On the center of cyclotomic quiver Hecke algebras of type $A$

Hu Jun (Beijing Institute of Technology)

National University of Singapore
December 15, 2017
Quiver Hecke algebras
The center of cyclotomic quiver Hecke algebras
The center of cyclotomic Hecke algebras
Brundan–Kleshche’s isomorphisms

Outline

1. Quiver Hecke algebras
2. The center of cyclotomic quiver Hecke algebras
3. The center of cyclotomic Hecke algebras
4. Brundan–Kleshche’s isomorphisms
1. Quiver Hecke algebras

2. The center of cyclotomic quiver Hecke algebras

3. The center of cyclotomic Hecke algebras

4. Brundan–Kleshchev’s isomorphisms

Hu Jun (Beijing Institute of Technology)

On the center of cyclotomic quiver Hecke algebras of type $A$
Outline

1. Quiver Hecke algebras
2. The center of cyclotomic quiver Hecke algebras
3. The center of cyclotomic Hecke algebras
4. Brundan–Kleshchev’s isomorphisms
Outline

1. Quiver Hecke algebras
2. The center of cyclotomic quiver Hecke algebras
3. The center of cyclotomic Hecke algebras
4. Brundan–Kleshchev’s isomorphisms
Let $\Gamma$ be a simply laced quiver (without loop) with vertex set $I$. Let $d_{i,j}$ be the number of arrows $i \rightarrow j$, where $i, j \in I$. Set $m_{i,j} = d_{i,j} + d_{j,i}$, $a_{i,i} := 2$, $a_{i,j} := -m_{i,j}$, $\forall i \neq j$. Then $C := (a_{i,j})$ is a symmetric generalised Cartan matrix.

Let $g$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the generalised Cartan matrix $C$. Let $\alpha_1, \cdots, \alpha_m$ be the simple roots, $\alpha_1^\vee, \cdots, \alpha_m^\vee$ be the simple coroots. Then $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$. Let $Q^+$ be the positive root lattice. For each $n \in \mathbb{N}$, set

$$Q_n^+ := \left\{ \sum_{i=1}^{m} c_i \alpha_i \in Q^+ \mid \sum_{i=1}^{m} c_i = n \right\}.$$
Let $\Gamma$ be a simply laced quiver (without loop) with vertex set $I$. Let $d_{i,j}$ be the number of arrows $i \to j$, where $i, j \in I$. Set $m_{i,j} = d_{i,j} + d_{j,i}$, $a_{i,i} := 2$, $a_{i,j} := -m_{i,j}$, $\forall \ i \neq j$. Then $C := (a_{i,j})$ is a symmetric generalised Cartan matrix.

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the generalised Cartan matrix $C$. Let $\alpha_1, \cdots, \alpha_m$ be the simple roots, $\alpha_1^\vee, \cdots, \alpha_m^\vee$ be the simple coroots. Then $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$. Let $Q^+$ be the positive root lattice. For each $n \in \mathbb{N}$, set

$$Q_n^+ := \left\{ \sum_{i=1}^{m} c_i \alpha_i \in Q^+ \mid \sum_{i=1}^{m} c_i = n \right\}.$$
Let $\Gamma$ be a simply laced quiver (without loop) with vertex set $I$. Let $d_{i,j}$ be the number of arrows $i \rightarrow j$, where $i, j \in I$. Set $m_{i,j} = d_{i,j} + d_{j,i}$, $a_{i,i} := 2$, $a_{i,j} := -m_{i,j}, \forall i \neq j$. Then $C := (a_{i,j})$ is a symmetric generalised Cartan matrix.

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the generalised Cartan matrix $C$. Let $\alpha_1, \cdots, \alpha_m$ be the simple roots, $\alpha_1^\vee, \cdots, \alpha_m^\vee$ be the simple coroots. Then $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$. Let $Q^+$ be the positive root lattice. For each $n \in \mathbb{N}$, set

$$Q^+_n := \left\{ \sum_{i=1}^{m} c_i \alpha_i \in Q^+ \mid \sum_{i=1}^{m} c_i = n \right\}. $$
Let $\Gamma$ be a simply laced quiver (without loop) with vertex set $I$. Let $d_{i,j}$ be the number of arrows $i \to j$, where $i, j \in I$. Set $m_{i,j} = d_{i,j} + d_{j,i}$, $a_{i,i} := 2$, $a_{i,j} := -m_{i,j}$, $\forall i \neq j$. Then $C := (a_{i,j})$ is a symmetric generalised Cartan matrix.

Let $\mathfrak{g}$ be the Kac–Moody algebra over $\mathbb{C}$ associated with the generalised Cartan matrix $C$. Let $\alpha_1, \cdots, \alpha_m$ be the simple roots, $\alpha_1^\vee, \cdots, \alpha_m^\vee$ be the simple coroots. Then $a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle$. Let $Q^+$ be the positive root lattice. For each $n \in \mathbb{N}$, set

$$Q^+_n := \left\{ \sum_{i=1}^{m} c_i \alpha_i \in Q^+ \mid \sum_{i=1}^{m} c_i = n \right\}.$$
For each $\beta \in Q_n^+$, we set

$$I^\beta := \{i = (i_1, \cdots, i_n) \in I^n | \sum_{j=1}^{n} \alpha_{ij} = \beta\}.$$ 

Let $u, v$ be two indeterminates. For any $i \neq j$, we define

$$Q_{i,j} := (-1)^{d_{i,j}}(u - v)^{m_{i,j}}.$$ 

Let $n \in \mathbb{N}$. Let $K$ be a field of characteristic $p \geq 0$. 

Hu Jun (Beijing Institute of Technology)
For each \( \beta \in Q_n^+ \), we set

\[
I^\beta := \{ \mathbf{i} = (i_1, \cdots, i_n) \in I^n \mid \sum_{j=1}^{n} \alpha_{ij} = \beta \}.
\]

Let \( u, v \) be two indeterminates. For any \( i \neq j \), we define

\[
Q_{i,j} := (-1)^{d_{i,j}} (u - v)^{m_{i,j}}.
\]

Let \( n \in \mathbb{N} \). Let \( K \) be a field of characteristic \( p \geq 0 \).
Definition (Khovanov-Lauda, Brundan–Kleshchev, Rouquier)

Let $R = K$ be a field. For each $\beta \in \mathbb{Q}_n^+$, let $R_{\beta}$ be the $K$-algebra defined by the generators:

$$\{e(i) | i \in I^\beta\} \cup \{\psi_r | 1 \leq r < n\} \cup \{y_r | 1 \leq r \leq n\}$$

and the following relations:

$$e(i)e(j) = \delta_{ij}e(i), \quad \sum_{i \in I^\beta} e(i) = 1, \quad y_r e(i) = e(i)y_r,$$

$$\psi_r e(i) = e(s_r \cdot i)\psi_r, \quad y_r y_s = y_s y_r.$$
Let $R = K$ be a field. For each $\beta \in \mathbb{Q}_n^+$, let $R_\beta$ be the $K$-algebra defined by the generators:

$$\{ e(i) | i \in I^\beta \} \cup \{ \psi_r | 1 \leq r < n \} \cup \{ y_r | 1 \leq r \leq n \}$$

and the following relations:

$$e(i)e(j) = \delta_{ij}e(i), \quad \sum_{i \in I^\beta} e(i) = 1, \quad y_re(i) = e(i)y_r,$$

$$\psi_r e(i) = e(s_{ir})\psi_r, \quad y_ry_s = y_sy_r.$$
Quiver Hecke algebras

Definition (Khovanov-Lauda, Brundan–Kleshchev, Rouquier)

Let $R = K$ be a field. For each $\beta \in Q_n^+$, let $R_{\beta}$ be the $K$-algebra defined by the generators:

$$\{ e(i) | i \in I^\beta \} \cup \{ \psi_r | 1 \leq r < n \} \cup \{ y_r | 1 \leq r \leq n \}$$

and the following relations:

$$e(i)e(j) = \delta_{ij}e(i), \quad \sum_{i \in I^\beta} e(i) = 1, \quad y_re(i) = e(i)y_r,$$

$$\psi_re(i) = e(s_r \cdot i)\psi_r, \quad y_ry_s = y_sy_r.$$
Quiver Hecke algebras

Definition (Khovanov-Lauda, Brundan–Kleshchev, Rouquier)

Let $R = K$ be a field. For each $\beta \in \mathbb{Q}_+^n$, let $\mathcal{R}_\beta$ be the $K$-algebra defined by the generators:

$$\{e(i) | i \in I^\beta\} \cup \{\psi_r | 1 \leq r < n\} \cup \{y_r | 1 \leq r \leq n\}$$

and the following relations:

$$e(i)e(j) = \delta_{ij}e(i), \quad \sum_{i \in I^\beta} e(i) = 1, \quad y_r e(i) = e(i)y_r,$$

$$\psi_r e(i) = e(s_r \cdot i)\psi_r, \quad y_ry_s = y_sy_r,$$
Quiver Hecke algebras

The center of cyclotomic quiver Hecke algebras
The center of cyclotomic Hecke algebras
Brundan–Kleshchev’s isomorphisms

