Structural Rationality in Dynamic Games

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Prelude: Credible Threats

(Out, (S, S))
Prelude: Credible Threats

“Bob will play S”

\[(Out, (S, S))\]

▶ Threat: On-path beliefs about off-path play
Prelude: Credible Threats

Threat: On-path beliefs about off-path play

Credible: Off-path beliefs

\[(\text{Out}, (S, S))\]
This Paper

Behavioral content of assumptions on beliefs

Testable implications of solution concepts

in dynamic games
Benchmark: Simultaneous-Move Games

- Luce-Raiffa: *elicit* beliefs via incentive-compatible *side bets*

- Also practical: e.g. Van Huyck, Battalio, and Beil, 1990; Nyarko and Schotter, 2002. (See also Aumann-Dreze, 2009)

Objective: *do the same for* **dynamic** games
Eliciting Bob’s beliefs in the subgame

If subgame reached, could offer side bets on $B$ vs. $S$

But in this SPE, the subgame is not reached
Eliciting Bob’s beliefs in the subgame

Could elicit Bob’s prior beliefs, then condition on “In”

But in this SPE, “In” has zero prior probability
Ex-ante conditional bets? (de Finetti)

$p$ close to 1; randomization picks game vs. bet payoff for Bob
Ex-ante conditional bets? (de Finetti)

Bob $p$

Ann $I$

In $B$

Out $2, (2, 0)$

$p$ close to 1; randomization picks game vs. bet payoff for Bob

▶ Now Bob’s bet is always observed
Ex-ante conditional bets? (de Finetti)

$p$ close to 1; randomization picks game vs. bet payoff for Bob

▶ Now Bob’s bet is always observed

▶ Sequential rationality: Bob is indifferent between $p$ and $b$
Ex-ante conditional bets? (de Finetti)

$p$ close to 1; randomization picks game vs. bet payoff for Bob

- Now Bob’s bet is always observed
- Sequential rationality: Bob is indifferent between $p$ and $b$
- $(Out, p, (S, S))$ a sequential equilibrium
The role of sequential rationality

Sequential rationality: Bob

- reacts optimally to surprises: e.g., if $In$, expect $S \Rightarrow$ play $S$
The role of sequential rationality

Sequential rationality: Bob

- reacts optimally to surprises: e.g., if In, expect S ⇒ play S
- but need not take into account potential future surprises
The role of sequential rationality

Sequential rationality: Bob

▶ reacts optimally to surprises: e.g., if \( \text{In} \), expect \( S \) ⇒ play \( S \)

▶ but need not take into account potential future surprises
  e.g., \( p \) sequentially rational despite Bob’s beliefs following \( \text{In} \)
## Structural Rationality

Every action choice

- takes into account **beliefs** at all **unexpected events**
- in a **principled** way
**Structural Rationality**

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Loosely inspired by evidence on strategy method (Selten, 1967)
### Structural Rationality

Every action choice
- takes into account **beliefs at all unexpected events**
- in a **principled** way

Loosely inspired by evidence on strategy method (Selten, 1967)

### Results:
- Implies sequential rationality *(generically equivalent)*
- Coincides with EU in simultaneous-move games
- Justifies the **elicitation** of all conditional beliefs
- Characterization via “minimally invasive” trembles
Dynamic games with perfect recall

Information sets (or nodes): $I, J \ldots \in \mathcal{I}_i$. Root: $\phi$, in every $\mathcal{I}_i$

Strategies $S_a = \{OutB, OutS, InB, InS\}; \ S_b = \{B, S\}$

Payoff function: $U_i(s_i, s_{-i})$; usual linear extension to $\Delta(S_{-i})$

Ann’s strategies allowing $J$: $S_a(J) = \{InB, InS\}$; $S_a(\phi) = S_a$, $S_a(J)$ are conditioning events
Dynamic games with perfect recall

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- Ann’s strategies allowing \( J \): \( S_a(J) = \{InB, InS\} \);
  \( S_a(\phi) = S_a, S_a(J) \) are conditioning events

This talk: “Nested Strategic Information” (paper generalizes)
Beliefs in Dynamic Games

