Schur-Weyl duality and dominant dimension

Ming Fang

Institute of Mathematics, Chinese Academy of Sciences

Representation Theory of Symmetric Groups and Related Algebras, 11-20 December 2017, Singapore
Let \( k \) be a field and \( E \) be an \( n \)-dimensional \( k \)-vector space. The symmetric group \( \Sigma_r \) acts on the \( r \)-fold tensor product \( E^\otimes r \) via:

\((e_1 \otimes \cdots \otimes e_r)\sigma = e_{1\sigma} \otimes \cdots \otimes e_{r\sigma}, \quad \forall \ e_1, \ldots, e_r \in E, \sigma \in \Sigma_r\)

The Schur algebra \( S_k(n, r) \) is known to be \( \text{End}_{\Sigma_r}(E^\otimes r) \), and the Schur functor \( \mathcal{F} \) is:

\[ \text{Hom}_{S_k(n, r)}(E^\otimes r, -): S_k(n, r)\text{-mod} \longrightarrow k\Sigma_r\text{-mod} \]
Notation

Let $k$ be a field and $E$ be an $n$-dimensional $k$-vector space. The symmetric group $\Sigma_r$ acts on the $r$-fold tensor product $E^{\otimes r}$ via:

$$(e_1 \otimes \cdots \otimes e_r)\sigma = e_{1\sigma} \otimes \cdots \otimes e_{r\sigma}, \quad \forall \ e_1, \ldots, e_r \in E, \sigma \in \Sigma_r$$

The Schur algebra $S_k(n, r)$ is known to be $\text{End}_{\Sigma_r}(E^{\otimes r})$, and the Schur functor $\mathcal{F}$ is:

$$\text{Hom}_{S_k(n, r)}(E^{\otimes r}, -) : S_k(n, r)\text{-mod} \longrightarrow k\Sigma_r\text{-mod}$$

Theorem (Schur-Weyl duality)

The canonical morphism $k\Sigma_r \rightarrow \text{End}_{S_k(n, r)}(E^{\otimes r})$ is always an epimorphism of $k$-algebras, and is an isomorphism if $n \geq r$. 

Motivation

Around 2000, Doty, Hemmer, Kleschev and Nakano compared the cohomologies of general linear and symmetric groups and obtained: If \( n \geq r \) and \( p = \text{char}(k) > 0 \), then the Schur functor \( \mathcal{F} \) induces

\[
\text{Ext}^i_{S_k(n,r)}(M, N) \cong \text{Ext}^i_{\Sigma_r}(\mathcal{F}(M), \mathcal{F}(N)) \quad 0 \leq i \leq p - 3
\]

for any \( M \) and \( N \) with \( N \) filtered by dual Weyl modules.

Question: Is \( p - 3 \) the best upper bound? What is its meaning?
In 2004, Doty, Erdmann and Nakano set up a general framework to study the cohomological behavior of general Schur functors: Given an idempotent $e$ in a finite dimensional $k$-algebra $A$, how are the extension groups preserved under the Schur functor

$$\text{Hom}_A(Ae, -) : A\text{-mod} \longrightarrow eAe\text{-mod}.$$ 

Question: Can we recover the result above for Schur algebras? What about other algebras?
In 2004, Doty, Erdmann and Nakano set up a general framework to study the cohomological behavior of general Schur functors: Given an idempotent \( e \) in a finite dimensional \( k \)-algebra \( A \), how are the extension groups preserved under the Schur functor

\[
\text{Hom}_A(Ae, -) : A\text{-mod} \longrightarrow eAe\text{-mod}.
\]

Question: Can we recover the result above for Schur algebras? What about other algebras?

In 2001, Koenig-Slungenård-Xi gave a computation free proof of the Schur-Weyl duality by showing that the Schur algebra \( S_k(n, r) \) has dominant dimension at least two if \( n \geq r \).

Question: What happens if the dominant dimension is large? What is the dominant dimension of \( S_k(n, r) \)?
Definition (Koenig-F 2011)

A finite dimensional $k$-algebra $A$ is called gendo-symmetric if it is an endomorphism ring of a generator over some symmetric algebra.
Definition (Koenig-F 2011)

A finite dimensional $k$-algebra $A$ is called gendo-symmetric if it is an endomorphism ring of a generator over some symmetric algebra.