Quiver Hecke algebras

Definition (Continued)

\[ \psi_r y_{r+1} e(i) = (y_r \psi_r + \delta_{ir,i_{r+1}})e(i), \quad y_{r+1} \psi_r e(i) = (\psi_r y_r + \delta_{ir,i_{r+1}})e(i), \]

\[ \psi_r y_s = y_s \psi_r, \quad \text{if } s \neq r, r + 1, \]

\[ \psi_r \psi_s = \psi_s \psi_r, \quad \text{if } |r - s| > 1, \]

\[ \psi_r^2 e(i) = Q_{ir,i_{r+1}}(y_r, y_{r+1}) e(i), \]

\[ (\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(i) = \]

\[ \frac{Q_{ir,i_{r+1}}(y_r, y_{r+1}) - Q_{ir,i_{r+1}}(y_{r+2}, y_{r+1})}{y_r - y_{r+2}} e(i). \]
Definition (Continued)

\[
\begin{align*}
\psi_r y_{r+1} e(i) &= (y_r \psi_r + \delta_{i_r,i_{r+1}}) e(i), \\
y_{r+1} \psi_r e(i) &= (\psi_r y_r + \delta_{i_r,i_{r+1}}) e(i), \\
\psi_r y_s &= y_s \psi_r, & \text{if } s \neq r, r+1, \\
\psi_r \psi_s &= \psi_s \psi_r, & \text{if } |r - s| > 1, \\
\psi_r^2 e(i) &= Q_{i_r,i_{r+1}} (y_r, y_{r+1}) e(i), \\
(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(i) &= \\
&= \delta_{i_r,i_{r+2}} \frac{Q_{i_r,i_{r+1}} (y_r, y_{r+1}) - Q_{i_r,i_{r+1}} (y_{r+2}, y_{r+1})}{y_r - y_{r+2}} e(i).
\end{align*}
\]
Definition (Continued)

\[
\begin{align*}
\psi_r y_{r+1} e(i) &= (y_r \psi_r + \delta_{ir,r+1}) e(i), \\
y_{r+1} \psi_r e(i) &= (\psi_r y_r + \delta_{ir,r+1}) e(i), \\
\psi_r y_s &= y_s \psi_r, & \text{if } s \neq r, r+1, \\
\psi_r \psi_s &= \psi_s \psi_r, & \text{if } |r - s| > 1, \\
\psi_r^2 e(i) &= Q_{ir,i_{r+1}} (y_r, y_{r+1}) e(i), \\
(\psi_r \psi_{r+1} \psi_r - \psi_{r+1} \psi_r \psi_{r+1}) e(i) &= \\
\delta_{ir,i_{r+2}} \frac{Q_{ir,i_{r+1}} (y_r, y_{r+1}) - Q_{ir,i_{r+1}} (y_{r+2}, y_{r+1})}{y_r - y_{r+2}} e(i).
\end{align*}
\]
There is a unique $\mathbb{Z}$-grading on $R_{\beta}$ such that

$$\deg e(i) = 0, \quad \deg y_r = 2, \quad \deg \psi_r e(i) = -a_{i_r, i_{r+1}}.$$

Let $P^+$ be the dominant weight lattice of $g$ and $\Lambda \in P^+$ be a dominant weight of level $\ell$.

**Definition**

The cyclotomic quiver Hecke algebra $R^n_\beta$ associated with $\beta, \Lambda$ is the quotient of $R_\beta$ by the two-sided ideal generated by

$$\sum_{i \in I_\beta} y_i^{(\Lambda, \alpha_i^\vee)} e(i).$$
There is a unique $\mathbb{Z}$-grading on $R_\beta$ such that

$$\deg e(i) = 0, \quad \deg y_r = 2, \quad \deg \psi_r e(i) = -a_{i_r, i_{r+1}}.$$ 

Let $P^+$ be the dominant weight lattice of $g$ and $\Lambda \in P^+$ be a dominant weight of level $\ell$.

**Definition**

The cyclotomic quiver Hecke algebra $R^\Lambda_\beta$ associated with $\beta, \Lambda$ is the quotient of $R_\beta$ by the two-sided ideal generated by

$$\sum_{i \in I_\beta} y_i^{(\Lambda, \alpha_i^\vee)} e(i).$$
Cyclotomic quiver Hecke algebras

There is a unique $\mathbb{Z}$-grading on $R_\beta$ such that

$$\deg e(i) = 0, \quad \deg y_r = 2, \quad \deg \psi_r e(i) = -a_{i_r, i_{r+1}}.$$ 

Let $P^+$ be the dominant weight lattice of $g$ and $\Lambda \in P^+$ be a dominant weight of level $\ell$.

**Definition**

The cyclotomic quiver Hecke algebra $R_\beta^\Lambda$ associated with $\beta, \Lambda$ is the quotient of $R_\beta$ by the two-sided ideal generated by

$$\sum_{i \in I_\beta} y_1 \langle \Lambda, \alpha_i^\vee \rangle e(i).$$
The algebras $R_\beta$ and $R_{\Lambda, \beta}$ play important roles in the categorification of the negative part of the quantised enveloping algebra $U_v(g)$ of $g$ and its irreducible integrable highest weight modules. For the references, see


Write $\beta = \sum_{i \in I} k_i \alpha_i \in Q^+_n$. Then $n = \sum_{i \in I} k_i$ and $|\{ i \in I | k_i \neq 0 \}| < \infty$.

We identify the finite set $\{ i \in I | k_i \neq 0 \}$ with $\{1, 2, \cdots, m\}$ and set $\lambda(\beta) := (k_1, k_2, \cdots, k_m)$, which is a composition of $n$. We define

$$\mathcal{S}(\beta) := \mathcal{S}_{\lambda(\beta)} = \mathcal{S}_{\{1,2,\cdots,k_m\}} \times \mathcal{S}_{\{k_1+1,k_1+2,\cdots,k_1+k_2\}} \times \cdots, \quad (2.1)$$

which is a Young subgroup of $\mathcal{S}_n$. 

Hu Jun (Beijing Institute of Technology)
Write $\beta = \sum_{i \in I} k_i \alpha_i \in Q_n^+$. Then $n = \sum_{i \in I} k_i$ and $|\{i \in I|k_i \neq 0\}| < \infty$.

We identify the finite set $\{i \in I|k_i \neq 0\}$ with $\{1, 2, \cdots, m\}$ and set $\lambda(\beta) := (k_1, k_2, \cdots, k_m)$, which is a composition of $n$. We define

$$\mathcal{G}(\beta) := \mathcal{G}_{\lambda(\beta)} = \mathcal{G}_{\{1,2,\cdots,k_m\}} \times \mathcal{G}_{\{k_1+1,k_1+2,\cdots,k_1+k_2\}} \times \cdots \overset{(2.1)}{\text{,}}$$

which is a Young subgroup of $\mathcal{G}_n$. 
Let $w_0$ denote the unique longest element in $S_n$ and $D(\beta)$ denote the set of minimal length right coset representatives of $S(\beta)$ in $S_n$. Let $d_{\beta,0}$ be the unique element in $w_0 S_{\lambda(\beta)} \cap D(\beta)$.

**Definition**

Let $\Sigma \subset S_n$ be a subset of $S_n$ which contains the identity element $1$. For any monomial $y_1^{c_1} \cdots y_n^{c_n} e(i) \in R^\Lambda_{\beta}$, we define

$$
\pi_\Sigma (y_1^{c_1} \cdots y_n^{c_n} e(i)) := \sum_{w \in \Sigma} y_w^{c_1(1)} \cdots y_w^{c_n(n)} e(wi). \quad (2.3)
$$
The center of quiver Hecke algebras

Lemma (Khovanov-Lauda, Rouquier)

The center \( Z(\mathcal{R}_\beta) \) of the quiver Hecke algebra \( \mathcal{R}_\beta \) is as follows:

\[
Z(\mathcal{R}_\beta) = K[y_1, \cdots, y_n, e(i) | i \in I^\beta] \mathbb{S}_n
\]

\[
= \pi_{\mathcal{D}(\beta)} \left( (K[y_1, \cdots, y_{k_1}] \mathbb{S}_{k_1} K[y_{k_1+1}, \cdots, y_{k_1+k_2}] \mathbb{S}_{\{k_1+1, \cdots, k_1+k_2\}} \cdots \right)
\]

\[
e(1^{k_1} 2^{k_2} \cdots m^{k_m}),
\]

where

\[
(1^{k_1} 2^{k_2} \cdots m^{k_m}) := (\underbrace{1, \cdots, 1}_{k_1 \text{ copies}}, \underbrace{2, \cdots, 2}_{k_2 \text{ copies}}, \cdots, \underbrace{m, \cdots, m}_{k_m \text{ copies}}) \in I^\beta.
\]
In particular,

\[ Z(R_\beta) \cong K[y_1, \cdots, y_{k_1}] \mathcal{S}_{k_1} \otimes_K K[y_{k_1+1}, \cdots, y_{k_1+k_2}] \mathcal{S}_{\{k_1+1, \cdots, k_1+k_2\}} \otimes \cdots, \]

and \( Z(R_\beta) \) is a Noetherian integral domain.
Two conjectures about the center

**Conjecture A**

*If the generalized Cartan matrix $C$ is symmetric then the center $Z(R_\beta)$ maps surjectively onto the center $Z(R_\Lambda)$ of $R_\Lambda$. In other words, $Z(R_\beta)$ is equal to the set of symmetric elements in $y_1e(i), \ldots, y_ne(i), e(i), i \in I_\beta$.*

If the generalized Cartan matrix $C$ is symmetric of finite type and the ground field is of characteristic 0, then Conjecture A was proved by Webster using earlier results of Shan-Varagnolo-Vasserot.
Two conjectures about the center

Conjecture A

If the generalized Cartan matrix $C$ is symmetric then the center $Z(R_\beta)$ maps surjectively onto the center $Z(R_\beta^\Lambda)$ of $R_\beta^\Lambda$. In other words, $Z(R_\beta^\Lambda)$ is equal to the set of symmetric elements in $y_1 e(i), \ldots, y_n e(i), e(i), i \in I^\beta$.

If the generalized Cartan matrix $C$ is symmetric of finite type and the ground field is of characteristic 0, then Conjecture A was proved by Webster using earlier results of Shan-Varagnolo-Vasserot.
Two conjectures about the center

Conjecture A

If the generalized Cartan matrix $C$ is symmetric then the center $Z(R_{\beta})$ maps surjectively onto the center $Z(R^\Lambda_{\beta})$ of $R^\Lambda_{\beta}$. In other words, $Z(R^\Lambda_{\beta})$ is equal to the set of symmetric elements in $y_1 e(i), \ldots, y_n e(i), e(i), i \in I^\beta$.

If the generalized Cartan matrix $C$ is symmetric of finite type and the ground field is of characteristic 0, then Conjecture A was proved by Webster using earlier results of Shan-Varagnolo-Vasserot.
Two conjectures about the center

Conjecture B

If the generalized Cartan matrix $C$ is symmetric then the dimension of the center $Z(R^\Lambda_\beta)$ is stable under base change, i.e., does not depend on the choice of the ground field $K$. 
Suppose that $\beta = \ell \alpha_0$. Then $R^\wedge_\beta$ is the cyclotomic nilHecke algebra of type $A$. In this case,

\begin{align*}
\{ \psi_w y_1^{a_1} \cdots y_n^{a_n} \mid 0 \leq a_i \leq \ell - i, \forall 1 \leq i \leq n, w \in S_n \} \quad (2.8)
\end{align*}

form a $K$-basis of $R^\wedge_\beta$. 

