B-spline Generated Frames and their Extensions to Locally Compact Abelian Groups

Say Song Goh

matgohss@nus.edu.sg

Department of Mathematics
National University of Singapore

(Joint Work with Ole Christensen)

Workshop on Spline Approximation and its Applications on Carl de Boor’s 80th Birthday
December 4 to 6, 2017
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
- Scaling partition of unity and B-spline generated wavelet frames for $L^2(\mathbb{R})$. 
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
- Scaling partition of unity and B-spline generated wavelet frames for $L^2(\mathbb{R})$.
- Frames on locally compact abelian groups.
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
- Scaling partition of unity and B-spline generated wavelet frames for $L^2(\mathbb{R})$.
- Frames on locally compact abelian groups.
- Gabor frames generated by weighted B-splines on dual group $\hat{G}$. 
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
- Scaling partition of unity and B-spline generated wavelet frames for $L^2(\mathbb{R})$.
- Frames on locally compact abelian groups.
- Gabor frames generated by weighted B-splines on dual group $\hat{G}$.
- Unitary extension principle on locally compact abelian groups.
Outline of Talk

- B-splines, wavelet frames and Gabor frames on $\mathbb{R}$.
- Scaling partition of unity and B-spline generated wavelet frames for $L^2(\mathbb{R})$.
- Frames on locally compact abelian groups.
- Gabor frames generated by weighted B-splines on dual group $\hat{G}$.
- Unitary extension principle on locally compact abelian groups.
- Tight wavelet frames generated by B-splines on original group $G$. 
A sequence \( \{f_n\}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if there exist \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{n \in I} |\langle f, f_n \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{H}.
\]
Frames

- A sequence \( \{f_n\}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if there exist \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \sum_{n \in I} |\langle f, f_n \rangle|^2 \leq B \| f \|^2, \quad f \in \mathcal{H}.
\]

- If the upper inequality holds, then \( \{f_n\}_{n \in I} \) is a Bessel sequence in \( \mathcal{H} \).
A sequence \( \{f_n\}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) is a frame for \( \mathcal{H} \) if there exist \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{n \in I} |\langle f, f_n \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{H}.
\]

If the upper inequality holds, then \( \{f_n\}_{n \in I} \) is a Bessel sequence in \( \mathcal{H} \).

If \( A = B = 1 \), then \( \{f_n\}_{n \in I} \) forms a tight frame for \( \mathcal{H} \).
A sequence \( \{ f_n \}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) is a **frame** for \( \mathcal{H} \) if there exist \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \sum_{n \in I} | \langle f, f_n \rangle |^2 \leq B \| f \|^2, \quad f \in \mathcal{H}.
\]

If the upper inequality holds, then \( \{ f_n \}_{n \in I} \) is a **Bessel sequence** in \( \mathcal{H} \).

If \( A = B = 1 \), then \( \{ f_n \}_{n \in I} \) forms a **tight frame** for \( \mathcal{H} \).

Two Bessel sequences \( \{ f_n \}_{n \in I} \) and \( \{ \tilde{f}_n \}_{n \in I} \) in \( \mathcal{H} \) are **dual frames** for \( \mathcal{H} \) if

\[
\sum_{n \in I} \langle f, f_n \rangle \langle g, \tilde{f}_n \rangle = \langle f, g \rangle, \quad f, g \in \mathcal{H}.
\]
For $N \in \mathbb{N}$, the uniform B-spline of order $N$ is defined as

$$\phi(x) := \chi_{[0,1]} \ast \cdots \ast \chi_{[0,1]}(x), \quad x \in \mathbb{R}.$$
For $N \in \mathbb{N}$, the uniform B-spline of order $N$ is defined as

$$\phi(x) := \chi_{[0,1]} \ast \cdots \ast \chi_{[0,1]}(x), \quad x \in \mathbb{R}. $$

$\phi$ satisfies the translation partition of unity property:

$$\sum_{j \in \mathbb{Z}} \phi(x - j) = 1, \quad x \in \mathbb{R}. $$
For $N \in \mathbb{N}$, the uniform B-spline of order $N$ is defined as

$$
\phi(x) := \chi_{[0,1]} \ast \cdots \ast \chi_{[0,1]}(x), \quad x \in \mathbb{R}.
$$

$\phi$ satisfies the translation partition of unity property:

$$
\sum_{j \in \mathbb{Z}} \phi(x - j) = 1, \quad x \in \mathbb{R}.
$$

$\phi$ is refinable:

$$
\hat{\phi}(\gamma) = 2^{-1/2} H(\gamma/2) \hat{\phi}(\gamma/2), \quad \gamma \in \mathbb{R},
$$

where $H \in L^\infty[0,1]$ and $\hat{\phi}$ is the Fourier transform of $\phi$. 
With B-splines $\phi$ as refinable functions, wavelet frames for $L^2(\mathbb{R})$ of the form 
\[ \{ \phi(\cdot - j) \}_{j \in \mathbb{Z}} \cup \{ 2^{k/2} \psi(m)(2^k \cdot - j) \}_{k \geq 0, j \in \mathbb{Z}, m=1,\ldots,\rho} \]
were constructed via the unitary extension principle, see [Ron & Shen, 1997], [Chui, He & Stöckler, 2002], [Daubechies, Han, Ron & Shen, 2003] etc.
With B-splines \( \phi \) as refinable functions, wavelet frames for \( L^2(\mathbb{R}) \) of the form 
\[
\{ \phi(\cdot - j) \}_{j \in \mathbb{Z}} \cup \{ 2^{k/2} \psi(m)(2^k \cdot - j) \}_{k \geq 0, j \in \mathbb{Z}, m=1,\ldots,\rho} 
\] were constructed via the unitary extension principle, see [Ron & Shen, 1997], [Chui, He & Stöckler, 2002], [Daubechies, Han, Ron & Shen, 2003] etc.

Using the translation partition of unity property of B-splines \( \phi \), Gabor frames for \( L^2(\mathbb{R}) \) of the form 
\[
\{ e^{2\pi ika \cdot \phi(\cdot - jb)} \}_{k, j \in \mathbb{Z}}
\] where \( a, b > 0 \), were constructed, see [Christensen, 2006], [Christensen & Kim, 2010] etc.
Wavelet and Gabor Frames for $L^2(\mathbb{R})$

- With B-splines $\phi$ as refinable functions, wavelet frames for $L^2(\mathbb{R})$ of the form $\{\phi(\cdot - j)\}_{j \in \mathbb{Z}} \cup \{2^{k/2} \psi(m)(2^k \cdot - j)\}_{k \geq 0, j \in \mathbb{Z}, m = 1, \ldots, \rho}$ were constructed via the unitary extension principle, see [Ron & Shen, 1997], [Chui, He & Stöckler, 2002], [Daubechies, Han, Ron & Shen, 2003] etc.

- Using the translation partition of unity property of B-splines $\phi$, Gabor frames for $L^2(\mathbb{R})$ of the form $\{e^{2\pi ika \cdot} \phi(\cdot - jb)\}_{k, j \in \mathbb{Z}}$, where $a, b > 0$, were constructed, see [Christensen, 2006], [Christensen & Kim, 2010] etc.

