Chain Stability in Trading Networks

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Commodity Markets vs. Matching Markets

Is buying a stock listed on the Singapore Exchange similar to applying for being admitted to study at the National University of Singapore?

**Commodity markets:**
- small trades in large markets for “standardized” commodities are arms-length and anonymous
- the market helps do the “price discovery” and price alone determines who gets what
- as long as you can afford it, you can choose what you want

**Matching markets:**
- individual “personalized” trades; the identity of the parties matters
- prices have a more limited role
- you can’t just choose what you want, you also have to be chosen: double coincidence of choice
From 2-Sided Matching to Trading Networks (TN)

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An Existential Question

Is the market in equilibrium?
How would we know...?
Cooperative solution concepts often rely on coordinated deviations by large groups of agents:

- How can such coalitions in fact form?
- Do all the agents in the economy need to consider all the possible deviations by all the possible coalitions?
An Existential Question

Competitive equilibria requires specifying prices for all possible goods and trades in the economy – even those that are not actually traded:

- Do the agents know what the “unobserved” prices are?
- Can agents tell if a particular set of realized contracts is consistent with competitive equilibrium?
An Existential Question

Is the market in equilibrium?
How would we know...

[Equilibrium] is characterised by an existence that precedes its essence. —J. P. Sartre
An Existential Question

Is the market in equilibrium?
How would we know...?

[Equilibrium] is characterised by an existence that precedes its essence. —J. P. Sartre

What the heck does that mean!?
An Existential Question

We might have more confidence in solution concepts that are easier to verify.

Or – even better – those that agents themselves can evaluate.¹

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¹Agents might test relatively simple perturbations.
Overview: Framework

We consider general trading networks with indivisible goods:

- agents can be both buyers and sellers (and trade in cycles!),
- discrete or continuous prices,
- quasi-linear or non-quasilinear utilities, and
- with or without indifferences in agents’ preferences,
Overview: Framework

We consider general trading networks with indivisible goods:
- agents can be both buyers and sellers (and trade in cycles!),
- discrete or continuous prices,
- quasi-linear or non-quasilinear utilities, and
- with or without indifferences in agents’ preferences,

\[ \sim \text{stability} = \text{no deviations by sets of agents} \]
\[ \sim \text{chain stability} = \text{no deviations by chains of agents:} \]

\[ f \rightarrow g \rightarrow h \]

\[ f \leftarrow g \rightarrow h \]

\[ f \rightarrow g \rightarrow j \]

\[ f \leftarrow g \rightarrow j \]
Overview: Results

Chain stability is asymptotically easier to check than stability.

Main Result

If all agents’ preferences are “jointly” fully substitutable and satisfy the Laws of Aggregate Supply and Demand, then

\[ \text{stability is equivalent to chain stability}, \]

and we further show that our assumptions are independent.

- When utility is fully transferable, if all agents’ preferences are fully substitutable, then an outcome is consistent with competitive equilibrium if and only if it is not blocked by a chain.
Setting: Trades and Contracts

Finite set of agents \( I \)

Finite set of bilateral trades \( \Omega \)

- each trade \( \omega \in \Omega \) has a seller \( s(\omega) \) and a buyer \( b(\omega) \)

Set of contracts \( X \equiv \Omega \times \mathbb{R} \)

- each contract \( x \in X \) is a pair \( (\omega, p_\omega) \)

A \textit{(feasible) outcome} is a set of contracts \( A \subseteq X \) which uniquely prices each trade in \( A \).

For each \( i \in I \) and \( Y \subseteq X \), we let \( Y_i \equiv Y_{i\rightarrow} \cup Y_{\rightarrow i} \), where \( Y_{i\rightarrow} \equiv \{ y \in Y : i = s(y) \} \) and \( Y_{\rightarrow i} \equiv \{ y \in Y : i = b(y) \} \).

\[ a(Y) \equiv \bigcup_{y \in Y} \{ b(y), s(y) \} \]
Preferences

The utility of agent $i$ over feasible sets $Y \subseteq X_i$ is $U_i$, where

- $U_i(Y) \in \mathbb{R} \cup \{-\infty\}$, and
- $U_i(\emptyset) \in \mathbb{R}$.

The choice correspondence of agent $i$ from a set of contracts $Y \subseteq X_i$ is:

$$C_i(Y) \equiv \{Z \subseteq Y : Z \text{ is feasible}; \forall \text{ feasible } Z' \subseteq Y, \ U_i(Z) \geq U_i(Z')\}.$$

A1. The preferences of agent $i$ are fully substitutable (FS).

A2. The preferences of agent $i$ satisfy the Laws of Aggregate Supply and Demand (LoASD).