Theorem (Hu-Liang, 2017)

The elements in the following set
Suppose that $\beta = \ell \alpha_0$. Then $R^\wedge_\beta$ is the cyclotomic nilHecke algebra of type $A$. In this case,

\begin{align*}
\{ \psi_w y_1^{a_1} \cdots y_n^{a_n} \mid 0 \leq a_i \leq \ell - i, \ \forall \ 1 \leq i \leq n, w \in S_n \} \tag{2.8}
\end{align*}

form a $K$-basis of $R^\wedge_\beta$. 

Theorem (Hu-Liang, 2017) 

*The elements in the following set* 

\begin{align*}
\{ \psi_w y_1^{a_1} \cdots y_n^{a_n} \mid 0 \leq a_i \leq \ell - i, \ \forall \ 1 \leq i \leq n, w \in S_n \} \tag{2.8}
\end{align*}

*form a $K$-basis of $R^\wedge_\beta$. 

Hu Jun (Beijing Institute of Technology)
The nilHecke algebra case

Suppose that $\beta = \ell \alpha_0$. Then for each $\mu$ with $\theta(\mu) = (k_1, \cdots, k_n)$, there is a unique element $z(\mu)$ living in the center of the nilHecke algebra $R_\beta$ such that

$$y_1^{\ell-k_1} \cdots y_n^{\ell-k_n} \psi_{w_0} = z(\mu) + \sum_{r=1}^{n-1} \psi_r h_r,$$

where $h_r \in R_\beta$ for each $r$. We define $z_\mu := \pi(z(\mu)) \in R_\beta^\wedge$.

Theorem (Hu-Liang, 2017)

The elements in the set $\{z_\mu\}_{\mu}$ form a $K$-basis of the center $Z := Z(R_\beta^\wedge)$ of $R_\beta^\wedge$. In particular, the center of $R_\beta^\wedge$ is the set of symmetric polynomials in $y_1, \cdots, y_n$. In this case, both Conjecture A and B hold.
Suppose that $\beta = \ell\alpha_0$. Then for each $\mu$ with $\theta(\mu) = (k_1, \cdots, k_n)$, there is a unique element $z(\mu)$ living in the center of the nilHecke algebra $R_\beta$ such that

$$y_1^{\ell-k_1} \cdots y_n^{\ell-k_n} \psi_{w_0} = z(\mu) + \sum_{r=1}^{n-1} \psi_r h_r,$$

where $h_r \in R_\beta$ for each $r$. We define $z_\mu := \pi(z(\mu)) \in R^\Lambda_\beta$.

**Theorem (Hu-Liang, 2017)**

The elements in the set $\{z_\mu\}_\mu$ form a $K$-basis of the center $Z := Z(R^\Lambda_\beta)$ of $R^\Lambda_\beta$. In particular, the center of $R^\Lambda_\beta$ is the set of symmetric polynomials in $y_1, \cdots, y_n$. In this case, both Conjecture A and B hold.
Suppose that $\beta = \ell\alpha_0$. Then for each $\mu$ with $\theta(\mu) = (k_1, \cdots, k_n)$, there is a unique element $z(\mu)$ living in the center of the nilHecke algebra $R_\beta$ such that

$$y_1^{\ell-k_1} \cdots y_n^{\ell-k_n} \psi_{w_0} = z(\mu) + \sum_{r=1}^{n-1} \psi_r h_r,$$

where $h_r \in R_\beta$ for each $r$. We define $z_\mu := \pi(z(\mu)) \in R^\Lambda_\beta$.

**Theorem (Hu-Liang, 2017)**

The elements in the set \( \{z_\mu\}_\mu \) form a $K$-basis of the center $Z := Z(R^\Lambda_\beta)$ of $R^\Lambda_\beta$. In particular, the center of $R^\Lambda_\beta$ is the set of symmetric polynomials in $y_1, \cdots, y_n$. In this case, both Conjecture A and B hold.
Suppose that \( \beta = \sum_{j=1}^{n} \alpha_{i_j} \), where \( \alpha_{i_1}, \ldots, \alpha_{i_n} \) are pairwise distinct. Then the dimensions of the center \( Z(\mathcal{R}_\beta^\Lambda) \) and the commutator subspace \( [\mathcal{R}_\beta^\Lambda, \mathcal{R}_\beta^\Lambda] \) are stable under base change. In this case, Conjecture B holds.
What about the center of the cyclotomic Hecke algebras?

We have the following two situations:

1) The non-degenerate case. In this case, we set $1 \neq q \in K^\times$, and let $e$ be the minimal positive integer $k$ such that $1 + q + q^2 + \cdots + q^{k-1} = 0$ in $K$; or set $e := 0$ if no such $k$ exists.

2) The degenerate case. In this case, we set $q = 1 \in K$, and let $e := p$ be the characteristic of the ground field $K$. 
What about the center of the cyclotomic Hecke algebras?

We have the following two situations:

1) The non-degenerate case. In this case, we set $1 \neq q \in K^\times$, and let $e$ be the minimal positive integer $k$ such that $1 + q + q^2 + \cdots + q^{k-1} = 0$ in $K$; or set $e := 0$ if no such $k$ exists.

2) The degenerate case. In this case, we set $q = 1 \in K$, and let $e := p$ be the characteristic of the ground field $K$. 
What about the center of the cyclotomic Hecke algebras?

We have the following two situations:

1) The non-degenerate case. In this case, we set $1 \neq q \in K^{\times}$, and let $e$ be the minimal positive integer $k$ such that $1 + q + q^2 + \cdots + q^{k-1} = 0$ in $K$; or set $e := 0$ if no such $k$ exists.

2) The degenerate case. In this case, we set $q = 1 \in K$, and let $e := p$ be the characteristic of the ground field $K$. 
What about the center of the cyclotomic Hecke algebras?

We have the following two situations:

1) The non-degenerate case. In this case, we set $1 \neq q \in K^\times$, and let $e$ be the minimal positive integer $k$ such that

$$1 + q + q^2 + \cdots + q^{k-1} = 0 \text{ in } K;$$

or set $e := 0$ if no such $k$ exists.

2) The degenerate case. In this case, we set $q = 1 \in K$, and let $e := p$ be the characteristic of the ground field $K$. 
The **non-degenerate affine Hecke algebra** $H_n(q)$ of type $A$ with Hecke parameter $q$ is defined by:

**Generators:** $T_1, \cdots, T_{n-1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1}$;

**Relations:**

$$(T_r + 1)(T_r - q) = 0, \quad T_r T_s = T_s T_r, \quad \text{if } |r - s| > 1,$$

$$X_r^{\pm 1} X_t^{\pm 1} = X_t^{\pm 1} X_r^{\pm 1}, \quad X_t^{-1} X_t = 1 = X_t X_t^{-1},$$

$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \quad T_r X_t = X_t T_r, \quad \text{if } t \neq r, r + 1,$$

$$X_{r+1}(T_r - q + 1) = T_r X_r,$$

where $1 \leq r < n, 1 \leq s < n - 1$ and $1 \leq t \leq n$. 

Hu Jun (Beijing Institute of Technology)
The **non-degenerate affine Hecke algebra** $\mathcal{H}_n(q)$ of type $A$ with Hecke parameter $q$ is defined by:

**Generators:** $T_1, \cdots, T_{n-1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1}$;

**Relations:**

\[
(T_r + 1)(T_r - q) = 0, \quad T_r T_s = T_s T_r, \quad \text{if } |r - s| > 1,
\]

\[
X_r^{\pm 1} X_t^{\pm 1} = X_t^{\pm 1} X_r^{\pm 1}, \quad X_t^{-1} X_t = 1 = X_t X_t^{-1},
\]

\[
T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \quad T_r X_t = X_t T_r, \quad \text{if } t \neq r, r+1,
\]

\[
X_{r+1}(T_r - q + 1) = T_r X_r,
\]

where $1 \leq r < n$, $1 \leq s < n - 1$ and $1 \leq t \leq n$. 
The **non-degenerate affine Hecke algebra** \( \mathcal{H}_n(q) \) of type \( A \) with Hecke parameter \( q \) is defined by:

generators: \( T_1, \cdots, T_{n-1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1} \);
relations:

\[
(T_r + 1)(T_r - q) = 0, \quad T_r T_s = T_s T_r, \quad \text{if } |r - s| > 1,
\]

\[
X_r^{\pm 1} X_t^{\pm 1} = X_t^{\pm 1} X_r^{\pm 1}, \quad X_t^{-1} X_t = 1 = X_t X_t^{-1},
\]

\[
T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \quad T_r X_t = X_t T_r, \quad \text{if } t \neq r, r + 1,
\]

\[
X_{r+1}(T_r - q + 1) = T_r X_r,
\]

where \( 1 \leq r < n \), \( 1 \leq s < n - 1 \) and \( 1 \leq t \leq n \).
Non-degenerate affine Hecke algebras

The **non-degenerate affine Hecke algebra** $\mathcal{H}_n(q)$ of type $A$ with Hecke parameter $q$ is defined by:

- **generators:** $T_1, \cdots, T_{n-1}, X_1^{\pm 1}, \cdots, X_n^{\pm 1}$;
- **relations:**

\[
(T_r + 1)(T_r - q) = 0, \quad T_r T_s = T_s T_r, \quad \text{if } |r - s| > 1, \\
X_r^{\pm 1} X_t^{\pm 1} = X_t^{\pm 1} X_r^{\pm 1}, \quad X_t^{-1} X_t = 1 = X_t X_t^{-1}, \\
T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \quad T_r X_t = X_t T_r, \quad \text{if } t \neq r, r + 1, \\
X_{r+1}(T_r - q + 1) = T_r X_r,
\]

where $1 \leq r < n$, $1 \leq s < n - 1$ and $1 \leq t \leq n$. 
Let $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+$.

**Definition**

The non-degenerate cyclotomic Hecke algebra $\mathcal{H}_n^\Lambda(q)$ of type A with Hecke parameter $q$ and cyclotomic parameters $q^{\kappa_1}, \cdots, q^{\kappa_\ell}$ is defined to be the quotient of $\mathcal{H}_n(q)$ by the two-sided ideal generated by

$$(X_1 - q^{\kappa_1}) \cdots (X_1 - q^{\kappa_\ell}).$$

Let $\pi : \mathcal{H}_n(q) \to \mathcal{H}_n^\Lambda(q)$ be the canonical map. Set $L_i := \pi(X_i)$ for each $i$. Then $\{L_1, \cdots, L_n\}$ are called the Jucys-Murphy operators of $\mathcal{H}_n^\Lambda(q)$. 
Non-degenerate cyclotomic Hecke algebras

Let $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+$. 