- Analogously, [Lemvig, 2009] constructed non-spline functions with the scaling partition of unity property that generate wavelet frames for $L^2(\mathbb{R})$. 
For $a > 1$, suppose that $h : \mathbb{R} \rightarrow \mathbb{C}$ has the scaling partition of unity property:

$$\sum_{k \in \mathbb{Z}} h(a^k \gamma) = 1, \quad \gamma \in \mathbb{R} \setminus \{0\}.$$
For $a > 1$, suppose that $h : \mathbb{R} \to \mathbb{C}$ has the scaling partition of unity property:

$$\sum_{k \in \mathbb{Z}} h(a^k \gamma) = 1, \quad \gamma \in \mathbb{R} \setminus \{0\}.$$ 

Assume there exists $C > 0$ such that $\sum_{k \in \mathbb{Z}} |h(a^k \gamma)| \leq C, \quad \gamma \in \mathbb{R} \setminus \{0\}.$
For $a > 1$, suppose that $h : \mathbb{R} \rightarrow \mathbb{C}$ has the scaling partition of unity property:

$$\sum_{k \in \mathbb{Z}} h(a^k \gamma) = 1, \quad \gamma \in \mathbb{R}\{0\}.$$ 

Assume there exists $C > 0$ such that $\sum_{k \in \mathbb{Z}} |h(a^k \gamma)| \leq C, \quad \gamma \in \mathbb{R}\{0\}$.

**Theorem (Christensen & Goh, 2017)**

The integral transform

$$(K_h f)(\gamma) := \int_{\mathbb{R}} f(u) h(\gamma/|u|) \, du, \quad \gamma \in \mathbb{R},$$

is well defined for every $f \in L^1(\mathbb{R})$, and

$$\sum_{k \in \mathbb{Z}} (K_h f)(a^k \gamma) = \int_{\mathbb{R}} f(u) \, du, \quad \gamma \in \mathbb{R}\{0\}.$$
Consider $a > 1$ and $S := [-1, -a^{-1}) \cup (a^{-1}, 1]$. 
Consider $a > 1$ and $S := [-1, -a^{-1}) \cup (a^{-1}, 1]$.

Let $h_1 := \chi_S$, and define $h_N$, $N \geq 2$, by

$$h_N(\gamma) := (K_{\chi_S} h_{N-1})(\gamma) = \int_{\mathbb{R}} h_{N-1}(u) \chi_S(\gamma/|u|) \, du, \quad \gamma \in \mathbb{R}.$$
Splines with Scaling Partition of Unity

- Consider $a > 1$ and $S := [-1, -a^{-1}) \cup (a^{-1}, 1]$.
- Let $h_1 := \chi_S$, and define $h_N$, $N \geq 2$, by

$$h_N(\gamma) := (K_{\chi_S} h_{N-1})(\gamma) = \int_{\mathbb{R}} h_{N-1}(u) \chi_S(\gamma/|u|) \, du, \quad \gamma \in \mathbb{R}.$$ 

For $N \geq 2$, $h_N$ is a nonnegative $C^{N-2}$-spline with knots at $\pm a^{-N}, \pm a^{-(N-1)}, \ldots, \pm 1$ and supported on $[-1, -a^{-N}] \cup [a^{-N}, 1]$. 
Consider $a > 1$ and $S := [-1, -a^{-1}) \cup (a^{-1}, 1]$.

Let $h_1 := \chi_S$, and define $h_N$, $N \geq 2$, by

$$h_N(\gamma) := (K_{\chi_S} h_{N-1})(\gamma) = \int_{\mathbb{R}} h_{N-1}(u) \chi_S(\gamma/|u|) \, du, \quad \gamma \in \mathbb{R}.$$ 

For $N \geq 2$, $h_N$ is a nonnegative $C^{N-2}$-spline with knots at $\pm a^{-N}, \pm a^{-(N-1)}, \ldots, \pm 1$ and supported on $[-1, -a^{-N}] \cup [a^{-N}, 1]$.

**Theorem**

Let $D_N := \int_{\mathbb{R}} h_N(\gamma) \, d\gamma$, $N \in \mathbb{N}$. Then for $N \geq 2$,

$$D_{N-1}^{-1} \sum_{k \in \mathbb{Z}} h_N(a^k \gamma) = 1, \quad \gamma \in \mathbb{R}\{0\},$$

and there exists $C_N > 0$ such that $C_N \leq D_{N-1}^{-2} \sum_{k \in \mathbb{Z}} |h_N(a^k \gamma)|^2 \leq 1, \quad \gamma \in \mathbb{R}\{0\}$. 
Theorem (Christensen & Goh, 2017)

With $a$ and $h_N$ as above, fix $b \in (0, a^{-(N-1)/2}]$ and define $\psi, \tilde{\psi} \in L^2(\mathbb{R})$ by $\hat{\psi} := h_N$ and

$$\tilde{\psi}(\gamma) := \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma), \quad \gamma \in \mathbb{R}. $$

Then $\{a^{k/2}\psi(a^k \cdot -bj)\}_{k,j \in \mathbb{Z}}$ and $\{a^{k/2}\tilde{\psi}(a^k \cdot -bj)\}_{k,j \in \mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$. 
Theorem (Christensen & Goh, 2017)

With $a$ and $h_N$ as above, fix $b \in (0, a^{-(N-1)/2}]$ and define $\psi, \tilde{\psi} \in L^2(\mathbb{R})$ by $\hat{\psi} := h_N$ and

$$\tilde{\psi}(\gamma) := \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma), \quad \gamma \in \mathbb{R}.$$ 

Then $\{a^{k/2}\psi(a^k \cdot -bj)\}_{k, j \in \mathbb{Z}}$ and $\{a^{k/2}\tilde{\psi}(a^k \cdot -bj)\}_{k, j \in \mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.

- The expressions $\hat{\psi}$ and $\tilde{\psi}$ are given explicitly in terms of the spline $h_N$. 

Say Song Goh (NUS)  
B-spline generated frames  
December 4 to 6, 2017  
8 / 23
Theorem (Christensen & Goh, 2017)

With \( a \) and \( h_N \) as above, fix \( b \in (0, a^{-(N-1)}/2] \) and define \( \psi, \tilde{\psi} \in L^2(\mathbb{R}) \) by \( \hat{\psi} := h_N \) and

\[
\tilde{\psi}(\gamma) := \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma), \quad \gamma \in \mathbb{R}.
\]

Then \( \{a^{k/2} \psi(a^k \cdot - bj)\}_{k,j \in \mathbb{Z}} \) and \( \{a^{k/2} \tilde{\psi}(a^k \cdot - bj)\}_{k,j \in \mathbb{Z}} \) are dual frames for \( L^2(\mathbb{R}) \).

