 1: $G(t) = a \cdot \left(\frac{1}{t} - \nu\right)$. 
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Example 1: \( G(t) = a \cdot (\frac{1}{t} - \nu) \).

- If \( \nu_a > 0 \) is equilibrium, then \( a^* = 1 \) is only a best response for \( t \) in the null set \( (0, 1/\nu_a) \) — \( [\nu_a > 0] \Rightarrow [\nu_a = 0] \).

- If \( \nu_a = 0 \) is equilibrium, then for all \( t \in T \), \( \frac{1}{t} > \nu_a \), so everyone should (apparently) play the action 1, making \( \nu_a = 1 \).

- For \( \epsilon \)-equilibria, any tiny set of people play \( a = 1 \).
But the Equilibria Involve

\[ V(a, \nu) := -a \cdot \nu, \ G(t) = a \cdot \frac{1}{t} + V(a, \nu), \text{ for any } \delta > 0, \text{ we have} \]
\[ \mu(\{t \in T : \|G(t) - V\| < \delta\}) = 1, \quad (2) \]
But the Equilibria Involve

\[ V(a, \nu) := -a \cdot \nu, \ G(t) = a \cdot \frac{1}{t} + V(a, \nu), \] for any \( \delta > 0 \), we have

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hence \( \int \| G(t) - V \| \, d\mu(t) = 0 \) even though \( f(t) := \| G(t) - V \|, \) is strictly positive on \( T \).
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hence \( \int \|G(t) - V\| \, d\mu(t) = 0 \) even though \( f(t) := \|G(t) - V\| \), is strictly positive on \( T \).

If \( \mu(\{ t : G(t) = V \}) = 1 \), then equilibria have

\[ \mu(\{ t : a(t) = 0 \}) = 1. \]
Approximate Equilibria

\( G(t) = a \cdot u(t, \nu) \) where

\[
\begin{align*}
  u(t, \nu) &= \begin{cases} 
    1 & \text{if } \nu \leq \frac{1}{2}, \\
    1 - t(\nu - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \nu \leq \frac{1}{2} + \frac{2}{t}, \text{ and} \\
    -1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu.
  \end{cases}
\end{align*}
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Large Population Games  Finitely Additive Probabilities  PFAs in Population Games  PFAs in Economic Models

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Maximal absolute slope for \( t \) is \( t \). \( \mu([t, \infty)) = 1 \) is “trying to” put mass 1 on infinitely steep utility functions.
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Maximal absolute slope for \( t \) is \( t \). \( \mu([t, \infty)) \equiv 1 \) is “trying to” put mass 1 on infinitely steep utility functions.

To represent steepness = \( \infty \), the domain, \( \Delta(\{0, 1\}) = [0, 1] \), must expand.
\( G(t) = a \cdot u(t, \nu) \) with

\[
\begin{cases}
1 & \text{if } \nu \leq \frac{1}{2}, \\
1 - t(\nu - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \nu \leq \frac{1}{2} + \frac{2}{t}, \text{ and}
\end{cases}
\]

\[-1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu.
\]

- \( \nu \leq \frac{1}{2} \) \( \Rightarrow \) \( (\forall t)[a^{br}(t) = 1] \) so \( \epsilon \)-best responses put mass at least \( 1 - \epsilon \) on \( a = 1 \).
$G(t) = a \cdot u(t, \nu)$ with

\[
\begin{cases}
1 & \text{if } \nu \leq \frac{1}{2}, \\
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-1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu.
\end{cases}
\]

- $[\nu \leq \frac{1}{2}] \Rightarrow (\forall t)[a^{br}(t) = 1]$ so $\epsilon$-best responses put mass at least $1 - \epsilon$ on $a = 1$. Therefore, $[\nu_a \leq \frac{1}{2} \text{ an } \epsilon\text{-equilibrium}] \Rightarrow [\nu_a \geq (1 - \epsilon)^2]$. 
\[G(t) = a \cdot u(t, \nu)\] with

\[u(t, \nu) = \begin{cases} 1 & \text{if } \nu \leq \frac{1}{2}, \\ 1 - t(\nu - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \nu \leq \frac{1}{2} + \frac{2}{t}, \text{ and} \\ -1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu. \end{cases}\]

\[\nu \leq \frac{1}{2} \Rightarrow (\forall t)[a^{br}(t) = 1] \text{ so } \epsilon\text{-best responses put mass at least } 1 - \epsilon \text{ on } a = 1. \text{ Therefore, } [\nu_a \leq \frac{1}{2} \text{ an } \epsilon\text{-equilibrium}] \Rightarrow [\nu_a \geq (1 - \epsilon)^2].\]

\[\nu > \frac{1}{2} \Rightarrow [\mu(\{t : \frac{1}{2} + \frac{2}{t} < \nu_a\}) = 1]. \text{ A mass 1 set of players loses utility of 1 by playing } a = 1, \text{ so } \epsilon\text{-best responses must put mass at least } 1 - \epsilon \text{ on } a = 0.\]
NO Approximate Equilibria

\[ \mathcal{G}(t) = a \cdot u(t, \nu) \]

with

\[
u(t, \nu) = \begin{cases} 
1 & \text{if } \nu \leq \frac{1}{2}, \\
1 - t(\nu - \frac{1}{2}) & \text{if } \frac{1}{2} \leq \nu \leq \frac{1}{2} + \frac{2}{t}, \text{ and} \\
-1 & \text{if } \frac{1}{2} + \frac{2}{t} \leq \nu.
\end{cases}
\]