**Definition**

The *non-degenerate cyclotomic Hecke algebra* $\mathcal{H}_n^{\Lambda}(q)$ of type $A$ with Hecke parameter $q$ and cyclotomic parameters $q^{\kappa_1}, \cdots, q^{\kappa_\ell}$ is defined to be the quotient of $\mathcal{H}_n(q)$ by the two-sided ideal generated by 

$$(X_1 - q^{\kappa_1}) \cdots (X_1 - q^{\kappa_\ell}).$$

Let $\pi : \mathcal{H}_n(q) \rightarrow \mathcal{H}_n^{\Lambda}(q)$ be the canonical map. Set $L_i := \pi(X_i)$ for each $i$. Then $\{L_1, \cdots, L_n\}$ are called the Jucys-Murphy operators of $\mathcal{H}_n^{\Lambda}(q)$. 
Let $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+$. 

**Definition**

The **non-degenerate cyclotomic Hecke algebra** $\mathcal{H}_n^\Lambda(q)$ of type $A$ with Hecke parameter $q$ and cyclotomic parameters $q^{\kappa_1}, \cdots, q^{\kappa_\ell}$ is defined to be the quotient of $\mathcal{H}_n(q)$ by the two-sided ideal generated by $(X_1 - q^{\kappa_1}) \cdots (X_1 - q^{\kappa_\ell})$.

Let $\pi : \mathcal{H}_n(q) \rightarrow \mathcal{H}_n^\Lambda(q)$ be the canonical map. Set $L_i := \pi(X_i)$ for each $i$. Then $\{L_1, \cdots, L_n\}$ are called the **Jucys-Murphy operators** of $\mathcal{H}_n^\Lambda(q)$.
The degenerate affine Hecke algebra $H_n$ of type $A$ is defined by:

generators: $s_1, \cdots, s_{n-1}, x_1, \cdots, x_n$;

relations:

$$(s_r + 1)(s_r - 1) = 0,$$

$s_r s_a = s_a s_r, \quad \text{if } |r - a| > 1,$

$x_r x_t = x_t x_r,$

$s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}, \quad s_r x_t = x_t s_r, \quad \text{if } t \neq r, r + 1,$

$x_{r+1} s_r = s_r x_r + 1,$

where $1 \leq r < n, 1 \leq a < n - 1$ and $1 \leq t \leq n.$
The **degenerate affine Hecke algebra** $H_n$ of type $A$ is defined by:

**Generators:** $s_1, \cdots, s_{n-1}, x_1, \cdots, x_n$;

**Relations:**

\[(s_r + 1)(s_r - 1) = 0, \quad s_r s_a = s_a s_r, \quad \text{if } |r - a| > 1,\]

\[x_r x_t = x_t x_r,\]

\[s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}, \quad s_r x_t = x_t s_r, \quad \text{if } t \neq r, r + 1,\]

\[x_{r+1} s_r = s_r x_r + 1,\]

where $1 \leq r < n$, $1 \leq a < n - 1$ and $1 \leq t \leq n$. 
The degenerate affine Hecke algebra $H_n$ of type $A$ is defined by:
generators: $s_1, \cdots, s_{n-1}, x_1, \cdots, x_n$;
relations:

$$(s_r + 1)(s_r - 1) = 0,$$

$s_r s_a = s_a s_r$, if $|r - a| > 1,$

$x_r x_t = x_t x_r,$

$s_a s_{a+1} s_a = s_{a+1} s_a s_a + 1,$ $s_r x_t = x_t s_r$, if $t \neq r, r + 1,$

$x_{r+1} s_r = s_r x_r + 1,$

where $1 \leq r < n$, $1 \leq a < n - 1$ and $1 \leq t \leq n.$
The **degenerate affine Hecke algebra** $H_n$ of type $A$ is defined by:

**generators:** $s_1, \cdots, s_{n-1}, x_1, \cdots, x_n$;

**relations:**

\[(s_r + 1)(s_r - 1) = 0, \quad s_r s_a = s_a s_r, \quad \text{if } |r - a| > 1,\]

\[x_r x_t = x_t x_r,\]

\[s_a s_{a+1} s_a = s_{a+1} s_a s_{a+1}, \quad s_r x_t = x_t s_r, \quad \text{if } t \neq r, r + 1,\]

\[x_{r+1} s_r = s_r x_r + 1,\]

where $1 \leq r < n$, $1 \leq a < n - 1$ and $1 \leq t \leq n$. 
Let $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in \mathcal{P}^+$. 

**Definition**

The **degenerate cyclotomic Hecke algebra** $H_n^\Lambda$ of type $A$ with cyclotomic parameters $\kappa_1, \cdots, \kappa_\ell$ is defined to be the quotient of $H_n$ by the two-sided ideal generated by

$$(x_1 - \kappa_1) \cdots (x_1 - \kappa_\ell).$$

Let $\pi : H_n \twoheadrightarrow H_n^\Lambda$ be the canonical map. Set $L_i := \pi(x_i)$ for each $i$. Then $\{L_1, \cdots, L_n\}$ are called the **Jucys-Murphy operators** of $H_n^\Lambda$. 

Hu Jun (Beijing Institute of Technology)
Let $\Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+$. 

**Definition**

The **degenerate cyclotomic Hecke algebra** $H^\Lambda_n$ of type $A$ with cyclotomic parameters $\kappa_1, \cdots, \kappa_\ell$ is defined to be the quotient of $H_n$ by the two-sided ideal generated by

$$(x_1 - \kappa_1) \cdots (x_1 - \kappa_\ell).$$

Let $\pi : H_n \to H^\Lambda_n$ be the canonical map. Set $L_i := \pi(x_i)$ for each $i$. Then $\{L_1, \cdots, L_n\}$ are called the Jucys-Murphy operators of $H^\Lambda_n$. 

---

Hu Jun (Beijing Institute of Technology)
Let \( \Lambda := \Lambda_{\kappa_1} + \cdots + \Lambda_{\kappa_\ell} \in P^+ \).

**Definition**

The **degenerate cyclotomic Hecke algebra** \( H_n^\Lambda \) of type A with cyclotomic parameters \( \kappa_1, \cdots, \kappa_\ell \) is defined to be the quotient of \( H_n \) by the two-sided ideal generated by

\[
(x_1 - \kappa_1) \cdots (x_1 - \kappa_\ell).
\]

Let \( \pi : H_n \rightarrow H_n^\Lambda \) be the canonical map. Set \( L_i := \pi(x_i) \) for each \( i \).

Then \( \{ L_1, \cdots, L_n \} \) are called the **Jucys-Murphy operators** of \( H_n^\Lambda \).
Faithful Polynomial Representations

Let \( \{ t_k | 1 \leq k \leq n \} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \mathcal{P}_n := K[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \) and \( P_n := K[t_1, \cdots, t_n] \). Clearly there is a natural left action of \( S_n \) on \( l^n, \mathcal{P}_n \) and \( P_n \) respectively.

For any \( f \in \mathcal{P}_n, \ g \in P_n, \ 1 \leq r < n \) and \( 1 \leq k \leq n \), we define

\[
\begin{align*}
X_k^{\pm 1} * f & : = t_k^{\pm 1} f, \\
T_r * f & : = (t_{r+1} - qt_r) \frac{s_r(f) - f}{t_{r+1} - t_r} + qf,
\end{align*}
\]

and

\[
\begin{align*}
x_k * g & : = t_k g, \\
s_r * g & : = -\frac{s_r(g) - g}{t_{r+1} - t_r} + s_r(g),
\end{align*}
\]
Let \( \{ t_k \mid 1 \leq k \leq n \} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \mathcal{P}_n := K[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \) and \( P_n := K[t_1, \cdots, t_n] \). Clearly there is a natural left action of \( S_n \) on \( I^n, \mathcal{P}_n \) and \( P_n \) respectively.

For any \( f \in \mathcal{P}_n, g \in P_n, 1 \leq r < n \) and \( 1 \leq k \leq n \), we define

\[
\begin{align*}
X_{k}^{\pm 1} \ast f & := t_{k}^{\pm 1}f, \\
T_r \ast f & := (t_{r+1} - qt_r) \frac{s_r(f) - f}{t_{r+1} - t_r} + qf,
\end{align*}
\]  

(3.3)

and

\[
\begin{align*}
x_k \ast g & := t_k g, \\
s_r \ast g & := -\frac{s_r(g) - g}{t_{r+1} - t_r} + s_r(g),
\end{align*}
\]  

(3.4)
Faithful Polynomial Representations

Let \( \{ t_k \mid 1 \leq k \leq n \} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( P_n := K[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \) and \( P_n := K[t_1, \cdots, t_n] \). Clearly there is a natural left action of \( S_n \) on \( I^n, P_n \) and \( P_n \) respectively. For any \( f \in P_n, g \in P_n, 1 \leq r < n \) and \( 1 \leq k \leq n \), we define

\[
\begin{align*}
X_k^{\pm 1} \ast f & := t_k^{\pm 1} f, \\
T_r \ast f & := (t_{r+1} - qt_r) \frac{s_r(f) - f}{t_{r+1} - t_r} + qf, \\
X_k \ast g & := t_k g, \\
s_r \ast g & := -\frac{s_r(g) - g}{t_{r+1} - t_r} + s_r(g),
\end{align*}
\tag{3.3}
\]

and

\[
\begin{align*}
X_k \ast g & := t_k g, \\
s_r \ast g & := -\frac{s_r(g) - g}{t_{r+1} - t_r} + s_r(g),
\end{align*}
\tag{3.4}
\]
Faithful Polynomial Representations

Let \( \{t_k | 1 \leq k \leq n\} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \mathcal{P}_n := K[t_1^{\pm 1}, \cdots, t_n^{\pm 1}] \) and \( P_n := K[t_1, \cdots, t_n] \). Clearly there is a natural left action of \( \mathcal{S}_n \) on \( I^n \), \( \mathcal{P}_n \) and \( P_n \) respectively.