- The expressions \( \hat{\psi} \) and \( \tilde{\psi} \) are given explicitly in terms of the spline \( h_N \).
- The frame generators \( \psi \) and \( \tilde{\psi} \) are bandlimited.
Idea of Proof

- For $\gamma \in \text{supp} \hat{\psi}$,

$$\hat{\psi}(\gamma) = \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma) = \frac{b}{D_{N-1}^2} \sum_{k \in \mathbb{Z}} h_N(a^k \gamma) = \frac{b}{D_{N-1}}.$$
Idea of Proof

- For $\gamma \in \text{supp} \widehat{\psi}$,

\[
\widehat{\psi}(\gamma) = \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma) = \frac{b}{D_{N-1}^2} \sum_{k \in \mathbb{Z}} h_N(a^k \gamma) = \frac{b}{D_{N-1}}.
\]

- Thus $\overline{\widehat{\psi}(\gamma)} \widehat{\psi}(\gamma) = \overline{\widehat{\psi}(\gamma)} \frac{b}{D_{N-1}}$ for all $\gamma \in \mathbb{R}$ and so
Idea of Proof

For $\gamma \in \text{supp} \hat{\psi}$,

$$\hat{\psi}(\gamma) = \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma) = \frac{b}{D_{N-1}^2} \sum_{k \in \mathbb{Z}} h_N(a^k \gamma) = \frac{b}{D_{N-1}}.$$ 

Thus $\hat{\psi}(\gamma)\hat{\psi}(\gamma) = \hat{\psi}(\gamma)\frac{b}{D_{N-1}}$ for all $\gamma \in \mathbb{R}$ and so

$$\sum_{k \in \mathbb{Z}} \hat{\psi}(a^k \gamma)\hat{\psi}(a^k \gamma) = \frac{b}{D_{N-1}} \sum_{k \in \mathbb{Z}} \hat{\psi}(a^k \gamma) = b, \quad \gamma \in \mathbb{R}\{0\}.$$
For $\gamma \in \text{supp} \hat{\psi}$,

$$\hat{\psi}(\gamma) = \frac{b}{D_{N-1}^2} \sum_{k=-N+1}^{N-1} h_N(a^k \gamma) = \frac{b}{D_{N-1}^2} \sum_{k \in \mathbb{Z}} h_N(a^k \gamma) = \frac{b}{D_{N-1}}.$$ 

Thus $\underline{\hat{\psi}(\gamma)} \hat{\psi}(\gamma) = \underline{\hat{\psi}(\gamma)} \frac{b}{D_{N-1}}$ for all $\gamma \in \mathbb{R}$ and so

$$\sum_{k \in \mathbb{Z}} \underline{\hat{\psi}(a^k \gamma)} \hat{\psi}(a^k \gamma) = \frac{b}{D_{N-1}} \sum_{k \in \mathbb{Z}} \underline{\hat{\psi}(a^k \gamma)} = b, \quad \gamma \in \mathbb{R}\{0\}.$$ 

Use sufficient condition for dual wavelet frames to complete the proof.
In practice, many functions of interest have domain in $\mathbb{R}^s$, $\mathbb{T}^s$, $\mathbb{Z}^s$, $\mathbb{Z}_N^s$ and their products.
In practice, many functions of interest have domain in $\mathbb{R}^s$, $\mathbb{T}^s$, $\mathbb{Z}^s$, $\mathbb{Z}_N^s$ and their products.

**Goal:** Extend B-splines and B-spline generated Gabor and wavelet frames to **locally compact abelian (LCA) groups**.
In practice, many functions of interest have domain in $\mathbb{R}^s$, $\mathbb{T}^s$, $\mathbb{Z}^s$, $\mathbb{Z}_N^s$ and their products.

Goal: Extend B-splines and B-spline generated Gabor and wavelet frames to \textbf{locally compact abelian (LCA) groups}.

General approaches on LCA groups would capture both the stationary and nonstationary case.
In practice, many functions of interest have domain in $\mathbb{R}^s, \mathbb{T}^s, \mathbb{Z}^s, \mathbb{Z}_N^s$ and their products.

**Goal:** Extend B-splines and B-spline generated Gabor and wavelet frames to **locally compact abelian (LCA) groups**.

General approaches on LCA groups would capture both the stationary and nonstationary case.

The more general abstract setup on LCA groups may also reveal new insight.
Let $G$ be an LCA group with group composition “$+$” and neutral element $0$. Let $G$ be a countable union of compact sets, metrizable and equipped with a *Haar measure* $\mu_G$. 

Examples:

- $\hat{\mathbb{R}} = \mathbb{R}$
- $\hat{\mathbb{T}} = \mathbb{Z}$
- $\hat{\mathbb{Z}} = \mathbb{T}$
- $\hat{\mathbb{Z}}^N = \mathbb{Z}^N$

Notation: $(x, \gamma) := \gamma(x)$, $x \in G$, $\gamma \in \hat{G}$. 

Say Song Goh (NUS)
Let $G$ be an LCA group with group composition “$+$” and neutral element $0$. Let $G$ be a countable union of compact sets, metrizable and equipped with a *Haar measure* $\mu_G$.

A **character** on $G$ is a continuous function $\gamma : G \rightarrow \mathbb{T}$ satisfying

$$\gamma(x + y) = \gamma(x)\gamma(y), \quad x, y \in G.$$ 

**Example:** For $G = \mathbb{R}$ and $b \in \mathbb{R}$, $\gamma_b(x) = e^{2\pi ibx}$, $x \in \mathbb{R}$. 


Let $G$ be an LCA group with group composition “$+$” and neutral element 0. Let $G$ be a countable union of compact sets, metrizable and equipped with a \textit{Haar measure} $\mu_G$.

A \textbf{character} on $G$ is a continuous function $\gamma : G \rightarrow \mathbb{T}$ satisfying

$$\gamma(x + y) = \gamma(x)\gamma(y), \quad x, y \in G.$$ 

\textbf{Example:} For $G = \mathbb{R}$ and $b \in \mathbb{R}$, $\gamma_b(x) = e^{2\pi ibx}$, $x \in \mathbb{R}$.

Under an appropriate topology, the set of characters, denoted by $\hat{G}$, also forms an LCA group called \textbf{dual group}, where for $x \in G$,

$$(\gamma + \gamma')(x) := \gamma(x)\gamma'(x), \quad \gamma, \gamma' \in \hat{G},$$

and $\hat{\hat{G}} = G$.

\textbf{Examples:} $\hat{\mathbb{R}} = \mathbb{R}$, $\hat{\mathbb{T}} = \mathbb{Z}$, $\hat{\mathbb{Z}} = \mathbb{T}$, $\hat{\mathbb{Z}_N} = \mathbb{Z}_N$. 
Let $G$ be an LCA group with group composition “$+$” and neutral element 0. Let $G$ be a countable union of compact sets, metrizable and equipped with a Haar measure $\mu_G$.

A character on $G$ is a continuous function $\gamma : G \rightarrow \mathbb{T}$ satisfying

$$\gamma(x + y) = \gamma(x)\gamma(y), \quad x, y \in G.$$ 

**Example:** For $G = \mathbb{R}$ and $b \in \mathbb{R}$, $\gamma_b(x) = e^{2\pi ibx}, \ x \in \mathbb{R}$.

Under an appropriate topology, the set of characters, denoted by $\hat{G}$, also forms an LCA group called dual group, where for $x \in G$,

$$(\gamma + \gamma')(x) := \gamma(x)\gamma'(x), \quad \gamma, \gamma' \in \hat{G},$$

and $\hat{\hat{G}} = G$.

**Examples:** $\hat{\mathbb{R}} = \mathbb{R}, \hat{\mathbb{T}} = \mathbb{Z}, \hat{\mathbb{Z}} = \mathbb{T}, \hat{\mathbb{Z}_N} = \mathbb{Z}_N$.

**Notation:** $(x, \gamma) := \gamma(x), \ x \in G, \ \gamma \in \hat{G}$. 

A lattice in an LCA group $G$ is a discrete subgroup $\Lambda$ for which $G/\Lambda$ is compact.

The annihilator $\Lambda \perp$ of a lattice $\Lambda$ is the lattice $\Lambda \perp := \{ \gamma \in \hat{G} : (\lambda, \gamma) = 1, \lambda \in \Lambda \}$.