Ann holds beliefs about $S_b$ at each infoset

Definition (Myerson, 1986; Ben-Porath 1997)

A **conditional probability system (CPS)** for $i$ is a collection

$\mu = \left\langle \mu(\cdot | S_{-i}(I)) \right\rangle_{I \in \mathcal{I}_i}$ such that

1. for all $I \in \mathcal{I}_i$, $\mu(\cdot | S_{-i}(I)) \in \Delta(S_{-i})$ and $\mu(S_{-i}(I)|S_{-i}(I)) = 1$

2. for all $I, J \in \mathcal{I}_i$ and $E \subseteq S_{-i}$ with $E \subseteq S_{-i}(I) \subseteq S_{-i}(J)$:

$$\mu(E|S_{-i}(J)) = \mu(E|S_{-i}(I)) \cdot \mu(S_{-i}(I)|S_{-i}(J)).$$

“Chain rule whenever possible”
Sequential Rationality

Definition (Sequential Rationality à la Reny - Rubinstein)

Fix a CPS $\mu$ for player $i$.

A strategy $s_i$ is sequentially rational (for $\mu$) iff, for all $I \in \mathcal{I}_i$ allowed by $s_i$, and all $t_i$ that also allow $I$,

$$U_i(s_i, \mu(\cdot|S_{-i}(I))) \geq U_i(t_i, \mu(\cdot|S_{-i}(I))).$$
Structural Rationality
Basic beliefs

Chain rule: if $S_{-i}(I) \subset S_{-i}(J)$ and $\mu(S_{-i}(I)|S_{-i}(J)) > 0$, beliefs at $I$ derived from beliefs at $J$

Definition

Fix a CPS $\mu$ for $i$.

$I \in \mathcal{I}_i$ is $\mu$-basic if $\mu(S_{-i}(I)|S_{-i}(J)) = 0$ for all $J \in \mathcal{I}_i$ with $S_{-i}(J) \supset S_{-i}(I)$

Belief $\mu(\cdot|S_{-i}(I))$ not derived from “earlier” beliefs
Basic beliefs

Chain rule: if $S_{-i}(I) \subset S_{-i}(J)$ and $\mu(S_{-i}(I)|S_{-i}(J)) > 0$, beliefs at $I$ derived from beliefs at $J$

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Belief $\mu(\cdot|S_{-i}(I))$ not derived from “earlier” beliefs

$S_{-i}(J) \supset S_{-i}(I)$, $\mu(S_{-i}(I)|S_{-i}(J)) = 0$ also suggest $J$ infinitely more likely than $I$
Definition (Structural Preferences over strategies)

Fix a CPS $\mu$ for $i$. Strategy $s_i$ is **structurally (weakly) preferred** to strategy $t_i$ ($s_i \succ^{\mu} t_i$) if, for every $\mu$-basic $I \in \mathcal{I}_i$ with

$$U(s_i, \mu(\cdot | S_{-i}(I))) < U(t_i, \mu(\cdot | S_{-i}(I))),$$

there is another $\mu$-basic $J \in \mathcal{I}_i$ with $S_{-i}(J) \supset S_{-i}(I)$ and

$$U(s_i, \mu(\cdot | S_{-i}(J))) > U(t_i, \mu(\cdot | S_{-i}(J))).$$

“$s_i$ infinitely more likely to be better than to be worse vs. $t_i$”
Structural Preferences

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“$s_i$ infinitely more likely to be better than to be worse vs. $t_i$”

“Break ties along each path”
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there is another $\mu$-basic $J \in I_i$ with $S_{-i}(J) \supset S_{-i}(I)$ and

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“$s_i$ infinitely more likely to be better than to be worse vs. $t_i$”

“Break ties along each path”

“Extensive-form analog of lexicographic preferences”
Definition (Structural Rationality)

Strategy $s_i$ is **structurally rational for** $\mu$ if there is no strategy $t_i$ such that $t_i \succ^\mu s_i$ (that is, $t_i \succ^\mu s_i$ and not $s_i \succ^\mu t_i$).