For any \( f \in \mathcal{P}_n \), \( g \in P_n \), \( 1 \leq r < n \) and \( 1 \leq k \leq n \), we define

\[
\begin{align*}
X_k^{\pm 1} \ast f &:= t_k^{\pm 1} f, \\
T_r \ast f &:= (t_{r+1} - q t_r) \left( \frac{s_r(f) - f}{t_{r+1} - t_r} \right) + q f, \\
\end{align*}
\]

and

\[
\begin{align*}
x_k \ast g &:= t_k g, \\
s_r \ast g &:= -\frac{s_r(g) - g}{t_{r+1} - t_r} + s_r(g),
\end{align*}
\]
Faithful Polynomial Representations

Lemma

The above rules extend uniquely to a faithful representation $\rho_q$ of $\mathcal{H}_n(q)$ on $P_n$ as well as a faithful representation $\rho_1$ of $H_n$ on $P_n$. 
Lemma

The elements in the following set

\[ \{ T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{Z} \} \]

are $K$-linearly independent and form a basis of $\mathcal{H}_n(q)$. Similarly, the elements in the following set

\[ \{ w X_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{N} \} \]

are $K$-linearly independent and form a basis of $H_n$. 
Standard Bases

Lemma

The elements in the following set

$$\left\{ T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{Z} \right\}$$

are $K$-linearly independent and form a basis of $\mathcal{H}_n(q)$. Similarly, the elements in the following set

$$\left\{ w X_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{N} \right\}$$

are $K$-linearly independent and form a basis of $H_n$. 
Standard Bases

Lemma

The elements in the following set

\[ \{ T_w X_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{Z} \} \]

are \( K \)-linearly independent and form a basis of \( \mathcal{H}_n(q) \). Similarly, the elements in the following set

\[ \{ wX_1^{a_1} \cdots X_n^{a_n} \mid w \in S_n, a_1, \cdots, a_n \in \mathbb{N} \} \]

are \( K \)-linearly independent and form a basis of \( H_n \).
Bernstein’s result and a conjecture

Lemma (Bernstein)

The center of $\mathcal{H}_n(q)$ is equal to the set of symmetric Laurent polynomials in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, while the center of $H_n$ is equal to the set of symmetric polynomials in $x_1, \ldots, x_n$.

Conjecture C

$\pi(Z(\mathcal{H}_n(q)))) = Z(\mathcal{H}_n^\Lambda(q))$, $\pi(Z(H_n)) = Z(H_n^\Lambda)$. 

Hu Jun (Beijing Institute of Technology)

On the center of cyclotomic quiver Hecke algebras of type $A$
Bernstein’s result and a conjecture

Lemma (Bernstein)

The center of $\mathcal{H}_n(q)$ is equal to the set of symmetric Laurent polynomials in $X_1^{±1}, \ldots, X_n^{±1}$, while the center of $H_n$ is equal to the set of symmetric polynomials in $x_1, \ldots, x_n$.

Conjecture C

$\pi(Z(\mathcal{H}_n(q))) = Z(\mathcal{H}_n^\Lambda(q))$, $\pi(Z(H_n)) = Z(H_n^\Lambda)$. 
Bernstein’s result and a conjecture

Lemma (Bernstein)

The center of $H_n(q)$ is equal to the set of symmetric Laurent polynomials in $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, while the center of $H_n$ is equal to the set of symmetric polynomials in $x_1, \ldots, x_n$.

Conjecture C

$\pi(Z(H_n(q))) = Z(H_n^\Lambda(q)), \quad \pi(Z(H_n)) = Z(H_n^\Lambda)$. 

Hu Jun (Beijing Institute of Technology)

On the center of cyclotomic quiver Hecke algebras of type A
Some history about the Conjecture C

Some special cases of Conjecture C were known to be true.

1. If $q = 1$, then Conjecture C was proved by Murphy (1983) in level one case and by Brundan (2008) in general case;

2. If $q \neq 1$ and $\ell = 1$, then Conjecture C was proved by Dipper and James (1987) in the semisimple case, and by Francis and Graham (2006) in general case.

3. If $q \neq 1$, $\ell > 1$ and $e = 0$, then Conjecture C was proved by Mcgerty (2012).
Some history about the Conjecture C

Some special cases of Conjecture C were known to be true.

1. If \( q = 1 \), then Conjecture C was proved by Murphy (1983) in level one case and by Brundan (2008) in general case;

2. If \( q \neq 1 \) and \( \ell = 1 \), then Conjecture C was proved by Dipper and James (1987) in the semisimple case, and by Francis and Graham (2006) in general case.

3. If \( q \neq 1 \), \( \ell > 1 \) and \( e = 0 \), then Conjecture C was proved by McGerty (2012).
Some special cases of Conjecture C were known to be true.

1. If $q = 1$, then Conjecture C was proved by Murphy (1983) in level one case and by Brundan (2008) in general case;

2. If $q \neq 1$ and $\ell = 1$, then Conjecture C was proved by Dipper and James (1987) in the semisimple case, and by Francis and Graham (2006) in general case.

3. If $q \neq 1$, $\ell > 1$ and $e = 0$, then Conjecture C was proved by McGerty (2012).
Some history about the Conjecture C

Some special cases of Conjecture C were known to be true.

1. If $q = 1$, then Conjecture C was proved by Murphy (1983) in level one case and by Brundan (2008) in general case;

2. If $q \neq 1$ and $\ell = 1$, then Conjecture C was proved by Dipper and James (1987) in the semisimple case, and by Francis and Graham (2006) in general case.

3. If $q \neq 1$, $\ell > 1$ and $e = 0$, then Conjecture C was proved by McGerty (2012).
Inverse limits of cyclotomic Hecke algebras

Let $\mathcal{H}_n \in \{\mathcal{H}_n^{\text{aff}}, H_n^{\text{aff}}\}$. For any $\Lambda, \Lambda' \in P^+$, we define $\Lambda > \Lambda'$ if $\Lambda - \Lambda' \in P^+$. Then $(P^+, >)$ becomes a directed poset. If $\Lambda > \Lambda'$ in $P^+$, then there is a canonical surjective homomorphism

$$\pi_{\Lambda, \Lambda'} : \mathcal{H}_n^\Lambda \to \mathcal{H}_n^{\Lambda'}$$

such that $\pi_{\Lambda'} = \pi_{\Lambda, \Lambda'} \circ \pi_\Lambda$, where

$$\pi_\Lambda : \mathcal{H}_n \to \mathcal{H}_n^\Lambda$$

is the canonical surjection.
Inverse limits of cyclotomic Hecke algebras

Let

\[ \tilde{\pi} : \mathcal{H}_n \rightarrow \varprojlim \mathcal{H}_n^n \]

be the induced homomorphism. We define \( \tilde{\mathcal{H}}_n \) to be the image of \( \tilde{\pi} \) in \( \varprojlim \mathcal{H}_n^n \), and for any \( 1 \leq k < n, 1 \leq r \leq n \), we set

\[ \hat{T}_k := \tilde{\pi}(T_k), \quad \hat{X}_r := \tilde{\pi}(X_r), \quad \hat{s}_k := \tilde{\pi}(s_k), \quad \hat{x}_r := \tilde{\pi}(x_r). \quad (3.9) \]
Inverse limits of cyclotomic Hecke algebras

Let

\[ \tilde{\pi} : \mathcal{H}_n \to \varprojlim \mathcal{H}_n^\Lambda \]

be the induced homomorphism. We define \( \mathcal{H}_n^\Lambda \) to be the image of \( \tilde{\pi} \) in \( \varprojlim \mathcal{H}_n^\Lambda \), and for any \( 1 \leq k < n, 1 \leq r \leq n \), we set

\[ \hat{T}_k := \tilde{\pi}(T_k), \quad \hat{X}_r := \tilde{\pi}(X_r), \quad \hat{s}_k := \tilde{\pi}(s_k), \quad \hat{x}_r := \tilde{\pi}(x_r). \]  

(3.9)
Lifting the idempotent \( e(i) \) to inverse limits

**Lemma**

Let \( i \in I^n \). Then there exists an idempotent \( 0 \neq \hat{e}(i) \in \varprojlim \mathcal{H}_n^\wedge \)

such that \( \text{pr}_\wedge(\hat{e}(i)) = e(i) \) for any \( \wedge \in P^+ \). Furthermore, for any \( z \in \mathcal{H}_n \), if

\[
\hat{e}(i)z = 0 \quad \text{or} \quad z\hat{e}(i) = 0 \quad \text{in} \quad \varprojlim \mathcal{H}_n^\wedge,
\]

then \( z = 0 \) in \( \mathcal{H}_n \).
Definition (Hu-Li)

In the non-degenerate case, let $\beta \in \mathbb{Q}_n^+$. We define the modified non-degenerate affine Hecke algebra $\hat{H}_{\beta}(q)$ of type A to be the $K$-subalgebra of $\varprojlim H_{\Lambda}^n(q)$ generated by the following elements:

$$\hat{T}_k \hat{e}(i), \hat{X}_r^{\pm 1} \hat{e}(i), \hat{e}(i), (\hat{X}_a - \hat{X}_b)^{-1} \hat{e}(i), i \in I^\beta, \quad (3.12)$$

where $1 \leq k < n$, $1 \leq r \leq n$, $1 \leq a < b \leq n$ with $i_a \neq i_b$. 
Lemma

In the non-degenerate case, the following relations hold:

\[ \hat{X}_k^{\pm 1} \hat{e}(i) = \hat{e}(i) \hat{X}_k^{\pm 1}, \quad \hat{e}(i) \hat{e}(j) = \delta_{ij} \hat{e}(i), \quad (3.14) \]

\[ \begin{align*}
\hat{e}(i) \hat{T}_r \hat{e}(i) &= (q - 1) \hat{e}(i) \hat{X}_{r+1} (\hat{X}_{r+1} - \hat{X}_r)^{-1} \hat{e}(i), \\
\hat{e}(i) \hat{T}_r \hat{X}_r \hat{e}(i) &= \hat{e}(i) \hat{X}_r \hat{T}_r \hat{e}(i), \\
\hat{e}(i) \hat{T}_r \hat{X}_{r+1} \hat{e}(i) &= \hat{e}(i) \hat{X}_{r+1} \hat{T}_r \hat{e}(i), \\
\end{align*} \]

if \( i \in I^\beta, i_r \neq i_{r+1}, \quad (3.15) \]

\[ \hat{e}(i) f \hat{e}(j) = 0, \quad \text{if } i, j \in I^\beta, i \neq j, f \in K[\hat{X}_1^{\pm 1}, \cdots, \hat{X}_n^{\pm 1}], \quad (3.16) \]
Modified affine Hecke algebras: non-degenerate case