Remark. Each gendo-symmetric algebra $A$ contains a unique (up to conjugate) idempotent $e \in A$ such that $Ae$ is a projective, injective and faithful left $A$-module, and any other faithful left $A$ modules must contain $Ae$ as a direct summand. Therefore, we also denote by $(A, e)$ a gendo-symmetric algebra.
Definition (Koenig-F 2010)

Let class $A$ denote the set of all finite dimensional $k$-algebras $A$ satisfying

1. $A$ is quasi-hereditary;
2. $A$ is gendo-symmetric;
3. $A$ has an anti-involution.

Examples: (quantized) Schur algebras $S_k(n,r)$ for $n \geq r$ and their blocks; cyclotomic Schur algebras and block algebras of BGG category $O$, etc.
Definition (Koenig-F 2010)

Let class $\mathcal{A}$ denote the set of all finite dimensional $k$-algebras $A$ satisfying

1. $A$ is quasi-hereditary;
2. $A$ is gendo-symmetric;
3. $A$ has an anti-involution.

Examples: (quantized) Schur algebras $S_k(n, r)$ for $n \geq r$ and their blocks; cyclotomic Schur algebras and block algebras of BGG category $\mathcal{O}$, etc.
Definition (Nakayama, Tachikawa)

The dominant dimension of a finite dimensional $k$-algebra $A$, denoted by $\text{domdim} A$, is the largest number $t$ or $\infty$ such that in a minimal injective resolution of $A$

$$0 \rightarrow A A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

all injective modules $I^0, \ldots, I^{t-1}$ are projective.
Definition (Nakayama, Tachikawa)

The dominant dimension of a finite dimensional $k$-algebra $A$, denoted by $\text{domdim } A$, is the largest number $t$ or $\infty$ such that in a minimal injective resolution of $A$

$$0 \to A \to I^0 \to I^1 \to \cdots$$

all injective modules $I^0, \ldots, I^{t-1}$ are projective.

Examples: The $k$-algebra $k[x, y]/(x^2, xy, yx, y^2)$ has dominant dimension zero;

The algebra of upper triangular $2 \times 2$-matrices has dominant dimension one.

Self-injective algebras, and in particular semisimple algebras have dominant dimension $\infty$. 
Dominant dimension encodes information about abundance of projective-injective modules:

a) If \( \text{domdim} \ A \geq 2 \), then \( A \) has a unique (up to conjugate) idempotent \( e \) such that \( Ae \) is projective, injective and faithful and any other faithful \( A \) modules contain \( Ae \) as a direct summand. Moreover,

\[
\text{End}_A(Ae) \cong eAe, \quad \text{End}_{eAe}(Ae) \cong A.
\]

b) Nakayama conjecture: \( \text{domdim} \ A = \infty \) if and only if \( A \) is self-injective.
Cohomological behavior of Schur functors

**Theorem (Koenig-F, 2010)**

Let \((A, e)\) be an algebra in the class \(A\). Then

1. \(\text{domdim } A\) is a positive even number;
2. \(\text{domdim } A = \text{domdim } R(A)\), the Ringel dual of \(A\);
3. Standard \(A\)-modules are torsionless, i.e., are submodules of projective \(A\)-modules.
4. The Schur functor \(eA \otimes_A - : A\text{-mod} \to eAe\text{-mod}\) induces

\[
\text{Ext}^i_A(M, N) \cong \text{Ext}^i_{eAe}(eM, eN), \quad 0 \leq i \leq \text{domdim } A/2 - 2
\]

for any \(M, N\) with \(N\) filtered by standard \(A\)-modules, and

the bound \(\text{domdim } A/2 - 2\) is optimal.
Theorem (Miyachi-F)

For an algebra \((A, e)\) in the class \(A\), the Schur functor induces:

\[
\text{HH}^i(A) \cong \text{HH}^i(eAe), \quad 0 \leq i \leq \text{domdim} A - 2
\]

and the best upper bound is either \(\text{domdim} A - 2\) or \(\text{domdim} A - 1\). Moreover, any homogeneous generator of the reduced Hochschild cohomology ring \(\text{HH}^*(eAe)\) has degree 0 or at least \(\text{domdim} A - 1\).
Dominant dimension when $n \geq r$

**Theorem (Keonig-F, 2010)**

If $n \geq r$ and $p = \text{char}(k)$, then

$$\text{domdim } S_k(n, r) = \begin{cases} 
2(p - 1), & 0 < p \leq r; \\
\infty, & \text{else.}
\end{cases}$$
Dominant dimension when $n \geq r$