Example: For $G = \mathbb{R}$ and $\Lambda = b\mathbb{Z}$ where $b > 0$, $\Lambda \perp = b^{-1}\mathbb{Z}$.

A Borel measurable relatively compact set $Q \subseteq G$ is a fundamental domain associated with the lattice $\Lambda$ if $G = \bigcup_{\lambda \in \Lambda} (\lambda + Q)$, $(\lambda + Q) \cap (\lambda' + Q) = \emptyset$, $\lambda \neq \lambda'$.

Example: For $G = \mathbb{R}$ and $\Lambda = b\mathbb{Z}$ where $b > 0$, $Q = [0, b)$.
A lattice in an LCA group $G$ is a discrete subgroup $\Lambda$ for which $G/\Lambda$ is compact.

The annihilator $\Lambda^\perp$ of a lattice $\Lambda$ is the lattice

$$\Lambda^\perp := \{ \gamma \in \hat{G} : (\lambda, \gamma) = 1, \lambda \in \Lambda \}$$

Example: For $G = \mathbb{R}$ and $\Lambda = b\mathbb{Z}$ where $b > 0$, $\Lambda^\perp = b^{-1}\mathbb{Z}$. 
A lattice in an LCA group $G$ is a discrete subgroup $\Lambda$ for which $G/\Lambda$ is compact.

The annihilator $\Lambda^\perp$ of a lattice $\Lambda$ is the lattice

$$
\Lambda^\perp := \{ \gamma \in \hat{G} : (\lambda, \gamma) = 1, \ \lambda \in \Lambda \}
$$

Example: For $G = \mathbb{R}$ and $\Lambda = b\mathbb{Z}$ where $b > 0$, $\Lambda^\perp = b^{-1}\mathbb{Z}$.

A Borel measurable relatively compact set $Q \subseteq G$ is a fundamental domain associated with the lattice $\Lambda$ if

$$
G = \bigcup_{\lambda \in \Lambda} (\lambda + Q), \quad (\lambda + Q) \cap (\lambda' + Q) = \emptyset, \ \lambda \neq \lambda'.
$$

Example: For $G = \mathbb{R}$ and $\Lambda = b\mathbb{Z}$ where $b > 0$, $Q = [0, b)$.
For \( \lambda \in G \), define the modulation operator \( \mathcal{M}_\lambda : L^2(\hat{G}) \to L^2(\hat{G}) \) by

\[
(\mathcal{M}_\lambda F)(\gamma) := (\lambda, \gamma)F(\gamma), \quad \gamma \in \hat{G}.
\]
Frames for $L^2(\hat{G})$

- For $\lambda \in G$, define the modulation operator $M_\lambda : L^2(\hat{G}) \rightarrow L^2(\hat{G})$ by
  $$(M_\lambda F)(\gamma) := (\lambda, \gamma)F(\gamma), \quad \gamma \in \hat{G}.$$ 

- $\{e^{2\pi ika \cdot} g(\cdot - jb)\}_{k,j \in \mathbb{Z}}$ forming a Gabor frame for $L^2(\mathbb{R})$ is equivalent to $\{M_{jb} \hat{g}(\cdot - ka)\}_{k,j \in \mathbb{Z}}$ forming a frame for $L^2(\mathbb{R})$. 

Idea: For extension to LCA groups, consider frames for $L^2(\hat{G})$ of the form $\{M_\lambda \Psi(m)\}_{k \in I, \lambda \in \Lambda^k, m = 1, \ldots, \rho}$. 

Say Song Goh (NUS)
Frames for $L^2(\hat{G})$

- For $\lambda \in G$, define the modulation operator $\mathcal{M}_\lambda : L^2(\hat{G}) \rightarrow L^2(\hat{G})$ by
  $$(\mathcal{M}_\lambda F)(\gamma) := (\lambda, \gamma) F(\gamma), \quad \gamma \in \hat{G}.$$  

- $\{e^{2\pi i k a \cdot g(\cdot - jb)}\}_{k,j \in \mathbb{Z}}$ forming a Gabor frame for $L^2(\mathbb{R})$ is equivalent to $\{\mathcal{M}_{jb} \hat{g}(\cdot - ka)\}_{k,j \in \mathbb{Z}}$ forming a frame for $L^2(\hat{\mathbb{R}})$.  

- $\{\phi(\cdot - j)\}_{j \in \mathbb{Z}} \cup \{2^{k/2} \psi(m)(2^k \cdot -j)\}_{k \geq 0, j \in \mathbb{Z}, m=1,\ldots,\rho}$ forming a wavelet frame for $L^2(\mathbb{R})$ is equivalent to
  $$\{\mathcal{M}_j \hat{\phi}\}_{j \in \mathbb{Z}} \cup \{\mathcal{M}_{j/2^k} 2^{-k/2} \psi(m)(\cdot /2^k)\}_{k \geq 0, j \in \mathbb{Z}, m=1,\ldots,\rho}$$

forming a frame for $L^2(\hat{\mathbb{R}})$.  

Idea: For extension to LCA groups, consider frames for $L^2(\hat{G})$ of the form $\{\mathcal{M}_\lambda \Psi(m)\}_{k \in I, \lambda \in \Lambda_k, m=1,\ldots,\rho_k}$.  

Say Song Goh (NUS)  
B-spline generated frames  
December 4 to 6, 2017  
13 / 23
Frames for $L^2(\hat{G})$

- For $\lambda \in G$, define the modulation operator $\mathcal{M}_\lambda : L^2(\hat{G}) \longrightarrow L^2(\hat{G})$ by
  $$(\mathcal{M}_\lambda F)(\gamma) := (\lambda, \gamma)F(\gamma), \quad \gamma \in \widehat{G}.$$ 

- $\{e^{2\pi ika}g(\cdot - jb)\}_{k,j \in \mathbb{Z}}$ forming a Gabor frame for $L^2(\mathbb{R})$ is equivalent to $\{\mathcal{M}_{jb}\hat{g}(\cdot - ka)\}_{k,j \in \mathbb{Z}}$ forming a frame for $L^2(\widehat{\mathbb{R}})$.

- $\{\phi(\cdot - j)\}_{j \in \mathbb{Z}} \cup \{2^{k/2}\psi(m)(2^k \cdot - j)\}_{k \geq 0, j \in \mathbb{Z}, m = 1, \ldots, \rho}$ forming a wavelet frame for $L^2(\mathbb{R})$ is equivalent to

  $\{\mathcal{M}_{j}\hat{\phi}\}_{j \in \mathbb{Z}} \cup \{\mathcal{M}_{j/2^k}2^{-k/2}\psi(m)(\cdot / 2^k)\}_{k \geq 0, j \in \mathbb{Z}, m = 1, \ldots, \rho}$

  forming a frame for $L^2(\widehat{\mathbb{R}})$.

- **Idea:** For extension to LCA groups, consider frames for $L^2(\widehat{G})$ of the form $\{\mathcal{M}_\lambda \psi_k^{(m)}\}_{k \in I, \lambda \in \Lambda_k, m = 1, \ldots, \rho_k}$. 
Let \( \Lambda \) be a lattice in \( G \) with associated fundamental domain \( Q \).

Example: For \( G = \mathbb{R} \) and \( \Lambda = \mathbb{Z} \), \( Q = [0, 1) \).
Let \( \Lambda \) be a lattice in \( G \) with associated fundamental domain \( Q \).

