Efficient Disposal Equilibria of Pseudomarkets

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Workshop on Game Theory
Institute for Mathematical Sciences
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Introduction

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Outcomes may need to be computed.
Plan of the Talk

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• Two open problems are described.
Hylland and Zeckhauser (1979) study a setting in which there are:
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- They propose equilibrium allocations of a market with currency endowments and goods that are probabilities of being assigned to each object.
Compact Consumption Sets

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- These papers do not allow free disposal.
Course Allocation

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• Budish and Cantillon (2012): versions of random priority used at Harvard.
  Budish and Kessler (2016): auction and market-like mechanisms used at Wharton.

• Budish, Che, Kojima, and Milgrom (2013) (henceforth BCKM) study probabilistic allocations of seats.
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  - They give conditions under which assignments of probabilities can be realized by distribution over pure assignments.
  - They also give a highly restricted existence theorem generalizing HZ.
Other Pseudomarket Papers

- Eisenberg and Gale (1959) and Eisenberg (1961): equilibrium of a pari-mutuel betting system.
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Of course there is also a vast literature on matching and school choice. In such models usually (not always!) both sides of the market are strategic.
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- There are compact production sets \( Y_1, \ldots, Y_n \subset \mathbb{R}^\ell \) that contain the origin.
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- There are compact production sets $Y_1, \ldots, Y_n \subset \mathbb{R}^\ell$ that contain the origin.
- There is an $m \times n$ matrix $\theta$ of nonnegative ownership shares such that $\sum_i \theta_{ij} = 1$ for all $j$. 
• If $u_i(x_i) = \max_{x_i' \in X_i} u_i(x_i')$, then agent $i$ is sated at $x_i \in X_i$ and $x_i$ is a bliss point for $i$. Otherwise $i$ is unsated at $x_i$. 
• If $u_i(x_i) = \max_{x_i' \in X_i} u_i(x_i')$, then agent $i$ is *sated* at $x_i \in X_i$ and $x_i$ is a *bliss point* for $i$. Otherwise $i$ is *unsated* at $x_i$.

• Let $X = \prod_i X_i$ and $Y = \prod_j Y_j$. 
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• Let \( X = \prod_i X_i \) and \( Y = \prod_j Y_j \).

• For each \( j \) and \( p \in \mathbb{R}^\ell \) let

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\pi_j(p) = \max_{y_j \in Y_j} \langle p, y_j \rangle, \quad M_j(p) = \arg\max_{y_j \in Y_j} \langle p, y_j \rangle.
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• For each \( i \) and \( p \in \mathbb{R}^\ell \), \( i \)’s total income is

\[
\mu_i(p) = \langle p, \omega_i \rangle + \sum_j \theta_{ij} \pi_j(p).
\]
Efficient Disposal Equilibrium

A triple \((p, x, y) \in \mathbb{R}_+^\ell \times X \times Y\) is an efficient disposal equilibrium (EDE) if:
A triple \((p, x, y) \in \mathbb{R}_+^l \times X \times Y\) is an efficient disposal equilibrium (EDE) if:

(a) For each \(i\) there is no \(x'_i \in X_i\) such that 
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\langle p, x'_i \rangle \leq \langle p, x_i \rangle \quad \text{and} \quad u_i(x'_i) > u_i(x_i),
\]
and there is no \(x'_i \in X_i\) such that 
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(b) For each \(i\), if \(i\) is unsated at \(x_i\), then \(\langle p, x_i \rangle \geq \mu_i(p)\).
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(b) For each \(i\), if \(i\) is unsated at \(x_i\), then \(\langle p, x_i \rangle \geq \mu_i(p)\).

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(b) For each $i$, if $i$ is unsated at $x_i$, then $\langle p, x_i \rangle \geq \mu_i(p)$.

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(d) $\sum_i x_i \leq \omega + \sum_j y_j$. 


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(c) For each \(j\), \(y_j \in M_j(p)\).

(d) \(\sum_i x_i \leq \omega + \sum_j y_j\).