Lemma

\[ \hat{e}(i) \hat{T}_r \hat{e}(j) = 0, \quad \text{if } i, j \in I^\beta, i \notin \{j, s_rj\}, \quad (3.18) \]

\[
\begin{align*}
\hat{e}(i)(\hat{T}_r - q)(\hat{T}_r + 1)\hat{e}(j) &= 0, \\
\hat{e}(i)\hat{T}_i \hat{T}_{i+1} \hat{T}_i \hat{e}(j) &= \hat{e}(i)\hat{T}_{i+1} \hat{T}_i \hat{T}_{i+1} \hat{e}(j), \\
\hat{e}(i)\hat{X}_k^{\pm 1} \hat{X}_k^{\pm 1} \hat{e}(j) &= \hat{e}(i)\hat{X}_k^{\pm 1} \hat{X}_k^{\pm 1} \hat{e}(j), \\
\hat{e}(i)\hat{X}_r \hat{X}_k^{-1} \hat{e}(i) &= \hat{e}(i) = \hat{e}(i)\hat{X}_r^{-1} \hat{X}_k \hat{e}(i), \\
\hat{e}(i)\hat{X}_{r+1} \hat{T}_r \hat{e}(j) &= \hat{e}(i)(\hat{T}_r \hat{X}_r + (q - 1)\hat{X}_{r+1})\hat{e}(j), \\
\end{align*}
\]

\[ \quad \text{if } i, j \in I^\beta, \quad (3.19) \]
Lemma

\[ \hat{e}(i) \hat{T}_a \hat{T}_k \hat{e}(j) = \hat{e}(i) \hat{T}_k \hat{T}_a \hat{e}(j), \text{ if } |a - k| > 1 \text{ and } i, j \in I^\beta, \] \hspace{1cm} (3.21)

\[ \hat{e}(i) \hat{T}_b \hat{X}_k \hat{e}(j) = \hat{e}(i) \hat{X}_k \hat{T}_b \hat{e}(j), \text{ if } k \neq b, b + 1 \text{ and } i, j \in I^\beta, \] \hspace{1cm} (3.22)

where \( 1 \leq k \leq n, 1 \leq r, a, b < n, 1 \leq i < n - 1. \)
Modified affine Hecke algebras: degenerate case

**Definition (Hu-Li)**

In the degenerate case, let $\beta \in \mathbb{Q}_n^+$. We define the modified degenerate affine Hecke algebra $\hat{H}_\beta$ of type $A$ to be the $K$-subalgebra of $\varprojlim \Lambda H_n^\Lambda$ generated by the following elements:

$$\hat{s}_k \hat{e}(i), \hat{x}_r^{\pm 1} \hat{e}(i), \hat{e}(i), (\hat{x}_a - \hat{x}_b)^{-1} \hat{e}(i), i \in I^\beta,$$

(3.24)

where $1 \leq k < n$, $1 \leq r \leq n$, $1 \leq a < b \leq n$ with $i_a \neq i_b$. 
Lemma

In the degenerate case, the following relations hold:

\[\hat{x}_k \hat{e}(i) = \hat{e}(i) \hat{x}_k, \quad \hat{e}(i) \hat{e}(j) = \delta_{ij} \hat{e}(i), \quad (3.26)\]

\[\begin{align*}
\hat{e}(i) \hat{s}_r \hat{e}(i) &= (\hat{x}_{r+1} - \hat{x}_r)^{-1} \hat{e}(i), \\
\hat{e}(i) \hat{s}_r \hat{x}_r \hat{e}(i) &= \hat{e}(i) \hat{x}_r \hat{s}_r \hat{e}(i), \\
\hat{e}(i) \hat{s}_r \hat{x}_{r+1} \hat{e}(i) &= \hat{e}(i) \hat{x}_{r+1} \hat{s}_r \hat{e}(i),
\end{align*}\]

if \(i \in I^\beta\), \(i_r \neq i_{r+1}\), \( (3.27)\)

\[\hat{e}(i) f \hat{e}(j) = 0, \quad \text{if} \ i, j \in I^\beta, i \neq j, f \in K[\hat{x}_1, \ldots, \hat{x}_n], \quad (3.28)\]

\[\hat{e}(i) \hat{s}_r \hat{e}(j) = 0, \quad \text{if} \ i, j \in I^\beta, i \notin \{j, s_rj\}, \quad (3.29)\]
Lemma

\[
\begin{align*}
\hat{e}(i)\hat{s}_r^2\hat{e}(j) &= \delta_{ij}\hat{e}(j), \\
\hat{e}(i)\hat{s}_i\hat{s}_{i+1}\hat{s}_i\hat{e}(j) &= \hat{e}(i)\hat{s}_{i+1}\hat{s}_i\hat{s}_{i+1}\hat{e}(j), \\
\hat{e}(i)\hat{x}_r\hat{x}_k\hat{e}(j) &= \hat{e}(i)\hat{x}_k\hat{x}_r\hat{e}(j), \\
\hat{e}(i)\hat{x}_{r+1}\hat{s}_r\hat{e}(j) &= \hat{e}(i)\hat{s}_r\hat{x}_r\hat{e}(j) + \delta_{ij}\hat{e}(j),
\end{align*}
\]

if \( i, j \in I^\beta \), \hspace{1cm} (3.31)

\[
\hat{e}(i)\hat{s}_a\hat{s}_k\hat{e}(j) = \hat{e}(i)\hat{s}_k\hat{s}_a\hat{e}(j), \text{ if } |a - k| > 1 \text{ and } i, j \in I^\beta, \hspace{1cm} (3.32)
\]

\[
\hat{e}(i)\hat{s}_b\hat{x}_k\hat{e}(j) = \hat{e}(i)\hat{x}_k\hat{s}_b\hat{e}(j), \text{ if } k \neq b, b + 1 \text{ and } i, j \in I^\beta, \hspace{1cm} (3.33)
\]

where \( 1 \leq k \leq n, 1 \leq r, a, b < n, 1 \leq i < n - 1 \).
For each $i \in I^\beta$, let $\{t_k(i)\mid 1 \leq k \leq n\}$ be a set of $n$ algebraically independent indeterminates over $K$. We define

$$\text{Pol}_{\beta} = \bigoplus_{i \in I^\beta} \text{Pol}_n(i),$$

(3.34)

where

$$\text{Pol}_n(i) := \begin{cases} K[t_1(i)^{\pm 1}, \ldots, t_n(i)^{\pm 1}], & \text{if } \hat{H}_\beta = \hat{H}_\beta(q); \\ K[t_1(i), \ldots, t_n(i)], & \text{if } \hat{H}_\beta = \hat{H}_\beta. \end{cases}$$

(3.35)
Let $\widetilde{\text{Pol}}_n(i)$ be the localisation of $\text{Pol}_n(i)$ with respect to the following multiplicatively closed closed subset

$$\{(t_r(i) - t_s(i))^k \mid 1 \leq r \neq s \leq n, k \in \mathbb{Z}^{\geq 0}\}.$$  \hspace{1cm} (3.36)

We set

$$\widetilde{\text{Pol}}_{\beta} := \bigoplus_{i \in I^\beta} \widetilde{\text{Pol}}_n(i).$$ \hspace{1cm} (3.37)
Let \( \{t_k | 1 \leq k \leq n\} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \widetilde{P}_n \) be the localisation of \( K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) if \( \hat{H}_\beta = \hat{H}_\beta(q) \), or the localisation of \( K[t_1, \ldots, t_n] \) if \( \hat{H}_\beta = \hat{H}_\beta \), with respect to the following multiplicatively closed subset
\[
\{(t_r - t_s)^k | 1 \leq r \neq s \leq n, k \in \mathbb{Z}_{\geq 0}\}.
\]

Let \( \theta_i : \widetilde{P}_n \cong \widetilde{\text{Pol}}_n(i) \) be the canonical isomorphism induced by the map \( t_k^{\pm 1} \mapsto t_k(i)^{\pm 1} \) for each \( 1 \leq k \leq n \). For each \( f \in \widetilde{P}_n \), we set
\[
f_i := \theta_i(f) \in \widetilde{\text{Pol}}_n(i). \quad (3.38)
\]
Let \( \{t_k| 1 \leq k \leq n\} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \tilde{\mathcal{P}}_n \) be the localisation of \( K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) if \( \hat{\mathcal{H}}_\beta = \hat{\mathcal{H}}_\beta(q) \), or the localisation of \( K[t_1, \ldots, t_n] \) if \( \hat{\mathcal{H}}_\beta = \hat{\mathcal{H}}_\beta \), with respect to the following multiplicatively closed subset

\[
\{ (t_r - t_s)^k \mid 1 \leq r \neq s \leq n, k \in \mathbb{Z}^{\geq 0} \}.
\]

Let \( \theta_i : \tilde{\mathcal{P}}_n \cong \tilde{\text{Pol}}_n(i) \) be the canonical isomorphism induced by the map \( t_k^{\pm 1} \mapsto t_k(i)^{\pm 1} \) for each \( 1 \leq k \leq n \). For each \( f \in \tilde{\mathcal{P}}_n \), we set

\[
f_i := \theta_i(f) \in \tilde{\text{Pol}}_n(i). \tag{3.38}
\]
A Faithful Representation

Let \( \{ t_k | 1 \leq k \leq n \} \) be a set of \( n \) algebraically independent indeterminates over \( K \). Let \( \tilde{P}_n \) be the localisation of \( K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) if \( \hat{H}_\beta = \hat{H}_\beta(q) \), or the localisation of \( K[t_1, \ldots, t_n] \) if \( \hat{H}_\beta = \hat{H}_\beta \), with respect to the following multiplicatively closed subset

\[
\{(t_r - t_s)^k \mid 1 \leq r \neq s \leq n, k \in \mathbb{Z}_{\geq 0}\}.
\]