A3. The preferences of agent $i$ are monotone substitutable (MS), i.e., we require that A1 and A2 jointly hold.
Full Substitutability

The preferences of agent $i$ are fully substitutable if:

1. for all $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$ and $Y_{i\rightarrow} = Z_{i\rightarrow}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{\rightarrow i} \setminus Y^*_{i\rightarrow}) \subseteq (Z_{\rightarrow i} \setminus Z^*_{i\rightarrow})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$;

2. for all $Y, Z \subseteq X_i$ such that $|C_i(Y)| = |C_i(Z)| = 1$, $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$ and $Y_{i\rightarrow} = Z_{i\rightarrow}$, for the unique $Y^* \in C_i(Y)$ and $Z^* \in C_i(Z)$, we have $(Y_{i\rightarrow} \setminus Y^*_{i\rightarrow}) \subseteq (Z_{i\rightarrow} \setminus Z^*_{i\rightarrow})$ and $Y^*_{i\rightarrow} \subseteq Z^*_{i\rightarrow}$. 
Full Substitutability: Discussion

Full substitutability

- rules out: large fixed costs, economies of scale, complementarities in production or consumption, capacity constraints;

- allows for: goods (homogeneous or heterogeneous) with diminishing marginal utilities of consumption and increasing marginal costs of production, etc.
The preferences of agent \( i \) satisfy the Law of Aggregate Demand if for all \( Y, Z \subseteq X_i \) such that \( |C_i(Y)| = |C_i(Z)| = 1 \), \( Y_i \subseteq Z_i \) and \( Y_i = Z_i \), for the unique \( Y^* \in C_i(Y) \) and \( Z^* \in C_i(Z) \), we have
\[
|Z^*_{\rightarrow i}| - |Y^*_{\rightarrow i}| \geq |Z_{i\rightarrow}| - |Y_{i\rightarrow}|.
\]

The preferences of agent \( i \) satisfy the Law of Aggregate Supply if for all \( Y, Z \subseteq X_i \) such that \( |C_i(Y)| = |C_i(Z)| = 1 \), \( Y_i \subseteq Z_i \) and \( Y_i = Z_i \), for the unique \( Y^* \in C_i(Y) \) and \( Z^* \in C_i(Z) \), we have
\[
|Z^*_{i\rightarrow}| - |Y^*_{i\rightarrow}| \geq |Z_{\rightarrow i}| - |Y_{\rightarrow i}|.
\]
Monotone-substitutability

1. For all finite sets $Y, Z \subseteq X_i$ such that $Y_{i\rightarrow} = Z_{i\rightarrow}$ and $Y_{\rightarrow i} \subseteq Z_{\rightarrow i}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that both $Y^*$ and $Z^*$ are consistent with

- the full substitutability condition (on the demand-side), i.e.,
  $$(Y_{\rightarrow i} \setminus Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \setminus Z^*_{\rightarrow i})$$
  and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$, and with

- the Law of Aggregate Demand, i.e.,
  $$|Z^*_{\rightarrow i}| - |Z^*_{\rightarrow i}| \geq |Y^*_{\rightarrow i}| - |Y^*_{\rightarrow i}|.$$

2. For all finite sets $Y, Z \subseteq X_i$ such that $Y_{\rightarrow i} = Z_{\rightarrow i}$ and $Y_{i\rightarrow} \subseteq Z_{i\rightarrow}$, for every $Y^* \in C_i(Y)$ there exists $Z^* \in C_i(Z)$ such that both $Y^*$ and $Z^*$ are consistent with

- the full substitutability condition (on the supply-side), i.e.,
  $$(Y_{\rightarrow i} \setminus Y^*_{\rightarrow i}) \subseteq (Z_{\rightarrow i} \setminus Z^*_{\rightarrow i})$$
  and $Y^*_{\rightarrow i} \subseteq Z^*_{\rightarrow i}$, and with

- the Law of Aggregate Supply, i.e.,
  $$|Z^*_{\rightarrow i}| - |Z^*_{\rightarrow i}| \geq |Y^*_{\rightarrow i}| - |Y^*_{\rightarrow i}|.$$
A non-empty set of contracts \( Z \) is a *chain* if its elements can be arranged in some order \( y^1, \ldots, y^{|Z|} \) such that \( s(y^l+1) = b(y^l) \) for all \( l \in \{1, 2, \ldots, |Z| - 1\} \). Note that “self-crossing” chains are allowed.
Stability and Chain Stability

A non-empty set of contracts $Z$ is a *chain* if its elements can be arranged in some order $y^1, \ldots, y^{|Z|}$ such that $s(y^l + 1) = b(y^l)$ for all $l \in \{1, 2, \ldots, |Z| - 1\}$. Note that “self-crossing” chains are allowed.