$\succ^\mu$ possibly incomplete, but transitive: existence guaranteed.
Structural preferences in action

Centipede.

\[ D_1 \succ_\mu A_1 D_2 \succ_\mu A_1 A_2 \]

\[ s_a \quad [\phi] \quad [I] \]

| \( D_1 \) | 2 | 2 |
| \( A_1 D_2 \) | 1 | 4 |
| \( A_1 A_2 \) | 1 | 3 |
Structural preferences in action

Centipede. $D_1 \succ^\mu A_1 D_2 \succ^\mu A_1 A_2$

$D_1$ also unique sequential best reply to $\mu$
Structural preferences in action

Extra power! $A_1D_2 \succeq^\mu A_1A_2 \succeq^\mu D_1$

Both $D_1$ and $A_1D_2$ sequential best replies to $\mu$
“Extensive-form analog of lexicographic preferences”

<table>
<thead>
<tr>
<th>Features of beliefs</th>
<th>Lexicographic</th>
<th>Structural</th>
</tr>
</thead>
<tbody>
<tr>
<td>Representation</td>
<td>LPS</td>
<td>CPS</td>
</tr>
<tr>
<td>Ordering of probabilities</td>
<td>arbitrary</td>
<td>set inclusion</td>
</tr>
<tr>
<td>Richness of ordering</td>
<td>complete</td>
<td>partial</td>
</tr>
<tr>
<td>Related to extensive form?</td>
<td>no</td>
<td>yes (CPS, basic events)</td>
</tr>
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Main Result 1: Structural implies Sequential

Theorem

Fix a CPS $\mu$ for player i. If $s_i \in S_i$ is structurally rational for $\mu$, then it is sequentially rational for $\mu$. 
Main Result 1: Structural implies Sequential

**Theorem**

Fix a CPS $\mu$ for player $i$. If $s_i \in S_i$ is structurally rational for $\mu$, then it is sequentially rational for $\mu$.

In static games, structural preferences coincide with EU. Aligned with experimental evidence!
Main Result 1: Structural implies Sequential

Theorem

Fix a CPS $\mu$ for player $i$. If $s_i \in S_i$ is structurally rational for $\mu$, then it is sequentially rational for $\mu$.

In static games, structural preferences coincide with EU. Aligned with experimental evidence!

Generic equivalence with sequential rationality
Main Result 2

Elicitation
Back to the Battle of the Sexes
Back to the Battle of the Sexes

<table>
<thead>
<tr>
<th>$s_b$</th>
<th>$S_a$</th>
<th>$S_a(J) = S_a(K)$</th>
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<tr>
<td>$pB$</td>
<td>$\frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0$</td>
<td>$\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot p$</td>
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Main Result 2: Eliciting Off-Path Beliefs (Bob)

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<th>Theorem (Elicitation – Bob’s beliefs in the subgame)</th>
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<td><strong>Fix</strong> Ann’s CPS $\mu$ and Bob’s CPS $\nu$ in the original game.</td>
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<tr>
<td>In the elicitation game, assume same beliefs about coplayer, independent of Chance’s move. Then, given these beliefs:</td>
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<td>- $s_a$ is <strong>structurally rational</strong> in the elicitation game iff $s_a$ is structurally rational in the original game</td>
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<td>- if $(s_b, b)$ [resp. $(s_b, p)$] is <strong>structurally rational</strong>, then $s_b$ is structurally rational and $\mu(S</td>
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Main Result 2: Eliciting Off-Path Beliefs (Bob)

Theorem (Elicitation – Bob’s beliefs in the subgame)

Fix Ann’s CPS $\mu$ and Bob’s CPS $\nu$ in the original game.

In the elicitation game, assume same beliefs about coplayer, independent of Chance’s move. Then, given these beliefs:

- $s_a$ is structurally rational in the elicitation game iff $s_a$ is structurally rational in the original game
- if $(s_b, b)$ [resp. $(s_b, p)$] is structurally rational, then $s_b$ is structurally rational and $\mu(S|S_{-i}(J)) \geq p$ (resp. $\leq p$)

- Initial, simultaneous choices reveal bound on Bob’s beliefs.
Main Result 2: Eliciting Off-Path Beliefs (Bob)

Theorem (Elicitation – Bob’s beliefs in the subgame)

Fix Ann’s CPS $\mu$ and Bob’s CPS $\nu$ in the original game.