Example: For \( G = \mathbb{R} \) and \( \Lambda = \mathbb{Z} \), \( Q = [0, 1) \).

For \( N \in \mathbb{N} \) and given weight functions \( w_1, \ldots, w_N \in L^2(Q) \), the weighted B-spline of order \( N \) is defined as

\[
\phi(x) := w_1 \chi_Q \ast \cdots \ast w_N \chi_Q(x), \quad x \in G.
\]
Let \( \Lambda \) be a lattice in \( G \) with associated fundamental domain \( Q \).

Example: For \( G = \mathbb{R} \) and \( \Lambda = \mathbb{Z} \), \( Q = [0, 1) \).

For \( N \in \mathbb{N} \) and given weight functions \( w_1, \ldots, w_N \in L^2(Q) \), the weighted B-spline of order \( N \) is defined as

\[
\phi(x) := w_1 \chi_Q \ast \cdots \ast w_N \chi_Q(x), \quad x \in G.
\]

For \( w_1 = \cdots = w_N = 1 \), this extension of uniform B-splines to LCA groups was proposed independently by [Dahlke, 1994] and [Tikhomirov, 1994].
Properties of Weighted B-splines

Theorem (Christensen & Goh, 2015)

(i) \( \{ \phi(\cdot - \lambda) \}_{\lambda \in \Lambda} \) is a Bessel sequence with bound \( \prod_{\ell=1}^{N} \| w_{\ell} \|_{L^2(Q)}^2 \).

(ii) \( \text{supp} \phi \subseteq Q + \cdots + Q = \overline{NQ} \).

(iii) If \( N \geq 2 \), then \( \phi \in C_c(G) \).

(iv) If \( w_{\ell} > 0 \) on \( Q \) for \( \ell = 1, \ldots, N \) and \( w_{\ell} = C \) for at least one index \( \ell \), then \( \phi \geq 0 \) on \( G \) and satisfies

\[
\sum_{\lambda \in \Lambda} \phi(x - \lambda) = \frac{1}{\mu_G(Q)} \prod_{\ell=1}^{N} \int_{Q} w_{\ell}(y) \, d\mu_G(y), \quad x \in G.
\]
Let \( \Lambda \) be a lattice in \( G \) and \( \Lambda^\perp \) its annihilator in \( \hat{G} \) with fundamental domain \( V \).

Consider weighted B-spline of order \( N \):

\[
\Phi := w_1 \chi_{\Omega} \ast \cdots \ast w_N \chi_{\Omega},
\]

where \( w_1, \ldots, w_N \in L^2(\Omega) \) with \( w_\ell > 0 \) on \( \Omega \) for \( \ell = 1, \ldots, N \) and \( w_\ell = C \) for at least one index \( \ell \).

Approach: Use partition of unity property of \( \Phi \) to construct Gabor frames.

Theorem (Christensen & Goh, 2015) If \( N \Omega \subseteq V \), then \( \{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda} \) is a frame for \( L^2(\hat{G}) \).

Remark: With an additional condition on \( \Omega \), a function \( \tilde{\Phi} \in L^2(\hat{G}) \) can be explicitly constructed from \( \Phi \) such that \( \{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda} \) and \( \{M_\lambda \tilde{\Phi}(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda} \) form dual frames for \( L^2(\hat{G}) \).
Construction of Gabor Frames

- Let $\Lambda$ be a lattice in $G$ and $\Lambda^\perp$ its annihilator in $\hat{G}$ with fundamental domain $V$.
- Let $\Gamma$ be a lattice in $\hat{G}$ with associated fundamental domain $\Omega$.

Consider weighted B-spline of order $N$: $\Phi := w_1 \chi_{\Omega} \ast \ldots \ast w_N \chi_{\Omega}$, where $w_1, \ldots, w_N \in L^2(\Omega)$ with $w_\ell > 0$ on $\Omega$ for $\ell = 1, \ldots, N$ and $w_\ell = c$ for at least one index $\ell$.

**Approach:** Use partition of unity property of $\Phi$ to construct Gabor frames.

**Theorem (Christensen & Goh, 2015)**

If $N_\Omega \subseteq V$, then $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ is a frame for $L^2(\hat{G})$.

**Remark:** With an additional condition on $\Omega$, a function $\tilde{\Phi} \in L^2(\hat{G})$ can be explicitly constructed from $\Phi$ such that $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ and $\{M_\lambda \tilde{\Phi}(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ form dual frames for $L^2(\hat{G})$. 
Construction of Gabor Frames

- Let $\Lambda$ be a lattice in $G$ and $\Lambda^\perp$ its annihilator in $\hat{G}$ with fundamental domain $V$.
- Let $\Gamma$ be a lattice in $\hat{G}$ with associated fundamental domain $\Omega$.
- Consider weighted B-spline of order $N$: $\Phi := w_1\chi_\Omega \ast \cdots \ast w_N\chi_\Omega$, where $w_1, \ldots, w_N \in L^2(\Omega)$ with $w_\ell > 0$ on $\Omega$ for $\ell = 1, \ldots, N$ and $w_\ell = C$ for at least one index $\ell$.

**Theorem (Christensen & Goh, 2015)**

If $N\Omega \subseteq V$, then $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ is a frame for $L^2(\hat{G})$.

**Remark**: With an additional condition on $\Omega$, a function $\tilde{\Phi} \in L^2(\hat{G})$ can be explicitly constructed from $\Phi$ such that $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ and $\{M_\lambda \tilde{\Phi}(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ form dual frames for $L^2(\hat{G})$. 
Construction of Gabor Frames

- Let $\Lambda$ be a lattice in $G$ and $\Lambda^\perp$ its annihilator in $\hat{G}$ with fundamental domain $V$.
- Let $\Gamma$ be a lattice in $\hat{G}$ with associated fundamental domain $\Omega$.
- Consider weighted B-spline of order $N$: $\Phi := w_1\chi_\Omega \ast \cdots \ast w_N\chi_\Omega$, where $w_1, \ldots, w_N \in L^2(\Omega)$ with $w_\ell > 0$ on $\Omega$ for $\ell = 1, \ldots, N$ and $w_\ell = C$ for at least one index $\ell$.
- **Approach**: Use partition of unity property of $\Phi$ to construct Gabor frames.
Construction of Gabor Frames

- Let $\Lambda$ be a lattice in $G$ and $\Lambda^\perp$ its annihilator in $\hat{G}$ with fundamental domain $V$.
- Let $\Gamma$ be a lattice in $\hat{G}$ with associated fundamental domain $\Omega$.
- Consider weighted B-spline of order $N$: $\Phi := w_1\chi_\Omega \ast \cdots \ast w_N\chi_\Omega$, where $w_1, \ldots, w_N \in L^2(\Omega)$ with $w_\ell > 0$ on $\Omega$ for $\ell = 1, \ldots, N$ and $w_\ell = C$ for at least one index $\ell$.
- Approach: Use partition of unity property of $\Phi$ to construct Gabor frames.