- \( [\nu \leq \frac{1}{2}] \Rightarrow (\forall t)[a^{br}(t) = 1] \) so \( \varepsilon \)-best responses put mass at least \( 1 - \varepsilon \) on \( a = 1 \). Therefore, \( [\nu_a \leq \frac{1}{2} \text{ an } \varepsilon\text{-equilibrium}] \Rightarrow [\nu_a \geq (1 - \varepsilon)^2] \).

- \( [\nu > \frac{1}{2}] \Rightarrow [\mu(\{t : \frac{1}{2} + \frac{2}{t} < \nu_a\}) = 1]. \) A mass 1 set of players loses utility of 1 by playing \( a = 1 \), so \( \varepsilon \)-best responses must put mass at least \( 1 - \varepsilon \) on \( a = 0 \). Therefore, \( [\nu_a > \frac{1}{2} \text{ an } \varepsilon\text{-equilibrium}] \Rightarrow [\nu_a \leq \varepsilon(1 - \varepsilon)]. \)
Equilibria with $\mu$

Now replace the spaces with their nonstandard extensions and analyze the dependence of equilibrium distributions of actions and utilities.
Equilibria with $^\ast\mu$

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Equilibrium involves everyone with $t < (\leq)t_c$ playing $a = 1$ where $F_1(t_c) = \frac{1}{2} + \frac{1}{t_c}$, using the quadratic formula on $t_c = \frac{1}{2} + \frac{1}{t_c}$ yields

$$t_c = \frac{1}{2} \left[ (N + \frac{1}{2}) + \sqrt{(N + \frac{1}{2})^2 + 4} \right],$$

which involves $t_c/(N + \frac{1}{2}) = 1 + \epsilon$ for an $\epsilon \simeq 0$. 
Agents in $[N, t_c]$, who have mass (a positive infinitesimal greater than) $\frac{1}{2}$, play $a = 1$, and their utility is distributed uniformly on $[0, 1]$, agents in $(t_c, N + 1]$ play $a = 0$ and receive utility 0. No strategy in the original game achieves this joint distribution of actions and utilities.
Observations

- Agents in \([N, t_c]\), who have mass (a positive infinitesimal greater than) \(\frac{1}{2}\), play \(a = 1\), and their utility is distributed uniformly on \([0, 1]\), agents in \((t_c, N + 1]\) play \(a = 0\) and receive utility 0. No strategy in the original game achieves this joint distribution of actions and utilities.

- Related, \(\nu = \frac{1}{2} + 1/t_c\) is NOT an element of \([0, 1]\), it is an element of \(*[0, 1]\). To find the equilibrium, the domain of the utility functions, \(\{0, 1\} \times [0, 1]\), was extended.
Now suppose $\mu_2$ the weak* standard part of $\frac{1}{4} U[0, N] + \frac{3}{4} U[0, N^2]$ for infinite $N$. Can solve for exact cutoff $t_c$, it satisfies $t_c/(N + \frac{1}{3}N^2) \approx 1$. 
Equilibrium Outcomes Depend on $\mu$

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Equilibrium outcomes: just over half of the agents, those in $[0, t_c]$ play $a = 1$, the rest play $a = 0$. Playing $a = 0$ yields utility 0. Half of the $a = 1$ agents receive utility 1 and half of them have utility uniformly distributed on $[0, 1]$. 
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Again, no strategy in the original game achieves this joint distribution of outcomes and actions.
Examples

Fishburn (1970). A society’s preference ordering, $\succeq_s$ satisfies Arrow assumptions iff for some pfa point mass $\eta$ we have

$$[x \succeq_s y] \iff (\exists E \subset T)[\eta(E) = 1 \text{ and } E = \{t \in T : x \succeq_t y\}].$$
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Examples

Kingman (1967). Pfa probabilities on the polynomials model jump process.
Examples

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Khan et al. (2016). Pfa population measures $\Rightarrow$ some population games have no equilibria.
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Khan et al. (2016). Pfa population measures $\Rightarrow$ some population games have no equilibria. Missing agents and their utility functions.
Possible Reactions?

So what to think of purely finitely additive probabilities?
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- Flawed (?fatally?) tool.
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So what to think of purely finitely additive probabilities?

- Flawed (?fatally?) tool.
- But $\mu$ finds the missing pieces.
Other Results in the Paper

- The equilibria of $\Gamma^*(\mu)$ are finitely approximable.
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- Can substitute compact Hausdorff spaces for the pieces of $\Gamma^*(\mu)$.
- The compactification of e.g. the unit ball in $C([0,1])$ is an incredibly cool Hausdorff space.
Anything Else?
Anything Else?

FINIS