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- Initial, simultaneous choices reveal bound on Bob’s beliefs.
- Analogous result in general games
Eliciting Ann’s initial beliefs

- Could offer Ann side bets at $\phi$ on Bob’s choices
- But in this SPE, Ann plays *Out*
- Incentives???
Elicitation and the strategy method

\[
\begin{align*}
&\text{Bob} \\
&\{Ou_{t}, b\} (2, 0), 2 \\
&\{Ou_{t}, p\} (2, p), 2 \\
&\{In_{B}, b\} (In_{B}, p) \\
&\{In_{S}, b\} (In_{S}, p)
\end{align*}
\]

\[
\begin{align*}
&\text{Ann} \\
&\{Ou_{t}, b\} (2, 1), 2 \\
&\{Ou_{t}, p\} (2, p), 2 \\
&\{In_{B}, b\} (In_{B}, p) \\
&\{In_{S}, b\} (In_{S}, p)
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&\text{Bob} \\
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Main Result 2: Eliciting On-path Beliefs (Ann)

**Theorem (Elicitation – Ann’s initial beliefs)**

*Fix Ann’s CPS $\mu$ and Bob’s CPS $\nu$ in the original game.*

*In the elicitation game, assume same beliefs about coplayers, independent of Chance’s move. Then:*

- $s_b$ is *structurally rational* in the elicitation game iff $s_b$ is *structurally rational* in the original game.

- if $(s_a, b)$ [resp. $(s_a, p)$] is *structurally rational*, then $s_a$ is *structurally rational* and $\mu(S|[[\phi]]) \geq p$ (resp. $\leq p$).

- Initial, simultaneous choices reveal bound on Ann’s beliefs.

- Again, analogous result for general games.
Main Result 3

Structural Rationality and Trembles
Ann’s CPS: $\mu(t|S_b) = 1$. Then $DT \succ^\mu U$.

Perturbation: $p_\epsilon(t) = 1 - \epsilon - \epsilon^2$, $p_\epsilon(m) = \epsilon$, $p_\epsilon(b) = \epsilon^2$.

Then $U_a(U, p_\epsilon) > U_a(DT, p_\epsilon)$
Ann’s CPS: $\mu(t|S_b) = 1$. Then $DT \sim^\mu U$.

Perturbation: $p_\epsilon(t) = 1 - \epsilon - \epsilon^2$, $p_\epsilon(m) = \epsilon^2$, $p_\epsilon(b) = \epsilon$.

Then $U_a(U, p_\epsilon) > U_a(DT, p_\epsilon)$
Perturbations and Spurious Beliefs (3)

▶ Ann’s CPS: $\mu(t|S_b) = \mu(m|S_b) = \frac{1}{2}$. Then $DT \succ^\mu U$.

▶ Perturbation: $p_\epsilon(t) = \frac{1}{2}$, $p_\epsilon(m) = \frac{1}{2} - \epsilon$, $p_\epsilon(b) = \epsilon$.

▶ Then $U_a(U, p_\epsilon) > U_a(DT, p_\epsilon)$
Main Result 3: Structural Rationality and Trembles

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<td>(s_i \in S_i) is structurally rational for (\mu) iff, for every (t_i \in S_i), there is a structural perturbation ((p^n)) of (\mu) such that (U(s_i, p^n) \geq U_i(t_i, p^b)) for all (n \geq 1.)</td>
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Conclusions

New optimality criterion: **Structural Rationality**

- Implies sequential rationality: the extensive form matters!
- Allows the elicitation of all conditional beliefs
- Also justifies the strategy method
- As a bonus, sometimes refines sequential rationality
- Characterization via “minimally invasive” *trembles*
- General games: Newcomb paradox, KW consistency
- Easy to add payoff uncertainty and higher-order beliefs
Papers