Theorem (Christensen & Goh, 2015)

If $N\Omega \subseteq V$, then $\{\mathcal{M}_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ is a frame for $L^2(\hat{G})$. 
Construction of Gabor Frames

- Let $\Lambda$ be a lattice in $G$ and $\Lambda^\perp$ its annihilator in $\hat{G}$ with fundamental domain $V$.
- Let $\Gamma$ be a lattice in $\hat{G}$ with associated fundamental domain $\Omega$.
- Consider weighted B-spline of order $N$: $\Phi := w_1\chi_\Omega \ast \cdots \ast w_N\chi_\Omega$, where $w_1, \ldots, w_N \in L^2(\Omega)$ with $w_\ell > 0$ on $\Omega$ for $\ell = 1, \ldots, N$ and $w_\ell = C$ for at least one index $\ell$.
- **Approach:** Use partition of unity property of $\Phi$ to construct Gabor frames.

**Theorem (Christensen & Goh, 2015)**

*If $N\Omega \subseteq V$, then $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ is a frame for $L^2(\hat{G})$.*

- **Remark:** With an additional condition on $\Omega$, a function $\tilde{\Phi} \in L^2(\hat{G})$ can be explicitly constructed from $\Phi$ such that $\{M_\lambda \Phi(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ and $\{M_\lambda \tilde{\Phi}(\cdot - \tau)\}_{\tau \in \Gamma, \lambda \in \Lambda}$ form dual frames for $L^2(\hat{G})$. 

Let $I = \{ k \}_{k=k_0}^{\infty}$ or $I = \{ k \}_{k=k_0}^{k_1}$, and $\{ \Lambda_k \}_{k \in I}$ be lattices in $G$ satisfying

$$\Lambda_{k_0} \subset \Lambda_{k_0+1} \subset \Lambda_{k_0+2} \subset \cdots.$$
General Setup for Wavelet Frames

- Let \( I = \{ k \}_{k=k_0}^{\infty} \) or \( I = \{ k \}_{k=k_0}^{k_1} \), and \( \{ \Lambda_k \}_{k \in I} \) be lattices in \( G \) satisfying
  \[
  \Lambda_{k_0} \subset \Lambda_{k_0+1} \subset \Lambda_{k_0+2} \subset \cdots .
  \]

- For \( k \in I \), let \( V_k \) be a fundamental domain associated with \( \Lambda_k \).

Example: For \( G = \mathbb{R}, k \in I = \mathbb{N} \cup \{ 0 \} \) and \( \Lambda_k = 2^{-k} \mathbb{Z} \),
\( \Lambda_k \perp = 2^k \mathbb{Z} \) and \( V_k = [0, 2^k) \).
General Setup for Wavelet Frames

- Let \( I = \{ k \}_{k=k_0}^{\infty} \) or \( I = \{ k \}_{k=k_0}^{k_1} \), and \( \{ \Lambda_k \}_{k \in I} \) be lattices in \( G \) satisfying
  \[
  \Lambda_{k_0} \subset \Lambda_{k_0+1} \subset \Lambda_{k_0+2} \subset \cdots.
  \]

- For \( k \in I \), let \( V_k \) be a fundamental domain associated with \( \Lambda_k^\perp \).
  Example: For \( G = \mathbb{R} \), \( k \in I = \mathbb{N} \cup \{0\} \) and \( \Lambda_k = 2^{-k} \mathbb{Z} \),
  \( \Lambda_k^\perp = 2^k \mathbb{Z} \) and \( V_k = [0, 2^k) \).

- Assume that for every compact set \( S \) in \( \hat{G} \), there exists \( K_1 \in I \) such that
  \[
  \mu_{\hat{G}}((\omega + S) \cap (\omega' + S)) = 0, \quad \omega \neq \omega', \ \omega, \omega' \in \Lambda_{K_1}^\perp.
  \]

  Intuition: The lattices \( \Lambda_k^\perp, k \in I \), are “expanding”.
General Setup for Wavelet Frames

- Let $I = \{k\}_{k=k_0}^{\infty}$ or $I = \{k\}_{k=k_0}^{k_1}$, and $\{\Lambda_k\}_{k \in I}$ be lattices in $G$ satisfying
  \[ \Lambda_{k_0} \subset \Lambda_{k_0+1} \subset \Lambda_{k_0+2} \subset \cdots. \]

- For $k \in I$, let $V_k$ be a fundamental domain associated with $\Lambda_k^\perp$.
  
  Example: For $G = \mathbb{R}$, $k \in I = \mathbb{N} \cup \{0\}$ and $\Lambda_k = 2^{-k} \mathbb{Z}$, $\Lambda_k^\perp = 2^k \mathbb{Z}$ and $V_k = [0, 2^k)$.

- Assume that for every compact set $S$ in $\hat{G}$, there exists $K_1 \in I$ such that
  \[ \mu_\hat{G}(((\omega + S) \cap (\omega' + S))) = 0, \quad \omega \neq \omega', \ \omega, \omega' \in \Lambda_{K_1}^\perp. \]

  Intuition: The lattices $\Lambda_k^\perp$, $k \in I$, are “expanding”.

- Approach: Use refinable functions to construct wavelet frames.
Let $\Phi_k, k \in I$, be nonstationary “refinable functions” in $L^2(\hat{G})$ such that for $k \in I$ and some periodic refinement mask $H_{k+1} \in L^\infty(V_{k+1})$,

$$\Phi_k(\gamma) = H_{k+1}(\gamma) \Phi_{k+1}(\gamma), \quad \text{a.e. } \gamma \in \hat{G}.$$ 

$[H_{k+1} \in L^\infty(V_{k+1})$ is periodic if $H_{k+1}(\omega + \gamma) = H_{k+1}(\gamma)$ for $\omega \in \Lambda_{k+1}^\perp, \gamma \in V_{k+1}].$
Assumptions on Refinable Functions

Let $\Phi_k, k \in I$, be nonstationary "refinable functions" in $L^2(\hat{G})$ such that for $k \in I$ and some periodic refinement mask $H_{k+1} \in L^\infty(V_{k+1}),$

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \quad \text{a.e. } \gamma \in \hat{G}. $$

$[H_{k+1} \in L^\infty(V_{k+1}) \text{ is periodic if } H_{k+1}(\omega + \gamma) = H_{k+1}(\gamma) \text{ for } \omega \in \Lambda_{k+1}^\perp, \gamma \in V_{k+1}.]$

Example: For $\hat{G} = \mathbb{R}$, $\hat{\phi}(\gamma) = 2^{-1/2}H(\gamma/2)\hat{\phi}(\gamma/2).$
Assumptions on Refinable Functions

- Let $\Phi_k, k \in I$, be nonstationary “refinable functions” in $L^2(\hat{G})$ such that for $k \in I$ and some periodic refinement mask $H_{k+1} \in L^\infty(V_{k+1})$,

  $$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \quad \text{a.e. } \gamma \in \hat{G}.$$  

  $[H_{k+1} \in L^\infty(V_{k+1})$ is periodic if $H_{k+1}(\omega + \gamma) = H_{k+1}(\gamma)$ for $\omega \in \Lambda_{k+1}^\perp, \gamma \in V_{k+1}].$

  Example: For $\hat{G} = \mathbb{R}$, $\hat{\phi}(\gamma) = 2^{-1/2}H(\gamma/2)\hat{\phi}(\gamma/2)$.

