Monoidal categories associated with strata of flag manifolds

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This is a joint work with
Masaki Kashiwara, Myungho Kim and Se-jin Oh (arXiv:1708.04428)
1.1. Quantum groups

- $I$ := an index set
- $A := (a_{ij})_{i,j \in I}$ (generalized Cartan matrix)
- $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ (root lattice)
- $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ (positive root lattice)
- $P$ := weight lattice
- $P^\vee$ := dual weight lattice
- $\Lambda_i \in P^+$ : fundamental weight (i.e., $\Lambda_i(h_j) = \delta_{ij}$)

**Definition** The quantum group $U_q(g)$ associated with $A$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in P^\vee$) satisfying certain defining relations.
We consider the unipotent quantum coordinate ring

\[ A_q(n) = \bigoplus_{\beta \in \mathbb{Q}_-} A_q(n)_\beta, \quad \text{where } A_q(n)_\beta := \text{Hom}_{\mathbb{Q}(q)}(U_q^+(g)_{-\beta}, \mathbb{Q}(q)). \]

**Note** \( A_q(n) \cong U_q^-(g) \) as a \( \mathbb{Q}(q) \)-algebra.

For \( \Lambda \in \mathbb{P}_+ \) and \( \mu, \zeta \in W\Lambda \) with \( \mu \preceq \zeta \),

\[ D(\mu, \zeta) := \text{unipotent quantum minor associated with } \mu, \zeta \]

It was known that

- \( D(\mu, \zeta) \) is a member of the upper global basis (dual canonical basis) of \( A_q(n) \).
- Let \( w, v \in W \) with \( v \preceq w \). Then

\[ D(w\Lambda, v\Lambda)D(w\Lambda', v\Lambda') = q^{-(v\Lambda, v\Lambda'-w\Lambda')}D(w(\Lambda+\Lambda'), v(\Lambda+\Lambda')). \]
If \( n := \langle h_i, \mu \rangle \geq 0 \), then
\[
\varepsilon_i(D(\mu, \zeta)) = 0 \quad \text{and} \quad e_i^{(n)}D(s_i\mu, \zeta) = D(\mu, \zeta).
\]

If \( \langle h_i, \mu \rangle \leq 0 \) and \( s_i\mu \preceq \zeta \), then \( \varepsilon_i(D(\mu, \zeta)) = -\langle h_i, \mu \rangle \).

If \( m := -\langle h_i, \zeta \rangle \geq 0 \), then
\[
\varepsilon_i^*(D(\mu, \zeta)) = 0 \quad \text{and} \quad e_i^{*(m)}D(\mu, s_i\zeta) = D(\mu, \zeta).
\]

If \( \langle h_i, \zeta \rangle \geq 0 \) and \( \mu \preceq s_i\zeta \), then \( \varepsilon_i^*(D(\mu, \zeta)) = \langle h_i, \zeta \rangle \).
1.2. Quiver Hecke algebras

For $\alpha \in \mathbb{Q}^+$ with $|\alpha| = m$, let

$$I^\alpha = \{ \nu = (\nu_1, \ldots, \nu_m) \in I^m \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_m} = \alpha \}$$

For $i, j \in I$, we take a homogeneous polynomial

$$Q_{i,j}(u, v) = \begin{cases} \sum_{-2(\alpha_i|\alpha_j) - 2d_i p - 2d_j q = 0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

such that $Q_{i,j}(u, v) = Q_{j,i}(v, u)$ and $t_{i,j;-a_{ij},0} \in k_0^\times$. 
Let $\alpha \in \mathbb{Q}^+$ with height $m$.

**Definition**  The *quiver Hecke algebra* $R(\alpha)$ is the associative graded $k$-algebra generated by
\[ e(\nu) \ (\nu \in I^\alpha), \ x_k \ (1 \leq k \leq m), \ \tau_t \ (1 \leq t \leq m - 1) \]
satisfying the following defining relations:

\[ e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \ \sum_{\nu \in I^\alpha} e(\nu) = 1, \ x_k e(\nu) = e(\nu)x_k, \ x_kx_l = x_lx_k, \]
\[ \tau_t e(\nu) = e(s_t(\nu))\tau_t, \ \tau_t \tau_s = \tau_s \tau_t \text{ if } |t - s| > 1, \]
\[ \tau_t^2 e(\nu) = \begin{cases} 0 & \text{if } \nu_t = \nu_{t+1}, \\ Q_{\nu_t,\nu_{t+1}}(x_t, x_{t+1})e(\nu) & \text{if } \nu_t \neq \nu_{t+1}, \end{cases} \]
\[ (\tau_t x_k - x_{s_t(k)} \tau_t)e(\nu) = \begin{cases} -e(\nu) & \text{if } k = t \text{ and } \nu_t = \nu_{t+1}, \\ e(\nu) & \text{if } k = t + 1 \text{ and } \nu_t = \nu_{t+1}, \\ 0 & \text{otherwise}, \end{cases} \]
\[ (\tau_{t+1} \tau_t \tau_{t+1} - \tau_t \tau_{t+1} \tau_t )e(\nu) = \begin{cases} \overline{Q}_{\nu_t, \nu_{t+1}}(x_t, x_{t+1}, x_{t+2})e(\nu) & \text{if } \nu_t = \nu_{t+2} \neq \nu_{t+1} , \\ 0 & \text{otherwise} , \end{cases} \]

where
\[ \overline{Q}_{i,j}(u, v, w) := \frac{Q_{i,j}(u, v) - Q_{i,j}(w, v)}{u - w} . \]