Now at http://faculty.wcas.northwestern.edu/~msi661

Sequential Rationality and Elicitation (this talk):
“Structural Preferences and Sequential Rationality”

Axiomatics:
“Foundations for Structural Preferences”

Ask me:

Forward induction
“Structural Preferences in Epistemic Game Theory”

THANK YOU!
Recall: $S_{-i}(I) =$ strategies of opponents reaching $I$

**Assumption (Nested strategic information)**

For every real player $i$ and infosets $I, J$ of $i$,

either $S_{-i}(I) \cap S_{-i}(J) = \emptyset$ or $S_{-i}(I) \subseteq S_{-i}(J)$ or $S_{-i}(J) \subseteq S_{-i}(I)$.

- Signalling games
- Games where a player moves only once on each path
- Games with centipede structure
- Ascending-clock auctions
- Event trees
Nested Strategic Information (2)

Rules out:

\[ S_{-i}(I) = \{tt', tb'\}; \quad S_{-i}(I') = \{tt', bt'\}. \] Not nested.
How about trembles? Removing actions?

Mechanical trembles: **no**

- Change the game (a fortiori if remove actions—e.g. $D_1$)
- Impact **strategic reasoning** (Reny, Ben-Porath, Bagwell)
- Also: **which** trembles (Binmore)? Details matter!
How about trembles? Removing actions?

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Belief perturbations (Kreps - Wilson, 1982): yes!

- Proposed approach also models infinitesimal probabilities
- Paper: novel (to me) implications of KW-style consistency
Structural Rationality for General Games
Non-nested strategic information: $[I] \not\supset [J], [J] \not\supset [I]$

$\mu(o|S_b) = 1; \quad \mu(t|[I]) = \mu(m|[I]) = \frac{1}{2}; \quad \mu(m|[J]) = \mu(b|[J]) = \frac{1}{2}$

$RB$ is "structurally rational:" see payoff given $\mu(\cdot|[J])$

Yet, $RB$ is not sequentially rational!
Step 1: Likelihood ordering

\[ J \supset [I], \mu([I][J]) = 0 \] suggests \( J \) “infinitely more likely” than \( I \)

Notice \( \mu([J][I]) > 0 \) (indeed, 1) because \( [J] \supset [I] \).

Generalize: even if \( [I], [J] \) not nested, \( \mu([J][I]) > 0 \), suggests \( [J] \) “not infinitely less likely” than \( [I] \)

Likelihood should be transitive. Hence:

**Definition (Likelihood ordering)**

\[ [J] \geq^\mu [I] \text{ iff there are } I_1, \ldots, I_L \in \mathcal{I}_i \text{ with } I_1 = I, I_L = J, \text{ and} \]

\[ \mu([I_{\ell+1}][I_{\ell}]) > 0 \quad \ell = 1, \ldots, L - 1. \]
Step 2: Basic event — back to the example

\[
\begin{align*}
\mu(o|S_b) &= 1; \quad \mu(t|I) = \mu(m|I) = \frac{1}{2}; \quad \mu(m|J) = \mu(b|J) = \frac{1}{2}
\end{align*}
\]

Definition of likelihood implies \( S_b >^\mu I =^\mu J \). Intuitive!

\( \mu(\cdot|I), \mu(\cdot|J) \) are updates of uniform prob on \([I] \cup [J] = \{ t, m, b \}\)

Take \([I] \cup [J]\) as basic event: prob uniquely identified from \( \mu! \)
Step 2: Basic events — definition

**Definition (CPS on general conditioning events)**

Fix a CPS $\mu$ for $i$ and consider $\geq \mu$. Let

$$G_i = \left\{ \bigcup_{k=1}^{K} [l_k] : K \in \mathbb{N}, \, , [l_k] = \mu [l_\ell] \, \forall \ell, k = 1, \ldots, K \right\}.$$  

The **extension** of $\mu$ is a CPS $\nu$ on $S_{\neg i}$ with conditioning events $G_i$ such that

$$\forall l \in I_i, \quad \nu(\cdot|[l]) = \mu(\cdot|[l]).$$

Note: $[l] \in G_i$ for all $l \in I_i$.