An outcome $A$ is **stable** if it is

1. *Individually rational*: for each $i \in I$, $A_i \in C_i(A)$;

2. *Unblocked*: There is no feasible nonempty blocking set $Z \subseteq X$ such that

   - $Z \cap A = \emptyset$ and
   - for each $i \in a(Z)$, for each of $i$’s choices $Y_i \in C_i(Z \cup A)$, we have $Z_i \subseteq Y_i$. 


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Main Equivalence

Theorem

*If all agents’ preferences are monotone-substitutable, then any chain stable outcome is stable.*
Main Equivalence

Theorem

Suppose that all agents’ preferences are monotone-substitutable.

Consider any outcome $A$ that is blocked by some nonempty set $Z$.

Then for some $M \geq 1$, we can partition $Z$ into a collection of $M$ chains $W^m$ such that $Z = \bigcup_{m=1}^{M} W^m$, $A$ is blocked by $W^1$, and for any $m \leq M - 1$, the set $A \cup W^1 \cup \cdots \cup W^m$ is blocked by chain $W^{m+1}$.
Lemma

For any feasible outcome $A$ blocked by a nonempty set $Z$, if $Z$ is not itself a chain, then there exists a nonempty chain $W \subseteq Z$ such that set $A$ is blocked by $Z \setminus W$ and set $A \cup (Z \setminus W)$ is blocked by $W$.

Intuitively, the goal is to “peel off” a chain, $W$, from the set $Z$ in such a way that the remaining set $Z \setminus W$ still blocks $A$.

Take any $y \in Z$ and consider its buyer, $j = b(y)$, and seller, $k = s(y)$. Since $y \in Z$, $y \in C_j(A \cup Z) \cap C_k(A \cup Z)$. 
Proof Sketch - I

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Take any y \in Z and consider its buyer, j = b(y), and seller, k = s(y). Since y \in Z, y \in C_j(A \cup Z) \cap C_k(A \cup Z).

What happens if we remove y from Z?
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For any feasible outcome $A$ blocked by a nonempty set $Z$, if $Z$ is not itself a chain, then there exists a nonempty chain $W \not\subset Z$ such that set $A$ is blocked by $Z \setminus W$ and set $A \cup (Z \setminus W)$ is blocked by $W$.

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What happens if we remove $y$ from $Z$?

1. if all contracts other than $y$ in $Z$ remain chosen by $j$ and $k$, $y$ is a trivial “peeled off” chain 😊;
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What happens if we remove $y$ from $Z$?

1. if all contracts other than $y$ in $Z$ remain chosen by $j$ and $k$, $y$ is a trivial “peeled off” chain 😊;
2. suppose not all contracts other than $y$ in $Z$ remain chosen by $j$ ...
Lemma

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If not all contracts other than $y$ in $Z$ remain chosen by $j$:

1. as the preferences of $j$ are FS, we know that all of the contracts in $Z_{\rightarrow j} \setminus \{y\}$ are in the choice of $j$ from $A \cup Z \setminus \{y\}$,
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2. by the LoAD, we know that there is exactly one contract in $Z_{j \rightarrow}$—say, $y'$—that is not in the choice of $j$ from $A \cup Z \setminus \{y\}$.
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If not all contracts other than $y$ in $Z$ remain chosen by $j$:

1. as the preferences of $j$ are FS, we know that all of the contracts in $Z \to j \setminus \{y\}$ are in the choice of $j$ from $A \cup Z \setminus \{y\}$,

2. by the LoAD, we know that there is exactly one contract in $Z_{j\to}$—say, $y'$—that is not in the choice of $j$ from $A \cup Z \setminus \{y\}$.

Thus, all contracts in $Z_j \setminus \{y, y'\}$ are in the choice of $j$ from $A \cup Z \setminus \{y, y'\}$—which is exactly the blocking condition for $j$. 
Lemma

For any feasible outcome $A$ blocked by a nonempty set $Z$, if $Z$ is not itself a chain, then there exists a nonempty chain $W \supseteq Z$ such that set $A$ is blocked by $Z \setminus W$ and set $A \cup (Z \setminus W)$ is blocked by $W$.

Of course, this condition may still not be satisfied for the seller of contract $y$ or the buyer of contract $y'$, in which case we continue expanding the “peeled off” chain in an analogous manner.