The \( \mathbb{Z} \)-grading on \( R(\alpha) \) is given by
\[ \deg(e(\nu)) = 0 , \deg(x_k e(\nu)) = (\alpha_{\nu_k} | \alpha_{\nu_k}) , \deg(\tau_t e(\nu)) = -(\alpha_{\nu_t} | \alpha_{\nu_{t+1}}) . \]
Convolution product

\[ M \circ N := R(\beta + \beta') e(\beta, \beta') \otimes R(\beta) \otimes R(\beta') (M \otimes N) \]

Functor \( E_i \) and \( F_i \) (\( i \in I \))

\[ E_i : R(\beta + \alpha_i)-\text{Mod} \to R(\beta)-\text{Mod} \]
\[ F_i : R(\beta)-\text{Mod} \to R(\beta + \alpha_i)-\text{Mod} \]

defined by \( E_i(N) = e(\alpha_i, \beta) N \) and \( F_i(M) = R(\alpha_i) \circ M \).

We set

\[ R(\beta)-\text{proj} := \text{category of f. g. projective graded } R(\beta)\text{-modules} \]
\[ R(\beta)-\text{mod} := \text{category of f. d. graded } R(\beta)\text{-modules} \]
[Khovanov-Lauda, Rouquier, Kang-Kashiwara, Webster]

\[ [R\text{-proj}] \simeq U^-_{\mathbb{Z}[q,q^{-1}]}(g), \quad [R\text{-mod}] \simeq A_q(n)_{\mathbb{Z}[q,q^{-1}]} \]

\[ [R^\Lambda\text{-proj}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda), \quad [R^\Lambda\text{-mod}] \simeq V_{\mathbb{Z}[q,q^{-1}]}(\Lambda)^* \]

[Brundan-Kleshchev, Rouquier]

In type A, \( R^\Lambda(n) \simeq \) cyclotomic Hecke algebras.

[Brundan, Hu, Kleshchev, Mathas, Ram, Wang, ...] In type A, Specht modules, graded cellular bases for \( R^\Lambda, \)

[Ariki-P.-Speyer] Specht modules for type C,

[Brundan, Kleshchev, McNamara, Ram, Tingley, Webster, ...] PBW bases theory, Convex orders, Cuspidal modules theory

[Kang-Kashiwara-Kim-Oh] Quantum cluster algebras

many results \( \cdots \)

**Remark** Quiver Hecke algebras are a vast generalization of Hecke algebras in the direction of categorification.
Preliminaries

Categories $\mathcal{C}_{w,v}$
Determinantal modules
Finite $ADE$ types

Quantum groups

- [Khovanov-Lauda, Rouquier, Kang-Kashiwara, Webster]
  
  $[R\text{-proj}] \cong U_{\mathbb{Z}[q,q^{-1}]}^-(g), \quad [R\text{-mod}] \cong A_q(n)_{\mathbb{Z}[q,q^{-1}]}$

  $[R^\Lambda\text{-proj}] \cong V_{\mathbb{Z}[q,q^{-1}]}(\Lambda), \quad [R^\Lambda\text{-mod}] \cong V_{\mathbb{Z}[q,q^{-1}]}(\Lambda)^*$

- [Brundan-Kleshchev, Rouquier]
  
  In type $A$, $R^\Lambda(n) \cong$ cyclotomic Hecke algebras.

- [Brundan, Hu, Kleshchev, Mathas, Ram, Wang, ...] In type $A$, Specht modules, graded cellular bases for $R^\Lambda$,

- [Ariki-P.-Speyer] Specht modules for type $C$,

- [Brundan, Kleshchev, McNamara, Ram, Tingley, Webster, ...] PBW bases theory, Convex orders, Cuspidal modules theory

- [Kang-Kashiwara-Kim-Oh] Quantum cluster algebras

- many results · · ·

Remark Quiver Hecke algebras are a vast generalization of Hecke algebras in the direction of categorification.

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Monoidal categories associated with strata of flag manifolds
1.3. Convex preorders

We review results in [Tingley-Webster, MV polytopes and KLR algebras, Compos. Math. 152 (2016)].

**Definition** A face is a decomposition of a subset $X$ of $V$ into three disjoint subsets $X = A_- \sqcup A_0 \sqcup A_+$ such that

$$(\text{span}_{R \geq 0} A_+ + \text{span}_{R} A_0) \cap \text{span}_{R \geq 0} A_- = \{0\},$$

$$(\text{span}_{R \geq 0} A_- + \text{span}_{R} A_0) \cap \text{span}_{R \geq 0} A_+ = \{0\}.$$
1.3. Convex preorders

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$$(\text{span}_{R \geq 0} A_- + \text{span}_R A_0) \cap \text{span}_{R \geq 0} A_+ = \{0\}.$$ 

Definition Let $X$ be a subset of $V \setminus \{0\}$.