Let \( \theta_i : \tilde{P}_n \cong \tilde{\text{Pol}}_n(i) \) be the canonical isomorphism induced by the map \( t_k^{\pm 1} \mapsto t_k(i)^{\pm 1} \) for each \( 1 \leq k \leq n \). For each \( f \in \tilde{P}_n \), we set

\[
f_i := \theta_i(f) \in \tilde{\text{Pol}}_n(i).
\]
A Faithful Representation

For any $i \in I^\beta, f \in \widetilde{P}_n$, $1 \leq r < n$ and $1 \leq k \leq n$, we define

\[
\begin{align*}
\hat{X}_k^{\pm 1} \cdot f_i & := t_k(i)^{\pm 1} f_i, \\
\hat{e}(j) \cdot f_i & := \delta_{ij} f_i, \quad \text{if } i, j \in I^\beta, \\
\hat{T}_r \hat{e}(i) \cdot f_i & := \left( \frac{t_{r+1} - qt_r}{t_{r+1} - t_r} s_r(f) \right)_{s_r i} + (q - 1) \frac{t_{r+1}(i)}{t_{r+1}(i) - t_r(i)} f_i,
\end{align*}
\]

and

\[
\begin{align*}
\hat{X}_k \cdot f_i & := t_k(i) f_i, \\
\hat{e}(j) \cdot f_i & := \delta_{ij} f_i, \quad \text{if } i, j \in I^\beta, \\
\hat{s}_r \hat{e}(i) \cdot f_i & := \left( \frac{t_{r+1} - t_r - 1}{t_{r+1} - t_r} s_r(f) \right)_{s_r i} + \frac{1}{t_{r+1}(i) - t_r(i)} f_i.
\end{align*}
\]
Lemma

Let $\hat{\mathcal{H}}_\beta \in \{\hat{\mathcal{H}}_\beta(q), \hat{\mathcal{H}}_\beta\}$. The above rules extend uniquely to a faithful representation $\rho$ of $\hat{\mathcal{H}}_\beta$ on $\tilde{\text{Pol}}_\beta$. 
Let $i \in I^\beta$. For each $w \in S_n$, we fix a reduced expression $s_{j_1} s_{j_2} \cdots s_{j_k}$ of $w$, and we define

$$\hat{w}_i := (\hat{e}(wi)\hat{s}_{j_1} \hat{e}(s_{j_1}wi))(\hat{e}(s_{j_1}wi)\hat{s}_{j_2} \hat{e}(s_{j_2}s_{j_1}wi)) \cdots (\hat{e}(s_{j_k}i)\hat{s}_{j_k} \hat{e}(i)),$$

$$\hat{T}_{w,i} := (\hat{e}(wi)\hat{T}_{j_1} \hat{e}(s_{j_1}wi))(\hat{e}(s_{j_1}wi)\hat{T}_{j_2} \hat{e}(s_{j_2}s_{j_1}wi)) \cdots (\hat{e}(s_{j_k}i)\hat{T}_{j_k} \hat{e}(i)).$$
Lemma

The elements in the following set

\[
\left\{ \hat{T}_{w,i} \hat{X}_1^{a_1} \cdots \hat{X}_n^{a_n} \prod_{\substack{1 \leq r < s \leq n \\mid i_r \neq i_s}} (\hat{X}_r - \hat{X}_s)^{-b_{r,s}} \hat{e}(i) \right\}
\]

form a $K$-basis of $\hat{H}_\beta(q)$.

Where $w \in \mathfrak{S}_n$, $i \in I^\beta$, $b_{r,s} \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{Z}$, $b_{r,s} > 0$ only if either $a_r = 0 \geq a_s$ or $a_r > 0 = a_s$. 

Hu Jun (Beijing Institute of Technology)

On the center of cyclotomic quiver Hecke algebras of type $A$
Lemma

The elements in the following set

\[
\left\{ \hat{w}_i \hat{x}_1^{a_1} \cdots \hat{x}_n^{a_n} \prod_{\substack{1 \leq r < s \leq n \atop i_r \neq i_s}} (\hat{x}_r - \hat{x}_s)^{-b_{r,s}} \hat{e}(i) \mid w \in \mathfrak{S}_n, i \in I^\beta, b_{r,s}, a_1, \cdots, a_n \in \mathbb{N}, b_{r,s} > 0 \text{ only if } a_s = 0 \right\}
\]

form a $K$-basis of $\hat{H}_\beta$. 
For any $1 \leq k \leq n$, $i \in I^\beta$ and $w \in \mathcal{S}_n$, we define

$$w(\hat{X}_k^{\pm 1} \hat{e}(i)) := \hat{X}_{w(k)}^{\pm 1} \hat{e}(wi), \ w(\hat{x}_k \hat{e}(i)) := \hat{x}_{w(k)} \hat{e}(wij).$$

This is well-defined and extends uniquely to an action of $\mathcal{S}_n$ on the set of polynomials in $\{\hat{X}_k^{\pm 1} \hat{e}(i)|1 \leq k \leq n, i \in I^\beta\}$ and on the set of polynomials in $\{\hat{x}_k \hat{e}(i)|1 \leq k \leq n, i \in I^\beta\}$ respectively.
For any $1 \leq k \leq n$, $i \in I^\beta$ and $w \in \mathcal{S}_n$, we define
\[
w(\hat{X}_k^{\pm 1} \hat{e}(i)) := \hat{X}_{w(k)}^{\pm 1} \hat{e}(wi), \quad w(\hat{x}_k \hat{e}(i)) := \hat{x}_{w(k)} \hat{e}(wi).
\]

This is well-defined and extends uniquely to an action of $\mathcal{S}_n$ on the set of polynomials in $\{\hat{X}_k^{\pm 1} \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta\}$ and on the set of polynomials in $\{\hat{x}_k \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta\}$ respectively.
For any $i, j \in l^n$, we write $i \sim j$ whenever $j = \sigma i$ for some $\sigma \in S_n$. Let $l^\beta / \sim$ be a set of representatives with respect to the equivalence relation “$\sim$”. Let $i \in l^\beta / \sim$, $1 \leq a < b \leq n$ with $i_a \neq i_b$.

We define

$$S_n(i, a, b) := \{\sigma \in S_n | \sigma i = i, \sigma(a) = a, \sigma(b) = b\}.$$
For any $i, j \in I^n$, we write $i \sim j$ whenever $j = \sigma i$ for some $\sigma \in S_n$. Let $I^\beta / \sim$ be a set of representatives with respect to the equivalence relation “$\sim$”. Let $i \in I^\beta / \sim$, $1 \leq a < b \leq n$ with $i_a \neq i_b$.

We define

$$S_n(i, a, b) := \{ \sigma \in S_n | \sigma i = i, \sigma(a) = a, \sigma(b) = b \}.$$
Let $\mathcal{D}_n(i, a, b)$ be a set of left coset representatives of $\mathcal{S}_n(i, a, b)$ in $\mathcal{S}_n$. For any $d \in \mathcal{D}_n(i, a, b)$, if $di = i$ then $(d(a), d(b)) \neq (b, a)$ because $i_a \neq i_b$.

**Lemma**

Let $\beta \in \mathbb{Q}^+_n$. The center $Z(\hat{H}_\beta(q))$ of $\hat{H}_\beta(q)$ is equal to

\[
\left\{ \left( \sum_{i \in I_\beta} \sum_{1 \leq a < b \leq n} \prod_{d \mid k \in \mathcal{D}_n(i, a, b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_i} \hat{e}(k) \right) f \mid a_i \in \mathbb{N}, \forall i \in I_\beta / \sim, f \text{ is a symmetric polynomial in } \{ \hat{X}_k^{\pm 1} \hat{e}(i), \hat{e}(i) \mid 1 \leq k \leq n, i \in I_\beta \} \right\}.
\]
Let $D_n(i, a, b)$ be a set of left coset representatives of $S_n(i, a, b)$ in $S_n$. For any $d \in D_n(i, a, b)$, if $di = i$ then $(d(a), d(b)) \neq (b, a)$ because $i_a \neq i_b$.

**Lemma**

Let $\beta \in Q_n^+$. The center $Z(\hat{H}_\beta(q))$ of $\hat{H}_\beta(q)$ is equal to

\[
\left\{ \left( \sum_{i \in I^\beta} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i, a, b)} (\hat{X}_{d(a)} - \hat{X}_{d(b)})^{-a_i} \hat{e}(k) \right) f \mid a_i \in \mathbb{N}, \forall i \in I^\beta/\sim, f \text{ is a symmetric polynomial in } \{ \hat{X}_k^{\pm 1} \hat{e}(i), \hat{e}(i) \mid 1 \leq k \leq n, i \in I^\beta \} \right\}.
\]
Lemma

Let $\beta \in Q^+_n$. The center $Z(\hat{H}_\beta)$ of $\hat{H}_\beta$ is equal to

$$\left\{ \left( \sum_{i \in I^\beta / \sim} \sum_{1 \leq a < b \leq n} \prod_{d \in D_n(i,a,b)} (\hat{x}_d(a) - \hat{x}_d(b))^{-a_i} \hat{e}(k) \right) f \bigg| a_i \in \mathbb{N}, \forall i \in I^\beta / \sim, \ f \ is \ a \ symmetric \ polynomial \ in \ \{ \hat{x}_k \hat{e}(i), \hat{e}(i) | 1 \leq k \leq n, i \in I^\beta \} \right\}.$$
Let $A$ be a ring with identity $1$ and $A_0$ a commutative subring of $A$. Let $e_1, \ldots, e_m$ be a complete set of pairwise orthogonal idempotents of $A$. Then $\sum_{i=1}^{m} e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ for any $i, j$. We assume further that $fe_i = e_i f$ for any $f \in A_0$ and $1 \leq i \leq m$.

For each $1 \leq i \leq m$, let $S(i)$ be a multiplicatively closed subset in $A_0$ such that $1 \in S(i)$ and for any $g, h \in A$, $s \in S(i)$,

$$se_i g = 0 \implies e_i g = 0, \quad he_i s = 0 \implies he_i = 0.$$ 

In particular, $0 \notin S(i)$. We set $S_i := S(i)e_i$ for each $i$. 
Let $A$ be a ring with identity 1 and $A_0$ a commutative subring of $A$. Let $e_1, \cdots, e_m$ be a complete set of pairwise orthogonal idempotents of $A$. Then $\sum_{i=1}^m e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ for any $i, j$. We assume further that $f e_i = e_i f$ for any $f \in A_0$ and $1 \leq i \leq m$.