- Assume that for every compact set $S$ in $\hat{G}$ and any $\epsilon > 0$, there exists $K_2 \in I$ such that for all $k \geq K_2, k \in I$,

  $$|\mu_{\hat{G}}(V_k)|\Phi_k(\gamma)|^2 - 1| \leq \epsilon, \quad \gamma \in S.$$
Assumptions on Refinable Functions

Let $\Phi_k, k \in I$, be nonstationary “refinable functions” in $L^2(\hat{G})$ such that for $k \in I$ and some periodic refinement mask $H_{k+1} \in L^\infty(V_{k+1})$,

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \text{ a.e. } \gamma \in \hat{G}.$$ 

$[H_{k+1} \in L^\infty(V_{k+1})$ is periodic if $H_{k+1}(\omega + \gamma) = H_{k+1}(\gamma)$ for $\omega \in \Lambda_{k+1}^\perp, \gamma \in V_{k+1}.]$ 

Example: For $\hat{G} = \mathbb{R}$, $\hat{\phi}(\gamma) = 2^{-1/2}H(\gamma/2)\hat{\phi}(\gamma/2)$.

Assume that for every compact set $S$ in $\hat{G}$ and any $\epsilon > 0$, there exists $K_2 \in I$ such that for all $k \geq K_2, k \in I$,

$$|\mu_{\hat{G}}(V_k)|\Phi_k(\gamma)|^2 - 1| \leq \epsilon, \quad \gamma \in S.$$ 

Example: For $\hat{G} = \mathbb{R}$, $\lim_{\gamma \to 0} \hat{\phi}(\gamma) = 1$. 

Construction of Wavelet Functions

- Find wavelet masks $G_{k+1}^{(m)} \in L^\infty(V_{k+1})$, $k \in I$, $m = 1, \ldots, \rho_k$, to define “wavelet functions”

$$
\psi_k^{(m)}(\gamma) := G_{k+1}^{(m)}(\gamma)\Phi_{k+1}(\gamma), \quad \gamma \in \hat{G},
$$

so that $\{M_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \cup \{M_\lambda \psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m = 1, \ldots, \rho_k}$ forms a tight frame for $L^2(\hat{G})$. 


Construction of Wavelet Functions

- Find wavelet masks $G_{k+1}^{(m)} \in L^\infty(V_{k+1}), k \in I, m = 1, \ldots, \rho_k$, to define "wavelet functions"

$$\psi_k^{(m)}(\gamma) := G_{k+1}^{(m)}(\gamma)\Phi_{k+1}(\gamma), \quad \gamma \in \hat{G},$$

so that $\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \cup \{\mathcal{M}_\lambda \psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_{k}, m = 1, \ldots, \rho_k}$ forms a tight frame for $L^2(\hat{G})$.

- For each $k \in I$, since $\Lambda_{k+1}^\perp \subset \Lambda_k^\perp$ and $|\Lambda_k^\perp / \Lambda_{k+1}^\perp| = \frac{\mu_{\hat{G}}(V_{k+1})}{\mu_{\hat{G}}(V_{k})} =: d_k$, choose $\{\nu_{k,\ell}\}_{\ell = 1, \ldots, d_k}$ in $\hat{G}$ such that $\nu_{k,1} = 0$ and

$$\Lambda_k^\perp = \bigcup_{\ell = 1}^{d_k} (\nu_{k,\ell} + \Lambda_{k+1}^\perp), \quad (\nu_{k,\ell} + \Lambda_{k+1}^\perp) \cap (\nu_{k,\ell'} + \Lambda_{k+1}^\perp) = \emptyset, \quad \ell \neq \ell'.$$
Construction of Wavelet Functions

- Find wavelet masks $G_{k+1}^{(m)} \in L^\infty(V_{k+1}), k \in I, m = 1, \ldots, \rho_k$, to define “wavelet functions”

$$\psi_k^{(m)}(\gamma) := G_{k+1}^{(m)}(\gamma)\Phi_{k+1}(\gamma), \quad \gamma \in \hat{G},$$

so that $\{\mathcal{M}_\lambda \Phi_{k_0}\}_{\lambda \in \Lambda_{k_0}} \cup \{\mathcal{M}_\lambda \psi_k^{(m)}\}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,\rho_k}$ forms a tight frame for $L^2(\hat{G})$.

- For each $k \in I$, since $\Lambda^\perp_{k+1} \subset \Lambda^\perp_k$ and $|\Lambda^\perp_k / \Lambda^\perp_{k+1}| = \frac{\mu_{\hat{G}}(V_{k+1})}{\mu_{\hat{G}}(V_k)} =: d_k$, choose $\{\nu_k,\ell\}_{\ell=1,\ldots,d_k}$ in $\hat{G}$ such that $\nu_{k,1} = 0$ and

$$\Lambda^\perp_k = \bigcup_{\ell=1}^{d_k} (\nu_k,\ell + \Lambda^\perp_{k+1}), \quad (\nu_k,\ell + \Lambda^\perp_{k+1}) \cap (\nu_k,\ell' + \Lambda^\perp_{k+1}) = \emptyset, \quad \ell \neq \ell'.$$

Example: For $\hat{G} = \mathbb{R}$ and $\Lambda^\perp_k = 2^k\mathbb{Z}$, $\nu_{k,1} = 0$, $\nu_{k,2} = 2^k$. 
Unitary Extension Principle (UEP) on LCA Groups

- For $k \in I$, define the $(\rho_k + 1) \times d_k$ matrix-valued function

$$P_k(\gamma) := \begin{pmatrix} H_{k+1}(\gamma + \nu_{k,1}) & \cdots & H_{k+1}(\gamma + \nu_{k,d_k}) \\ G^{(1)}_{k+1}(\gamma + \nu_{k,1}) & \cdots & G^{(1)}_{k+1}(\gamma + \nu_{k,d_k}) \\ \vdots & \ddots & \vdots \\ G^{(\rho_k)}_{k+1}(\gamma + \nu_{k,1}) & \cdots & G^{(\rho_k)}_{k+1}(\gamma + \nu_{k,d_k}) \end{pmatrix}, \quad \gamma \in V_k.$$
For $k \in I$, define the $(\rho_k + 1) \times d_k$ matrix-valued function

$$P_k(\gamma) := \begin{pmatrix}
H_{k+1}(\gamma + \nu_k, 1) & \cdots & H_{k+1}(\gamma + \nu_k, d_k) \\
G_{k+1}^{(1)}(\gamma + \nu_k, 1) & \cdots & G_{k+1}^{(1)}(\gamma + \nu_k, d_k) \\
\vdots & \ddots & \vdots \\
G_{k+1}^{(\rho_k)}(\gamma + \nu_k, 1) & \cdots & G_{k+1}^{(\rho_k)}(\gamma + \nu_k, d_k)
\end{pmatrix}, \quad \gamma \in V_k.$$

**Theorem (Christensen & Goh, 2016)**

For $k \in I$, with $\Phi_k, \psi_k^{(1)}, \ldots, \psi_k^{(\rho_k)}$ as above, if

$$P_k(\gamma)^*P_k(\gamma) = d_k I_{d_k}, \quad \text{a.e. } \gamma \in V_k,$$

then

$$\{M_\lambda \psi_{k_0} \}_{\lambda \in \Lambda_{k_0}} \cup \{M_\lambda \psi_{k}^{(m)} \}_{k \geq k_0, \lambda \in \Lambda_k, m=1,\ldots,\rho_k}$$

forms a tight frame for $L^2(\hat{G})$. 
Assume that for $k \in I$, $|\Lambda_{k+1}/\Lambda_k| = 2$. So there exist $\eta_k \in \Lambda_{k+1}/\Lambda_k$ and $\nu_k \in \Lambda_{k}^{\perp}/\Lambda_{k+1}^{\perp}$ such that

$$\Lambda_{k+1} = \Lambda_k \cup (\eta_k + \Lambda_{k}), \quad \Lambda_{k}^{\perp} = \Lambda_{k+1}^{\perp} \cup (\nu_k + \Lambda_{k+1}^{\perp}).$$
B-splines on LCA Groups