(i) A convex preorder $\preceq$ on $X$ is a total preorder on $X$ such that, for any $\preceq$-equivalence class $C$, the triple

$$(\{x \in X \mid x \prec C\}, C, \{x \in X \mid x \succ C\})$$

is a face.

(ii) A convex preorder $\preceq$ on $X$ is called a convex order if every $\preceq$-equivalence class is of the form $X \cap l$ for some line $l$. 

Note $\preceq$: a convex preorder on $X \subset V \setminus \{0\}$.

- If $\alpha, \beta, \gamma \in X$ with $\alpha + \beta = \gamma$ and $\alpha \prec \gamma$, then $\gamma \prec \beta$.
- If $\alpha, \beta, \gamma \in X$ with $\alpha + \beta = \gamma$ and $\gamma \prec \beta$, then $\alpha \prec \gamma$. 
Note: a convex preorder on $X \subset V \setminus \{0\}$.

- If $\alpha, \beta, \gamma \in X$ with $\alpha + \beta = \gamma$ and $\alpha \prec \gamma$, then $\gamma \prec \beta$.
- If $\alpha, \beta, \gamma \in X$ with $\alpha + \beta = \gamma$ and $\gamma \prec \beta$, then $\alpha \prec \gamma$.

We set

$$w := s_{i_1}s_{i_2} \cdots s_{i_l} \in W,$$

$$\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \text{ for } k = 1, \ldots, l.$$ 

Then we have $\Delta_+ \cap w\Delta_- = \{\beta_1, \ldots, \beta_l\}$.

**Proposition** There is a convex preorder $\preceq$ on $\Delta_+$ such that

$$\beta_1 \prec \beta_2 \prec \cdots \prec \beta_l \prec \gamma$$

for any $\gamma \in \Delta_+ \cap w\Delta_+$.

**Notation** $\preceq^w := a$ convex order which refines the above convex preorder
Example type $A_3$, $1 - 2 - 3$

- $j = s_2 s_1 s_3 s_2 s_3 s_1 \in W$
  \[ \alpha_3 \succ j \alpha_1 \succ j \alpha_1 + \alpha_2 + \alpha_3 \succ j \alpha_2 + \alpha_3 \succ j \alpha_1 + \alpha_2 \succ j \alpha_2 \]

- $i = s_3 s_2 s_1 s_3 s_2 s_3 \in W$
  \[ \alpha_1 \succ i \alpha_1 + \alpha_2 \succ i \alpha_2 \succ i \alpha_1 + \alpha_2 + \alpha_3 \succ i \alpha_2 + \alpha_3 \succ i \alpha_3 \]
2. Categories $\mathcal{C}_{w,v}$

We assume that $A$ is arbitrary.

**Definition** For $M \in R(\beta)$-$\text{Mod}$, we define

$$W(M) := \{ \gamma \in Q_+ \cap (\beta - Q_+) \mid e(\gamma, \beta - \gamma)M \neq 0 \},$$

$$W^*(M) := \{ \gamma \in Q_+ \cap (\beta - Q_+) \mid e(\beta - \gamma, \gamma)M \neq 0 \}.$$

Then we have

- $W^*(M) = \beta - W(M)$
- For $R$-modules $M$ and $N$,

We fix a convex order $\preceq$ on

$$\mathbb{Z}_{>0}\Delta_{+} := \{ k\beta \mid k \in \mathbb{Z}_{>0}, \beta \in \Delta_{+} \}.$$ 

**Definition** A simple $R(\beta)$-module $L$ is $\preceq$-cuspidal if

(a) $\beta \in \mathbb{Z}_{>0}\Delta_{+},$

(b) $W(L) \subset \text{span}_{\mathbb{R}_{\geq 0}}\{ \gamma \in \Delta_{+} \mid \gamma \preceq \beta \}.$
We fix a convex order $\preceq$ on
\[ \mathbb{Z}_{>0}\Delta_+ := \{ k\beta \mid k \in \mathbb{Z}_{>0}, \beta \in \Delta_+ \}. \]

**Definition** A simple $R(\beta)$-module $L$ is $\preceq$-cuspidal if
(a) $\beta \in \mathbb{Z}_{>0}\Delta_+$,
(b) $W(L) \subset \text{span}_{\mathbb{R}_{\geq 0}} \{ \gamma \in \Delta_+ \mid \gamma \preceq \beta \}$.

It was shown in [Tingley-Webster] that, for a simple $R$-module $L$, there exists a unique sequence $(L_1, L_2, \ldots, L_h)$ of $\preceq$-cuspidal modules (up to isomorphisms) such that
\[ -\text{wt}(L_1) \succ -\text{wt}(L_2) \succ \cdots \succ -\text{wt}(L_h), \]
\[ L \simeq \text{hd}(L_1 \circ L_2 \circ \cdots \circ L_h). \]

The sequence
\[ \varnothing(L) := (L_1, L_2, \ldots, L_h) \]
is called the $\preceq$-cuspidal decomposition of $L$. 