(e) For all \(h\), if \(\sum_i x_{ih} < \omega_h + \sum_j y_{jh}\), then \(p_h = 0\).
The Main Result

Let $e = (1, \ldots, 1) \in \mathbb{R}^\ell$, $V_0 = \{ x \in \mathbb{R}^\ell : \langle e, x \rangle = 0 \}$. 
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Let \( e = (1, \ldots, 1) \in \mathbb{R}^\ell \), \( V_0 = \{ x \in \mathbb{R}^\ell : \langle e, x \rangle = 0 \} \).

**Theorem:** If, for each \( i \) there is an \( x_i^0 \in X_i \) such that 
\( x_i^0 \leq \omega_i \), \( X_i \subset x_i^0 + V_0 \), and \( x_i^0 \) is in the interior 
(relative to \( x_i + V_0 \)) of \( X_i \), then for any \( \alpha \in \mathbb{R}^m_+ \) there 
is an EDE \( (p, x, y) \) such that

\[
\langle p, x_i \rangle - \mu_i(p) = \frac{\alpha_i}{\sum_{i' \in U} \alpha_{i'}} \left( \sum_{i'' \in S} \mu_{i''}(p) - \langle p, x_{i''} \rangle \right)
\]

for all \( i \in U \), where \( U \) is the set of \( i \) that are unsated 
at \( x_i \) and \( S = \{1, \ldots, m\} \setminus U \) is the set of \( i \) that are 
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for all $i \in U$, where $U$ is the set of $i$ that are unsated at $x_i$ and $S = \{1, \ldots, m\} \setminus U$ is the set of $i$ that are sated at $x_i$.

This generalizes all prior existence results.
Approaches to the Proof

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Prices and excess demand:

- The natural space of prices is $V_0$, which has two problems:
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  - Aggregate demand may be less valuable than aggregate supply because of satiation.
Our Methods

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- For each $j$ let $\tilde{Y}_j = \{0\} \times Y_j$. 
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  • $\tilde{Z}$ is upper hemicontinuous.
  • $\langle \tilde{p}, \tilde{z} \rangle = 0$ if $\tilde{z} \in \tilde{Z}(\tilde{p})$ (all income is spent).
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• $\langle \tilde{p}, \tilde{z} \rangle = 0$ if $\tilde{z} \in \tilde{Z}(\tilde{p})$ (all income is spent).
• Of $\tilde{p}_0 = \varepsilon$, then $\tilde{z}_0 > 0$ for all $\tilde{z} \in \tilde{Z}(\tilde{p})$. 
• For a small $\varepsilon > 0$ let

$$S_\varepsilon = \{ \tilde{p} = (\tilde{p}_0, p) \in V : \|\tilde{p}\| = 1 \text{ and } \tilde{p}_0 \geq \varepsilon \}.$$ 

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  • Of $\tilde{p}_0 = \varepsilon$, then $\tilde{z}_0 > 0$ for all $\tilde{z} \in \tilde{Z}(\tilde{p})$.
  • Thus $\tilde{Z}$ is an uhc vector field correspondence that is inward pointing on the boundary of $S_\varepsilon$, so the (generalized) Poincaré-Hopf theorem gives a $\tilde{p}^* \in S_\varepsilon$ such that $0 \in \tilde{Z}(\tilde{p}^*)$. 
A Technical Finesse

Recall that a *polyhedron* in $\mathbb{R}^\ell$ is an intersection of finitely many closed half spaces, and a *polytope* is a bounded polyhedron.
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**Proposition:** If $P_1$ and $P_2$ are polyhedra in $\mathbb{R}^\ell$, $Q = \{ q \in \mathbb{R}^\ell = (P_1 + q) \cap P_2 \neq \emptyset \}$, and $I : Q \to \mathbb{R}^\ell$ is the correspondence $I(q) = (P_1 + q) \cap P_2$, then $I$ is continuous.
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We take a sequence of expanded economies given by a sequence of endowments of the artificial good that go to zero and a sequence of polyhedra $X_i^k \subset X_i$ such that $X_i^k \to X_i$ and $X_i^k \cap \overline{X}_i \to \overline{X}_i$. 
Concluding Remarks

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The traditional concerns of general equilibrium theory are (mostly) meaningful and conceptually pertinent in relation to pseudomarkets, so one can easily produce a host of original and meaningful problems for further research.