- Assume that for $k \in I$, $|\Lambda_{k+1}/\Lambda_k| = 2$. So there exist $\eta_k \in \Lambda_{k+1}/\Lambda_k$ and $\nu_k \in \Lambda_k^\perp/\Lambda_{k+1}^\perp$ such that

$$\Lambda_{k+1} = \Lambda_k \cup (\eta_k + \Lambda_k), \quad \Lambda_k^\perp = \Lambda_{k+1}^\perp \cup (\nu_k + \Lambda_{k+1}^\perp).$$

- It can be shown that $(\eta_k, \nu_k) = -1$ for $k \in I$. 
Assume that for $k \in I$, $|\Lambda_{k+1}/\Lambda_k| = 2$. So there exist $\eta_k \in \Lambda_{k+1}/\Lambda_k$ and $\nu_k \in \Lambda_k^\perp/\Lambda_{k+1}^\perp$ such that

$$\Lambda_{k+1} = \Lambda_k \cup (\eta_k + \Lambda_k), \quad \Lambda_k^\perp = \Lambda_{k+1}^\perp \cup (\nu_k + \Lambda_{k+1}^\perp).$$

It can be shown that $(\eta_k, \nu_k) = -1$ for $k \in I$.

Let $Q_k$ be a fundamental domain associated with $\Lambda_k$. For $N \in \mathbb{N}$, the B-spline of order $N$ at level $k$ in $G$ is defined as

$$\phi_k(x) := \mu_G(Q_k)^{-N+1/2} \chi_{Q_k} * \cdots * \chi_{Q_k}(x), \quad x \in G.$$
Assume that for \( k \in I \), \(|\Lambda_{k+1}/\Lambda_k| = 2\). So there exist \( \eta_k \in \Lambda_{k+1}/\Lambda_k \) and \( \nu_k \in \Lambda_k^\perp/\Lambda_{k+1}^\perp \) such that

\[
\Lambda_{k+1} = \Lambda_k \cup (\eta_k + \Lambda_k) , \quad \Lambda_k^\perp = \Lambda_{k+1}^\perp \cup (\nu_k + \Lambda_{k+1}^\perp) .
\]

It can be shown that \((\eta_k, \nu_k) = -1\) for \( k \in I \).

Let \( Q_k \) be a fundamental domain associated with \( \Lambda_k \).
For \( N \in \mathbb{N} \), the B-spline of order \( N \) at level \( k \) in \( G \) is defined as

\[
\phi_k(x) := \mu_G(Q_k)^{-N+1/2} \chi_{Q_k} * \cdots * \chi_{Q_k}(x) , \quad x \in G .
\]

Taking Fourier transform \( \Phi_k(\gamma) := \int_G \phi_k(x)(-x, \gamma) d\mu_G(x) \), this gives

\[
\Phi_k(\gamma) = \mu_G(Q_k)^{-N+1/2} \left( \int_{Q_k} (-x, \gamma) d\mu_G(x) \right)^N , \quad \gamma \in \hat{G} .
\]
The functions $\Phi_k$, $k \in I$, satisfy

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \quad \gamma \in \hat{G},$$

where $H_{k+1} \in L^\infty(V_{k+1})$ is given by

$$H_{k+1}(\gamma) = 2^{-N+1/2}(1 + (-\eta_k, \gamma))^N, \gamma \in \hat{G}.$$
The functions $\Phi_k, k \in I$, satisfy

$$ \Phi_k(\gamma) = H_{k+1}(\gamma) \Phi_{k+1}(\gamma), \quad \gamma \in \hat{G}, $$

where $H_{k+1} \in L^\infty(V_{k+1})$ is given by

$$ H_{k+1}(\gamma) = 2^{-N+1/2}(1 + (-\eta_k, \gamma))^N, \gamma \in \hat{G}. $$

Example: For $\hat{G} = \mathbb{R}$,

$$ 1 + (-\eta_k, \gamma) = 1 + e^{-2\pi i \gamma/2^{k+1}} = 2e^{-\pi i \gamma/2^{k+1}} \cos(\pi \gamma/2^{k+1}). $$
The functions $\Phi_k, \ k \in I$, satisfy

$$\Phi_k(\gamma) = H_{k+1}(\gamma)\Phi_{k+1}(\gamma), \quad \gamma \in \hat{G},$$

where $H_{k+1} \in L^\infty(V_{k+1})$ is given by

$$H_{k+1}(\gamma) = 2^{-N+1/2}(1 + (-\eta_k, \gamma))^N, \gamma \in \hat{G}.$$ 

Example: For $\hat{G} = \mathbb{R}$,

$$1 + (-\eta_k, \gamma) = 1 + e^{-2\pi i \gamma/2^{k+1}} = 2e^{-\pi i \gamma/2^{k+1}} \cos(\pi \gamma/2^{k+1}).$$

Find: Wavelet masks $G_{k+1}^{(1)}, \ldots G_{k+1}^{(\rho_k)} \in L^\infty(V_{k+1})$ such that

$$P_k(\gamma) := \begin{pmatrix}
H_{k+1}(\gamma) & H_{k+1}(\gamma + \nu_k) \\
G_{k+1}^{(1)}(\gamma) & G_{k+1}^{(1)}(\gamma + \nu_k) \\
\vdots & \vdots \\
G_{k+1}^{(\rho_k)}(\gamma) & G_{k+1}^{(\rho_k)}(\gamma + \nu_k)
\end{pmatrix}, \quad \gamma \in V_k,$$

satisfies the UEP condition $P_k(\gamma)^*P_k(\gamma) = 2I_2$, a.e. $\gamma \in V_k$. 
Let $N = 2M$, i.e. $\phi_k$ is an even order B-spline on $G$. 
Let $N = 2M$, i.e. $\phi_k$ is an even order B-spline on $G$.

Define $G_{k+1}^{(1)}, \ldots, G_{k+1}^{(2M)} \in L^\infty(V_{k+1})$ by

$$G_{k+1}^{(m)}(\gamma) := 2^{-2M+1/2} \sqrt{\binom{2M}{m}} (1 + (-\eta_k, \gamma))^{2M-m}(1 - (-\eta_k, \gamma))^m, \gamma \in \hat{G}.$$ 

Then $P_k(\gamma)^* P_k(\gamma) = 2I_2$, a.e. $\gamma \in V_k$. 
Let $N = 2^M$, i.e. $\phi_k$ is an even order B-spline on $G$.

Define $G_{k+1}^{(1)}, \ldots, G_{k+1}^{(2^M)} \in L^\infty(V_{k+1})$ by

$$G_{k+1}^{(m)}(\gamma) := 2^{-2^{M+1}/2} \sqrt{\binom{2^M}{m}} (1 + (-\eta_k, \gamma))^{2^M-m}(1 - (-\eta_k, \gamma))^m, \quad \gamma \in \hat{G}.$$

Then $P_k(\gamma)^* P_k(\gamma) = 2I_2$, a.e. $\gamma \in V_k$.

This extends the B-spline wavelet masks on $\mathbb{R}$ in [Ron & Shen, 1997] to LCA groups and gives localized tight wavelet frames on $G$. 