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**Example** type $A_3, \ 1 - 2 - 3$

$\mathbf{\triangleright} \ j = s_2 s_1 s_3 s_2 s_3 s_1$

$\alpha_3 \succ j \alpha_1 \succ j \alpha_1 + \alpha_2 + \alpha_3 \succ j \alpha_2 + \alpha_3 \succ j \alpha_1 + \alpha_2 \succ j \alpha_2$

$\preceq^j$-cuspidal modules corresponding to $\Delta_+$

$L(3) \quad L(1) \quad L(213) \quad L(23) \quad L(21) \quad L(2)$

$\mathbf{\triangleright} \ i = s_3 s_2 s_1 s_3 s_2 s_3$

$\alpha_1 \succ i \alpha_1 + \alpha_2 \succ i \alpha_2 \succ i \alpha_1 + \alpha_2 + \alpha_3 \succ i \alpha_2 + \alpha_3 \succ i \alpha_3$

$\preceq^i$-cuspidal modules corresponding to $\Delta_+$

$L(1) \quad L(21) \quad L(2) \quad L(321) \quad L(32) \quad L(3)$

(Here, $L(ijk) := \tilde{f}_i \tilde{f}_j \tilde{f}_k 1 \in R$-mod)
Definition $w, v \in W$.

- $\mathcal{C}_w :=$ full subcategory of $R$-mod whose objects $M$ satisfy
  \[ W(M) \subset \text{span}_{R \geq 0} (\Delta_+ \cap w\Delta_-) \]

- $\mathcal{C}_{*, v} :=$ full subcategory of $R$-mod whose objects $N$ satisfy
  \[ W^*(N) \subset \text{span}_{R \geq 0} (\Delta_+ \cap v\Delta_+) \]

- $\mathcal{C}_{w, v} = \mathcal{C}_w \cap \mathcal{C}_{*, v}$
Definition \( w, v \in W \).

\( \mathcal{C}_w := \) full subcategory of \( R\text{-mod} \) whose objects \( M \) satisfy

\[
W(M) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap w\Delta_-)
\]

\( \(
\mathcal{C}_{*,v} := \) full subcategory of \( R\text{-mod} \) whose objects \( N \) satisfy

\[
W^*(N) \subset \text{span}_{\mathbb{R}_{\geq 0}}(\Delta_+ \cap v\Delta_+)
\]

\( \mathcal{C}_{w,v} = \mathcal{C}_w \cap \mathcal{C}_{*,v} \)

Remark: \( \mathcal{C}_w, \mathcal{C}_{*,v} \) and \( \mathcal{C}_{w,v} \) are stable under taking subquotients, extensions, convolution products and grading shifts. In particular, \( K_0(\mathcal{C}_w), K_0(\mathcal{C}_{*,v}) \) and \( K_0(\mathcal{C}_{w,v}) \) are \( \mathbb{Z}[q, q^{-1}] \)-algebras.
Categories $\mathcal{C}_{w,v}$

$w := s_{i_1} s_{i_2} \cdots s_{i_\ell}$

$\preceq_w := \text{a convex order on } \Delta_+ \text{ associated with } w$

$L := \text{a simple } R\text{-module with}$

$\varnothing(L) := (L_1, L_2, \ldots, L_h), \quad \gamma_k := -\text{wt}(L_k) \quad \text{for } k = 1, \ldots, h.$

$\beta_\ell := s_{i_1} \cdots s_{i_{\ell-1}}(\alpha_{i_\ell})$

**Proposition**

- $L \in \mathcal{C}_w \iff \beta_\ell \succeq \gamma_k \text{ for any } k,$
- $L \in \mathcal{C}_{\ast, w} \iff \gamma_k \succeq \beta_\ell \text{ for any } k.$
Categories $\mathcal{C}_{w,v}$

$w := s_{i_1} s_{i_2} \cdots s_{i_\ell}$

$\preceq^w := \text{a convex order on } \Delta_+ \text{ associated with } w$

$L := \text{a simple } R\text{-module with}$

$\varnothing(L) := (L_1, L_2, \ldots, L_h), \quad \gamma_k := -\text{wt}(L_k) \quad \text{for } k = 1, \ldots, h.$

$\beta_\ell := s_{i_1} \cdots s_{i_{\ell-1}} (\alpha_{i_\ell})$

Proposition

$\triangleright L \in \mathcal{C}_w \iff \beta_\ell \succeq \gamma_k$ for any $k$,

$\triangleright L \in \mathcal{C}_*,w \iff \gamma_k \succ \beta_\ell$ for any $k$.

Note

$K_0(\mathcal{C}_w) = A_q(n(w))_{\mathbb{Z}[q,q^{-1}]}$.

i.e. $\mathcal{C}_w = \text{the monoidal category giving a monoidal categorification of } A_q(n(w))_{\mathbb{Z}[q,q^{-1}]}$ given by Kang-Kashiwara-Kim-Oh.
**Example** type $A_3$, $w = s_2 s_1 s_3 s_2 s_3$, $v = s_3 s_2$

$a_3 \succ j \succ a_1 \succ a_1 + a_2 + a_3 \succ j \succ a_2 + a_3 \succ j \succ a_1 + a_2 \succ j \succ a_2$

$L \in C_w \iff \vartheta(L) = (L(1)^{t_1}, L(213)^{t_2}, L(23)^{t_3}, L(21)^{t_4}, L(2)^{t_5})$
Example type $A_3$, $w = s_2s_1s_3s_2s_3$, $v = s_3s_2$