At some point, the process has to stop (because set $Z$ is finite), giving us the required chain $W$ such that $A$ is blocked by $Z \setminus W$. 
Why do we care? (I)

Theorem

*If all agents’ preferences are monotone–substitutable, then stability is equivalent to chain stability.*

- General result (covers many important settings).\(^2\)
- Refinement in quasilinear case:
  - Full substitutability $\implies$ [chain stability $\sim$ CE].
- (Partially) addresses our existential question!

\(^2\)Sartre: “Existence is an imperfection.”
Why do we care? (II)

* Chain stability is much easier to check than stability!

**Theorem**

The density of the set of chains (in the set of potential blocking sets) goes to 0 as $|\Omega| \to \infty$.\(^a\)

\(^a\)Pretty cool proof!

Simplicity gain is stronger with additional structure:

- **Supply Chain** $\leadsto$ the total number of chains is polynomial; total number of sets is exponential.
- **Exchange Economy** $\leadsto$ constant bound.
- **Quasilinear Case** $\leadsto$ polynomial-time algorithm for finding chain blocks due to Candogan–Epitropou–Vohra (2016).
Quantifying the Simplicity Gain

Theorem

For any sequence of economies \((I, \Omega^m)_{m=1}^\infty\) such that \(|\Omega^m| = m\) for all \(m\), for any \(Y\), we have that

\[
\frac{\mathcal{C}^m(Y)}{\mathcal{B}^m(Y)} = O \left( \frac{\sqrt{\log_2 m}}{\sqrt{m}} \right).
\]

In particular, the ratio of [the set of chains of trades corresponding to possible blocking chains] to [the set of sets of trades corresponding to possible blocking sets] \(\frac{\mathcal{C}^m(Y)}{\mathcal{B}^m(Y)} \to 0\) as \(m \to \infty\).
Competitive Equilibrium with Personalized Prices

Each agent $i$ has a quasilinear utility function $U_i(\cdot)$ over each arrangement $[\Psi, p] \in \Omega \times \mathbb{R}^{|\Omega|}$, 

$$ U_i([\Psi; p]) = u_i(\Psi_i) + \sum_{\psi \in \Psi_i} p_\psi - \sum_{\psi \in \Psi \setminus \Psi_i} p_\psi. $$

For any vector of prices $p \in \mathbb{R}^{|\Omega|}$, the demand of $i$ is

$$ D_i(p) = \arg \max_{\Psi \subseteq \Omega_i} U_i([\Psi, p]). $$

An arrangement $[\Psi, p]$ is a competitive equilibrium (CE) if for each $i \in I$, 

$$ \Psi_i \in D_i(p). $$
For a more restricted environment, in which prices are continuous and unrestricted (i.e., the set of contracts $X$ is $X = \Omega \times \mathbb{R}$) and agents’ preferences are fully substitutable and quasilinear in prices:

**Corollary (using Hatfield et al. (2013))**

Suppose that the set of contracts is $X = \Omega \times \mathbb{R}$, and that all agents’ preferences are fully substitutable and quasilinear in prices. Then, an outcome is consistent with competitive equilibrium if and only if it is chain stable.
Examples: No LoASD

Agent $i$’s preferences do not satisfy LoAD.

\[
\{w, x, y, z\} > \{w, x, y\} > \{w, x, z\} > \{w, y, z\} > \{w, x\} > \{w, y\} > \{w, z\} > \{w\} > \emptyset.
\]

\[
\{w, x, y, z\} > \{x, y, z\} > \{x, y\} > \{x, z\} > \{y, z\} > \{x\} > \{y\} > \{z\} > \emptyset.
\]

$\emptyset$ is not stable (blocked by $\{w, x, y, z\}$) but it is chain stable.
Examples: No FS

\[ \{x\} > \emptyset \]

\[ \{y\} > \emptyset \]

Agent \( i \)'s preferences do not satisfy FS.

\( \emptyset \) is not stable (blocked by \( \{x, y\} \)) but it is chain stable.
Examples: No “Crossing” Chains

\{x^1, y^1\} > \emptyset \quad \{x^2, y^2\} > \emptyset

\{x^1, y^1, x^2, y^2\} > \{x^1, y^2\} > \{x^2, y^1\} > \emptyset

\emptyset \text{ is not chain stable (blocked by “self-crossing” chain } \{x^1, y^1, x^2, y^2\}\).

If attention is restricted to chains that do not cross themselves, \emptyset \text{ is robust to deviations by such chains.}
If all agents’ preferences are “jointly” fully substitutable and satisfy the Laws of Aggregate Supply and Demand, then:

**stability is equivalent to chain stability.**

- And chain stability is easier to check than stability!

Or as Sartre would say:

“Little flashes of sun *(chains)*
on the surface of a cold, dark sea *(all blocking sets).*”