Existence and uniqueness of basis: later, or ask me.
**Step 3: General Structural Preferences**

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<th>Definition (Structural Preferences over strategies)</th>
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<td>Fix a CPS $\mu$ for player $i$ that admits an extension $\mu$. Strategy $s_i$ is <strong>structurally (weakly) preferred</strong> to strategy $t_i$ ($s_i \succeq^\mu t_i$) if, for every $F \in \mathcal{G}_i$ with</td>
</tr>
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<td>$\int U(s_i, s_{-i}) d\nu(s_{-i}</td>
</tr>
<tr>
<td>there is $G \in \mathcal{G}_i$ with $G \succeq^\nu F$ and</td>
</tr>
<tr>
<td>$\int U(s_i, s_{-i}) d\nu(s_{-i}</td>
</tr>
</tbody>
</table>

Same as before, but using **extension $\nu$** instead of $\mu$. 
Structural preferences in action

\[ \mu(o|S_b) = 1; \quad \mu(t|[I]) = \mu(m|[I]) = \frac{1}{2}; \quad \mu(m|[J]) = \mu(b|[J]) = \frac{1}{2} \]

Likelihood: \( S_b \succ^\mu [I], S_b \succ^\mu [J], [I] = \mu [J] \)

\( G_a = \{ S_b, [I], [J], [I] \cup [J] \} \). Extension: \( \nu(\cdot|[I] \cup [J]) \) uniform

Basic events for \( \nu \): \( S_b, [I] \cup [J] \)

\( RT \succ^\mu RB \succ^\mu LT' \succ^\mu LB' \). \( RT \) structurally rational; unique
Congruent CPSs and Extensions
A Newcombe Paradox for CPSs

CPS: $\mu(o|S_b) = 1; \mu(b|[l]) = 1, \mu(c|[l']) = 1$

Set of sequential best replies: $LT, RT$.

Kreps-Wilson consistency, Myerson complete CPSs: 
\{LT, RT\} cannot be the set of sequential best replies

Indeed $\mu$ does not admit an extension!
Main Result 3: Congruent CPSs

\( \mu \) is **congruent** if, for every \((F_m)_{n=1}^{N}\) with \(\mu(F_{n+1}|F_n) > 0\), \(n = 1, \ldots, N - 1\), and every \(E \subseteq F_1 \cap F_N\),

\[
\mu(E|F_1) \cdot \prod_{n=1}^{N-1} \frac{\mu(F_n \cap F_{n+1}|F_{n+1})}{\mu(F_n \cap F_{n+1}|F_n)} = \mu(E|F_N)
\]

Congruence implies the Chain Rule: take \(F_1 \subset F_2\).

---

**Theorem**

The following are equivalent:

- \( \mu \) is congruent
- \( \mu \) is generated by taking limits of strictly positive probabilities
- \( \mu \) admits an extension, which is unique
Structural preferences in action: Extra Power!

\[ S_b ⊃ [l] = \{ a \}; \mu(d|S_b) = 1; \mu(a|[l]) = 1. \]

<table>
<thead>
<tr>
<th>( s_a )</th>
<th>( S_b )</th>
<th>([l] = { a } )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>2</td>
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</tr>
<tr>
<td>( A_1 D_2 )</td>
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<td>1</td>
</tr>
<tr>
<td>( A_1 A_2 )</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

\( D_1 ≻^\mu A_1 D_2 ≻^\mu A_1 A_2 \)
Structural preferences in action: Extra Power!

Yet $A_1 D_2$ sequentially rational: at $l$, no longer care about $D_1$
Structural preferences in action: Extra Power!

\[ s_b \supset [l] = \{a\}; \mu(d|S_b) = 1; \mu(a|[l]) = 1. \]

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\(D_1 \succ^\mu A_1D_2\) reflects ex-ante view: at \(\phi\), can still choose \(D_1\)
Structural preferences in action: Extra Power!

$S_b \supset [l] = \{a\}; \mu(d|S_b) = 1; \mu(a|[l]) = 1.$

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$(D_1 \succ^\nu A_1 D_2$ for any CPS $\nu$ — not just this $\mu$.)