- $j = s_2s_1s_3s_2s_3$, $\beta_5 = s_2s_1s_3s_2(\alpha_3) = \alpha_1$

$$\alpha_3 \triangleright j \triangleright \alpha_1 \triangleright j \alpha_1 + \alpha_2 + \alpha_3 \triangleright j \triangleright \alpha_2 + \alpha_3 \triangleright j \triangleright \alpha_1 + \alpha_2 \triangleright j \triangleright \alpha_2$$

$$L \in \mathcal{C}_w \iff \varnothing(L) = (L(1)^{\circ t_1}, L(213)^{\circ t_2}, L(23)^{\circ t_3}, L(21)^{\circ t_4}, L(2)^{\circ t_5})$$

- $i = s_3s_2s_1s_3s_2s_3$, $\beta_2 = s_3(\alpha_2) = \alpha_2 + \alpha_3$

$$\alpha_1 \triangleright i \triangleright \alpha_1 + \alpha_2 \triangleright i \triangleright \alpha_2 \triangleright i \triangleright \alpha_1 + \alpha_2 + \alpha_3 \triangleright i \triangleright \alpha_2 + \alpha_3 \triangleright i \triangleright \alpha_3$$

$$L \in \mathcal{C}_{*,v} \iff \varnothing(L) = (L(1)^{\circ t_1}, L(21)^{\circ t_2}, L(2)^{\circ t_3}, L(321)^{\circ t_4})$$
Example type $A_3$, $w = s_2 s_1 s_3 s_2 s_3$, $v = s_3 s_2$

- $j = s_2 s_1 s_3 s_2 s_3 s_1$, $\beta_5 = s_2 s_1 s_3 s_2 (\alpha_3) = \alpha_1$

\[
\alpha_3 \succ j \alpha_1 \succ j \alpha_1 + \alpha_2 + \alpha_3 \succ j \alpha_2 + \alpha_3 \succ j \alpha_1 + \alpha_2 \succ j \alpha_2
\]

$L \in C_w \iff \mathcal{O}(L) = (L(1)^{t_1}, L(213)^{t_2}, L(23)^{t_3}, L(21)^{t_4}, L(2)^{t_5})$

- $i = s_3 s_2 s_1 s_3 s_2 s_3$, $\beta_2 = s_3(\alpha_2) = \alpha_2 + \alpha_3$

\[
\alpha_1 \succ i \alpha_1 + \alpha_2 \succ i \alpha_2 \succ i \alpha_1 + \alpha_2 + \alpha_3 \succ i \alpha_2 + \alpha_3 \succ i \alpha_3
\]

$L \in C_{*,v} \iff \mathcal{O}(L) = (L(1)^{t_1}, L(21)^{t_2}, L(2)^{t_3}, L(321)^{t_4})$

- $C_{w,v} = C_w \cap C_{*,v}$
We revisit the unipotent quantum coordinate ring $A_q(n)_{\mathbb{Z}[q,q^{-1}]}$.

**Definition**

- $A_w :=$ the $\mathbb{Z}[q,q^{-1}]$-linear subspace of $A_q(n)_{\mathbb{Z}[q,q^{-1}]}$ spanned by $x \in A_q(n)_{\mathbb{Z}[q,q^{-1}]}$ such that
  \[ e_{i_1} \cdots e_{i_l} x = 0 \]
  for any sequence $(i_1, \ldots, i_l) \in I^\beta$ with $\beta \in Q_+ \cap wQ_+ \setminus \{0\}$,

- $A_{*,v} :=$ the $\mathbb{Z}[q,q^{-1}]$-linear subspace of $A_q(n)_{\mathbb{Z}[q,q^{-1}]}$ spanned by $x \in A_q(n)_{\mathbb{Z}[q,q^{-1}]}$ such that
  \[ e_{i_1}^* \cdots e_{i_l}^* x = 0 \]
  for any sequence $(i_1, \ldots, i_l) \in I^\beta$ with $\beta \in Q_+ \cap vQ_- \setminus \{0\}$,

- $A_{w,v} := A_w \cap A_{*,v} \subset A_q(n)_{\mathbb{Z}[q,q^{-1}]}$
Remark

\( G \) : reductive group over \( \mathbb{C} \)
\( N \) and \( N^- \) : unipotent radicals of \( B \) and \( B^- \)
\( n \) and \( n^- \) : Lie algebras of \( N \) and \( N^- \)
\( w, \nu \in W \)

Note that \( \mathbb{C}[N] \) is isomorphic to the dual of \( U(n) \). We set

\[
N'(w) = N \cap (wNw^{-1}) \quad \text{and} \quad N(\nu) := N \cap (\nu N^- \nu^{-1}).
\]

Then the doubly-invariant algebra

\[
N'(w)\mathbb{C}[N]^{N(\nu)} = \{ f | f(nxm) = f(x), x \in N, m \in N'(w), n \in N(\nu) \}
\]

\[
= \{ f | U(n)_\beta f U(n)_\gamma = 0, \text{ for all } \beta \in Q_+ \cap wQ_+ \setminus \{0\}, \gamma \in Q_+ \cap \nu Q_- \setminus \{0\} \}
\]