**Theorem (Keonig-F, 2010)**

*If $n \geq r$ and $p = \text{char}(k)$, then*

$$\text{domdim } S_k(n, r) = \begin{cases} 2(p - 1), & 0 < p \leq r; \\ \infty, & \text{else.} \end{cases}$$

**Theorem (Miyachi-F)**

*If $n \geq r$ and $\ell$ is the quantum characteristic, then*

$$\text{domdim } S_q(n, r) = \begin{cases} 2(\ell - 1), & \ell \leq r; \\ \infty, & \text{else.} \end{cases}$$
Theorem (Hu, Miyachi, Koenig, F)

If \( n \geq r \) and \( q \) is a root of one in the field \( k \) of characteristic \( p \).
Then for each block \( B_{\tau, w} \) of \( S_q(n, r) \),

\[
gldim B_{\tau, w} = 2(\ell w - d_{\ell, p}(\ell w)),
\]

\[
domdim B_{\tau, w} = \begin{cases} 
2(\ell - 1), & w \neq 0; \\
\infty, & w = 0.
\end{cases}
\]
Dominant dimension when $n < r$

**Theorem (Hu-F)**

For all $n$ and $r$, the endomorphism ring $\text{End}_{S_k(n,r)}(E^\otimes r)$ is a gendo-symmetric algebra, hence has dominant dimension at least two. For any partition $\lambda$ and $\mu$ of $r$, the dimension of $\text{Hom}_{S_k(n,r)}(S^\lambda E, D^\mu E)$ is independent of the characteristic of the ground field.
Theorem (F, 2014)

For all \( n \) and \( r \), there exists a \( S_k(n, r) \)-bimodule morphism

\[
\Theta : A_k(n, r) \otimes A_k(n, r) \to A_k(n, r)
\]

defined over \( \mathbb{Z} \), which defines an associative multiplication on \( A_k(n, r) \). Moreover, if \( p = \text{char}(k) > 0 \), then the following are equivalent

(1) \( \Theta \) is an epimorphism;

(2) \( A_k(n, r) = D_p(n, r) \);

(3) \( r \leq n(p - 1) \).

In particular, if \( r \leq n(p - 1) \), then \( S_k(n, r) \) is gendo-symmetric, hence has dominant dimension at least two, and given by

\[
\text{domdim} \, S_k(n, r) = \max\{d \mid H_i(B_\bullet) = 0, 0 \leq i \leq d\} + 1
\]

where \( B_\bullet \) is the bar complex associated to \( A_k(n, r) \) and \( \Theta \).
Theorem (Kerner-Yamagata-F, 2017)

Let $A$ be a finite dimensional $k$-algebra, and $D(A)$ the $k$-dual of $A$. Let $V = \text{Hom}_A(D(A), A)$. Then $\text{domdim } A \geq 2$ if and only if

$$D(A) \otimes_A V \otimes_A D(A) \cong D(A)$$

as $A$-bimodules, and $A$ is an endomorphism algebra of a generator over some self-injective algebra if and only if $\text{domdim } A \geq 2$ and $D(A) \otimes_A V \cong V \otimes_A D(A)$ as $A$-bimodules.
Characterization of dominant dimension

**Theorem (Koenig-F, 2011,2016)**

Let $A$ be a finite dimensional $k$-algebra and $D(A)$ be the $k$-dual of $A$. Then $A$ is a gendo-symmetric algebra if and only if $D(A) \otimes_A D(A) \cong D(A)$ as $A$-bimodules. Moreover, let $\mu$ be the induced multiplication on $D(A)$ and $C_\bullet$ be the associated bar complex, then

$$\text{domdim } A = \max \{ d \mid H_i(C_\bullet) = 0, 0 \leq i \leq d \} + 1.$$
Theorem (Hu-Koenig-F)

Let $A$ be a $k$-algebra with an anti-automorphism fixing simples. Then for any algebra $B$ derived equivalent to $A$,

$$\text{gldim} B \geq \text{gldim} A.$$ 

In particular, if both $A$ and $B$ have anti-automorphisms fixing simples, then $\text{gldim} A = \text{gldim} B$.

If furthermore, both $A$ and $B$ have dominant dimension at least one, then $\text{domdim} A = \text{domdim} B$. 

**Derived invariance**