Therefore, \( A_{w,\nu} \) = a quantum deformation of \( N'(w)\mathbb{C}[N]^{N(\nu)} \).
Theorem  Under the categorification, for $w, v \in W$, we have

- $K_0(C_w) = A_w$,
- $K_0(C_{\ast,v}) = A_{\ast,v}$,
- $K_0(C_{w,v}) = A_{w,v}$,

(Idea of Proof) Key properties

Cuspidal decomposition $\leadsto$ unmixed

$M \in C_w \iff W(M) \cap \Delta_+ \subset \Delta_+ \cap w\Delta_-$
Theorem  Under the categorification, for $w, v \in W$, we have

\[ K_0(C_w) = A_w, \]
\[ K_0(C_*,v) = A_*,v, \]
\[ K_0(C_w,v) = A_{w,v}, \]

(Idea of Proof) Key properties

Cuspidal decomposition $\sim$ unmixed

$M \in C_w \iff W(M) \cap \Delta_+ \subset \Delta_+ \cap w\Delta_-$

Corollary  The subspaces $A_w, A_*,v$ and $A_{w,v}$ are subalgebras of $A_q(n) \mathbb{Z}[q,q^{-1}]$. 
3. Determinantial modules

We assume that $A$ is arbitrary. We take

$$\Lambda \in P_+$$

$w, v \in W$ with $w \geq v$

$w = s_{i_1} \cdots s_{i_p}$ and $v = s_{j_1} \cdots s_{j_q}$

**Definition** Determinantial modules

$$M(w\Lambda, \Lambda) := \tilde{F}_{i_1}^{m_1} \cdots \tilde{F}_{i_p}^{m_p} 1, \quad \text{where } m_k = \langle h_k, s_{i_{k+1}} \cdots s_{i_p} \Lambda \rangle,$$

$$M(w\Lambda, v\Lambda) := \tilde{E}_{j_1}^{* \max} \cdots \tilde{E}_{j_q}^{* \max} M(w\Lambda, \Lambda).$$

Note that

- $M(w\Lambda, v\Lambda)$ is a self-dual simple module.
- It does not depend on the choice of reduced expressions.
Let $\Lambda, \Lambda' \in P_+, \, w, \nu \in W$ with $w \geq \nu$ and $\lambda, \mu \in W\Lambda$ with $\lambda \leq \mu$.

**Proposition**

- $[M(\lambda, \mu)] = \text{unipotent quantum minor } D(\lambda, \mu)$. Thus, it’s in the upper global basis (dual canonical basis).
- $M(w\Lambda, \nu\Lambda) \circ M(w\Lambda', \nu\Lambda') \simeq q^{-(\nu\Lambda, \nu\Lambda' - w\Lambda')} M(w(\Lambda + \Lambda'), \nu(\Lambda + \Lambda'))$
  Thus, $M(\lambda, \mu)$ is real.
- Let $\lambda, \lambda', \lambda'' \in W\Lambda$ with $\lambda \leq \lambda' \leq \lambda''$. Then there is an epimorphism

$$M(\lambda, \lambda') \circ M(\lambda', \lambda'') \twoheadrightarrow M(\lambda, \lambda'').$$
Lemma $\Lambda \in P_+, \lambda, \mu \in W\Lambda$ with $\lambda \preceq \mu$

$u_\lambda :=$ the extremal weight vector of weight $\lambda$ in $V(\Lambda)$.

$\triangleright$ If $\beta \in W(M(\lambda, \mu))$, then

$$\lambda + \beta \in \text{wt}(U^+_q(g)u_\lambda) \subset \text{wt}(V(\Lambda)).$$

$\triangleright$ If $\gamma \in W^*(M(\lambda, \mu))$, then

$$\mu - \gamma \in \text{wt}(U^+_q(g)u_\lambda) \subset \text{wt}(V(\Lambda)).$$

Note $U^+_q(g)u_\lambda =$ Demazure module associated with $\lambda$ in $V(\Lambda)$
$w, v \in W$ with $v \leq w$

Fix $\underline{w} = s_{i_1} s_{i_2} \cdots s_{i_l}$ of $w \in W$. For $i = 1, \ldots, l$, we set

$$w_{\leq k} := s_{i_1} \cdots s_{i_k}, \quad w_{\geq k+1} := s_{i_{k+1}} \cdots s_{i_l}.$$ 

We also define

(i) $v_{\geq k} = (v_{\leq k-1})^{-1} v,$

(ii) $v_{\leq k} = \begin{cases} v_{\leq k-1} s_{i_k} & \text{if } s_{i_k} v_{\geq k} < v_{\geq k}, \\ v_{\leq k-1} & \text{if } s_{i_k} v_{\geq k} > v_{\geq k}. \end{cases}$

Here we set $w_{\leq 0} = v_{\leq 0} = id \in W$.

**Note**

- $w_{\leq l} = w$ and $v_{\leq l} = v$
- $\ell(v) = \ell(v_{\leq k-1}) + \ell(v_{\geq k})$
- $v_{\leq k} \leq w_{\leq k}$ and $v_{\geq k} \leq w_{\geq k}$
Determinantial modules

\[ \Lambda, \Lambda' \in P_+, \ w, \nu \in W \text{ with } \nu \leq w, \ w := s_{i_1} \cdots s_{i_l}. \]

**Theorem** For \( k = 0, 1, \ldots, l, \)

\[ M(w \leq_k \Lambda, \nu \leq_k \Lambda) \in C_{w, \nu} \]

*(Idea of Proof)* properties of \( wt(Demazure \ module) \)
\( \Lambda, \Lambda' \in P_+, \ w, \nu \in W \) with \( \nu \leq w \), \( w := s_{i_1} \cdots s_{i_l} \).

**Theorem** For \( k = 0, 1, \ldots, l \),

\[
M(w_{\leq k} \Lambda, \nu_{\leq k} \Lambda) \in C_{w, \nu}
\]

**Idea of Proof** properties of \( wt(\text{Demazure module}) \)

**Theorem** Suppose that \( A \) is symmetric and \( R \) is symmetric.

- For \( 1 \leq k \leq j \leq l \),
  
  \[
  M(w_{\leq j} \Lambda, \nu_{\leq j} \Lambda) \circ M(w_{\leq k} \Lambda', \nu_{\leq k} \Lambda') \text{ is simple,}
  \]

- For \( 1 \leq k \leq j \leq l \),
  
  \[
  \Lambda(M(w_{\leq k} \Lambda, \nu_{\leq k} \Lambda), M(w_{\leq j} \Lambda', \nu_{\leq j} \Lambda')) = (w_{\leq k} \Lambda + \nu_{\leq k} \Lambda, \nu_{\leq j} \Lambda' - w_{\leq j} \Lambda'),
  \]

  where \( \Lambda(M, N) = \text{degree of the } R\text{-matrix} : M \circ N \rightarrow N \circ M \).

**Idea of Proof** properties of \( R\)-matrices

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Monoidal categories associated with strata of flag manifolds
**Remark** Results on cluster algebras in
[Leclerc, Cluster str. on strata of flag varieties, Adv. Math., 2016]

\[ G: ADE \text{ type, } w, \nu \in W \text{ with } \nu \leq w, \overline{w} = s_{i_1} \cdots s_{i_l}. \]

\[ R_{w,\nu} = \text{Schubert cell } C_w \cap \text{opposite Schubert cell } C^\nu \]

- \[ \mathbb{C}[R_{w,\nu}] \simeq \text{a certain localization of } N'(w)\mathbb{C}[N]^{N(\nu)} \]
- \[ \exists \text{ Frobenious subcategory } C_{w,\nu} \text{ of } \text{mod}(\Lambda) \text{ such that } \]
  \[ \text{Span}_\mathbb{C}\{\varphi_M \mid M \in C_{w,\nu}\} = N'(w)\mathbb{C}[N]^{N(\nu)} \]
  \[ \text{where } \varphi_M \in \mathbb{C}[N]: \text{cluster character of } M \]
- \[ \exists \text{ subalgebra of } N'(w)\mathbb{C}[N]^{N(\nu)} \text{ having a cluster structure with an initial cluster coming from } \]
  \[ D(w_{\leq k}(\varpi_{k+1}), \nu_{\leq k}(\varpi_{k+1})). \]

**Note** A quantization of \( \mathbb{C}[R_{w,\nu}] \) was studied by Lenagan-Yakimov in the aspect of quantum cluster algs.
Remark Results on cluster algebras in [Leclerc, Cluster str. on strata of flag varieties, Adv. Math., 2016]

\( G: \) ADE type, \( w, v \in W \) with \( v \leq w \), \( \underline{w} = s_{i_1} \cdots s_{i_l} \).

\( R_{w, v} = \) Schubert cell \( C_w \cap \) opposite Schubert cell \( C^v \)

\( \triangleright \) \( \mathbb{C}[R_{w, v}] \cong \) a certain localization of \( N'(w)\mathbb{C}[\mathcal{N}]^{N(v)} \)

\( \triangleright \exists \) Frobenious subcategory \( C_{w, v} \) of \( \text{mod}(\Lambda) \) such that

\[ \text{Span}_\mathbb{C}\{\varphi_M \mid M \in C_{w, v}\} = N'(w)\mathbb{C}[\mathcal{N}]^{N(v)} \]

where \( \varphi_M \in \mathbb{C}[\mathcal{N}] : \) cluster character of \( M \)

\( \triangleright \exists \) subalgebra of \( N'(w)\mathbb{C}[\mathcal{N}]^{N(v)} \) having a cluster structure with an initial cluster coming from \( D(w \leq k(\varpi_{k+1}), v \leq k(\varpi_{k+1})) \).

Note A quantization of \( \mathbb{C}[R_{w, v}] \) was studied by Lenagan-Yakimov in the aspect of quantum cluster algs.

Conjecture \( C_{w, v} \) gives a monoidal categorification of a quantization of the cluster algebra arising from \( R_{w, v} \).
4. Finite ADE types

We assume that $A$ is of finite ADE type and $k$ is of characteristic 0.

It was known that

- any quiver Hecke algebra is isomorphic to a symmetric quiver Hecke algebra.
- [Rouquier, Varagnolo-Vasserot] simple $R$-modules correspond to the upper global basis (dual canonical basis).

We define

- $\mathcal{C}_w$ the full subcategory of $R$-mod whose objects $M$ satisfy $\varepsilon_i(M) = 0$
- $\mathcal{C}_w^*$ the full subcategory of $R$-mod whose objects $M$ satisfy $\varepsilon_i^*(M) = 0$
It was proved by S. Kato that there exist reflection functors

$$\mathcal{T}_i : R_i(\beta)\text{-mod} \xrightarrow{\sim} iR(s_i\beta)\text{-mod},$$

$$\mathcal{T}^*_i : iR(\beta)\text{-mod} \xrightarrow{\sim} R_i(s_i\beta)\text{-mod},$$

which are equivalences of categories.

The functors $\mathcal{T}_i$ and $\mathcal{T}^*_i$ are counterparts of the Saito crystal reflections

$$T_i : B_i(\infty) \to iB(\infty), \quad b \mapsto \tilde{f}_i^* \varphi_i(b) \sim \varepsilon_i(b) b,$$

$$T^*_i : iB(\infty) \to B_i(\infty), \quad b \mapsto \tilde{f}_i^* \varphi_i^*(b) \sim \varepsilon_i^*(b) b.$$

**Note** $\mathcal{T}_i(L(b)) \simeq L(T_i(b))$ and $\mathcal{T}^*_i(L(b')) \simeq L(T^*_i(b'))$

where $L(b)$ is the simple $R$-module corresponding to $b$. 

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$w_0 := s_1 \cdots s_\ell$ of the longest $w_0 \in \mathcal{W}$,

$\prec_{w_0} :=$ a convex order corresponding to $w_0$

$L_k := \prec_{w_0}$-cuspidal module cor. to $\beta_k = s_1 \cdots s_{i_k-1}(\alpha_{i_k})$

Then, we have

$L_k \simeq L(T_{i_1}^* T_{i_2}^* \cdots T_{i_{k-1}}^* (f_{i_k})) \simeq T_{i_1}^* T_{i_2}^* \cdots T_{i_{k-1}}^* L(i_k)$

**Lemma** Let $i \in I$, $w \in \mathcal{W}$, and let $M$ be a simple $R$-module.

(i) Suppose that $M \in \mathcal{C}_w$. Then we have

(a) if $s_i w < w$ and $\varepsilon_i^*(M) = 0$, then $T_i(M) \in \mathcal{C}_{s_i w}$,
(b) if $s_i w > w$, then $T_i^*(M) \in \mathcal{C}_{s_i w}$.

(ii) Suppose that $M \in \mathcal{C}_{*,w}$. Then we have

(a) if $s_i w < w$, then $T_i(M) \in \mathcal{C}_{*,s_i w}$,
(b) if $s_i w > w$ and $\varepsilon_i(M) = 0$, then $T_i^*(M) \in \mathcal{C}_{*,s_i w}$.
Theorem Let $w, v \in W$ with $v \leq w$, $s_i w > w$ and $s_i v > v$. Then the restrictions of the functors

$$T_i|_{C_{s_i w, s_i v}} : C_{s_i w, s_i v} \to C_{w, v}, \quad T_i^*|_{C_{w, v}} : C_{w, v} \to C_{s_i w, s_i v}$$

give equivalences of the categories.
Theorem Let \( w, \nu \in W \) with \( \nu \leq w \), \( s_iw > w \) and \( s_iv > \nu \). Then the restrictions of the functors

\[
T_i|_{C_{s_iw,s_iv}} : C_{s_iw,s_iv} \xrightarrow{\sim} C_{w,\nu}, \quad T_i^*|_{C_{w,\nu}} : C_{w,\nu} \xrightarrow{\sim} C_{s_iw,s_iv}
\]

give equivalences of the categories.

Corollary Let \( w, u, \nu \in W \) with \( w = \nu u \), \( \ell(w) = \ell(\nu) + \ell(u) \). Then there is an equivalence of the categories

\[
C_{w,\nu} \simeq C_u.
\]

Note [Kang-Kashiwara-Kim-Oh]

\( C_u \) gives a monoidal categorification of \( A_q(n(u))\mathbb{Z}[q,q^{-1}] \).
**Conjecture** $\mathcal{C}_{w,v}$ gives a monoidal categorification of $A_{w,v}$.

**Note** If $\mathcal{T}_i$ and $\mathcal{T}_i^*$ are monoidal functors, then it implies that the conjecture is true.
Conjecture $\mathcal{C}_{w,v}$ gives a monoidal categorification of $A_{w,v}$.

Note If $\mathcal{T}_i$ and $\mathcal{T}_i^*$ are monoidal functors, then it implies that the conjecture is true.

Remark

- S. Kato extended the reflection functors $\mathcal{T}_i$ and $\mathcal{T}_i^*$ to symmetric cases and proved the monoidality (arXiv:1711.09085).
- P. McNamara also proved the monoidality independently (arXiv:1712.00173).

Thus, the conjecture is proved. Thanks!!
Conjecture $\mathcal{C}_{w,v}$ gives a monoidal categorification of $A_{w,v}$.

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Thus, the conjecture is proved. Thanks!!

Further Study (work in progress)

We expect that $\mathcal{C}_{w,v}$ gives a monoidal categorification of a quantization of the cluster algebra arising from $R_{w,v}$.
THANK YOU