For each $1 \leq i \leq m$, let $S(i)$ be a multiplicatively closed subset in $A_0$ such that $1 \in S(i)$ and for any $g, h \in A$, $s \in S(i)$,

$$se_i g = 0 \implies e_i g = 0, \quad he_i s = 0 \implies he_i = 0.$$  

In particular, $0 \not\in S(i)$. We set $S_i := S(i) e_i$ for each $i$.
Let $A$ be a ring with identity 1 and $A_0$ a commutative subring of $A$. Let $e_1, \cdots, e_m$ be a complete set of pairwise orthogonal idempotents of $A$. Then $\sum_{i=1}^m e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ for any $i, j$. We assume further that $fe_i = e_i f$ for any $f \in A_0$ and $1 \leq i \leq m$.

For each $1 \leq i \leq m$, let $S(i)$ be a multiplicatively closed subset in $A_0$ such that $1 \in S(i)$ and for any $g, h \in A$, $s \in S(i)$,

$$se_i g = 0 \implies e_i g = 0, \quad he_i s = 0 \implies he_i = 0.$$

In particular, $0 \notin S(i)$. We set $S_i := S(i)e_i$ for each $i$. 
Let $A$ be a ring with identity 1 and $A_0$ a commutative subring of $A$. Let $e_1, \cdots, e_m$ be a complete set of pairwise orthogonal idempotents of $A$. Then $\sum_{i=1}^{m} e_i = 1$ and $e_i e_j = \delta_{ij} e_i$ for any $i, j$. We assume further that $fe_i = e_i f$ for any $f \in A_0$ and $1 \leq i \leq m$.

For each $1 \leq i \leq m$, let $S(i)$ be a multiplicatively closed subset in $A_0$ such that $1 \in S(i)$ and for any $g, h \in A$, $s \in S(i)$,

$$se_i g = 0 \implies e_i g = 0, \quad he_i s = 0 \implies he_i = 0.$$ 

In particular, $0 \notin S(i)$. We set $S_i := S(i)e_i$ for each $i$. 
With the assumptions as above, and assume further that the subsets \( \{ S_i \}_{i=1}^m \) satisfy that for any \( 1 \leq i, j \leq m \) and any \( a \in e_j A e_i, s \in S_i \) and \( t \in S_j \), there exist some \( b, c \in e_j A e_i, u \in S_j \) and \( v \in S_i \), such that \( ua = bs, av = tc \). Then there exists a ring \( A[S_1, \cdots, S_m] \) together with a ring homomorphism \( \varphi : A \rightarrow A[S_1, \cdots, S_m] \) satisfying that:

(G1) \( \varphi \) is injective; and

(G2) \( \forall 1 \leq i \leq m, s \in S_i, \varphi(s) \in A_0 e_i \) is invertible in \( e_i A[S_1, \cdots, S_m] e_i \) (with identity element \( e_i \)); and
Lemma

With the assumptions as above, and assume further that the subsets \{S_i\}_{i=1}^m satisfy that for any \(1 \leq i, j \leq m\) and any \(a \in e_j A e_i\), \(s \in S_i\) and \(t \in S_j\), there exist some \(b, c \in e_j A e_i\), \(u \in S_j\) and \(v \in S_i\), such that \(ua = bs\), \(av = tc\). Then there exists a ring \(A[S_1, \cdots, S_m]\) together with a ring homomorphism \(\varphi : A \rightarrow A[S_1, \cdots, S_m]\) satisfying that:

(G1) \(\varphi\) is injective; and

(G2) \(\forall 1 \leq i \leq m, s \in S_i, \varphi(s) \in A_0 e_i\) is invertible in \(e_i A[S_1, \cdots, S_m] e_i\) (with identity element \(e_i\)); and
Lemma

With the assumptions as above, and assume further that the subsets $\{S_i\}_{i=1}^m$ satisfy that for any $1 \leq i, j \leq m$ and any $a \in e_jAe_i$, $s \in S_i$ and $t \in S_j$, there exist some $b, c \in e_jAe_i$, $u \in S_j$ and $v \in S_i$, such that $ua = bs$, $av = tc$. Then there exists a ring $A[S_1, \ldots, S_m]$ together with a ring homomorphism $\varphi : A \to A[S_1, \ldots, S_m]$ satisfying that:

(G1) $\varphi$ is injective; and

(G2) $\forall 1 \leq i \leq m, s \in S_i, \varphi(s) \in A_0e_i$ is invertible in $e_iA[S_1, \ldots, S_m]e_i$ (with identity element $e_i$); and
Generalised Ore localisation (general setting)

**Lemma**

*With the assumptions as above, and assume further that the subsets \( \{ S_i \}_{i=1}^m \) satisfy that for any \( 1 \leq i, j \leq m \) and any \( a \in e_j A e_i, s \in S_i \) and \( t \in S_j \), there exist some \( b, c \in e_j A e_i, u \in S_j \) and \( v \in S_i \), such that \( ua = bs, av = tc \). Then there exists a ring \( A[S_1, \cdots, S_m] \) together with a ring homomorphism \( \varphi : A \to A[S_1, \cdots, S_m] \) satisfying that:

\[(G1) \quad \varphi \text{ is injective;} \quad \text{and} \]

\[(G2) \quad \forall 1 \leq i \leq m, s \in S_i, \varphi(s) \in A_0 e_i \text{ is invertible in } e_i A[S_1, \cdots, S_m] e_i \text{ (with identity element } e_i) ; \quad \text{and} \]

Hu Jun (Beijing Institute of Technology)
Lemma

With the assumptions as above, and assume further that the subsets \( \{ S_i \}_{i=1}^m \) satisfy that for any \( 1 \leq i, j \leq m \) and any \( a \in e_jAe_i, s \in S_i \) and \( t \in S_j \), there exist some \( b, c \in e_jAe_i, u \in S_j \) and \( v \in S_i \), such that \( ua = bs \), \( av = tc \). Then there exists a ring \( A[S_1, \cdots, S_m] \) together with a ring homomorphism \( \varphi : A \rightarrow A[S_1, \cdots, S_m] \) satisfying that:

1. (G1) \( \varphi \) is injective; and
2. (G2) \( \forall 1 \leq i \leq m, s \in S_i, \varphi(s) \in A_0e_i \) is invertible in \( e_iA[S_1, \cdots, S_m]e_i \) (with identity element \( e_i \)); and
Lemma (Continued)

(68) each element in $A[S_1, \ldots, S_m]$ has the form

$$\sum_{1 \leq i, j \leq m} a_{ijk} \varphi(f_{ijk})^{-1},$$

where $a_{ijk} \in A, f_{ijk} \in S_j$. and such that it has the universal property that for any ring homomorphism $\psi : A \to B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \ldots, S_m] \to B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Lemma (Continued)

\[(G3)\) each element in \(A[S_1, \cdots, S_m]\) has the form

\[
\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.
\]

where \(a_{i,j,k} \in A, f_{i,j,k} \in S_j\).

and such that it has the universal property that for any ring homomorphism \(\psi : A \to B\) such that \(\psi(se_i)\) is invertible in \(\psi(e_i)B\psi(e_i)\) for every \(s \in S(i)\) and \(1 \leq i \leq m\), then there is a unique ring homomorphism \(\sigma : A[S_1, \cdots, S_m] \to B\) such that \(\sigma \varphi = \psi\). Moreover, if \(\psi\) is injective then \(\sigma\) is injective too.
(G3) each element in $A[S_1, \cdots, S_m]$ has the form

$$\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.$$ 

where $a_{i,j,k} \in A$, $f_{i,j,k} \in S_j$.

and such that it has the universal property that for any ring homomorphism $\psi : A \to B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \cdots, S_m] \to B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Lemma (Continued)

(G3) each element in $A[S_1, \ldots, S_m]$ has the form

$$
\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.
$$

where $a_{i,j,k} \in A$, $f_{i,j,k} \in S_j$.

and such that it has the universal property that for any ring homomorphism $\psi : A \rightarrow B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \ldots, S_m] \rightarrow B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Lemma (Continued)

(G3) each element in $A[S_1, \cdots, S_m]$ has the form

$$\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.$$ 

where $a_{i,j,k} \in A$, $f_{i,j,k} \in S_j$.

and such that it has the universal property that for any ring homomorphism $\psi : A \rightarrow B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \cdots, S_m] \rightarrow B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Lemma (Continued)

(G3) each element in \( A[S_1, \cdots, S_m] \) has the form

\[
\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.
\]

where \( a_{i,j,k} \in A, f_{i,j,k} \in S_j \).

and such that it has the universal property that for any ring homomorphism \( \psi : A \rightarrow B \) such that \( \psi(se_i) \) is invertible in \( \psi(e_i)B\psi(e_i) \) for every \( s \in S(i) \) and \( 1 \leq i \leq m \), then there is a unique ring homomorphism \( \sigma : A[S_1, \cdots, S_m] \rightarrow B \) such that \( \sigma \varphi = \psi \). Moreover, if \( \psi \) is injective then \( \sigma \) is injective too.
Lemma (Continued)

(G3) each element in $A[S_1, \cdots, S_m]$ has the form

$$\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.$$ 

where $a_{i,j,k} \in A$, $f_{i,j,k} \in S_j$.

and such that it has the universal property that for any ring homomorphism $\psi : A \to B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \cdots, S_m] \to B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Lemma (Continued)

(G3) each element in $A[S_1, \cdots, S_m]$ has the form

$$\sum_{1 \leq i, j \leq m} e_i a_{i,j,k} \varphi(f_{i,j,k})^{-1}.$$  

where $a_{i,j,k} \in A$, $f_{i,j,k} \in S_j$.

and such that it has the universal property that for any ring homomorphism $\psi : A \to B$ such that $\psi(se_i)$ is invertible in $\psi(e_i)B\psi(e_i)$ for every $s \in S(i)$ and $1 \leq i \leq m$, then there is a unique ring homomorphism $\sigma : A[S_1, \cdots, S_m] \to B$ such that $\sigma \varphi = \psi$. Moreover, if $\psi$ is injective then $\sigma$ is injective too.
Let $\tilde{\mathcal{H}}_\beta(q)$ be the generalised Ore localization of $\widehat{\mathcal{H}}_\beta(q)$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ (\hat{X}_r - q^b \hat{X}_s)^{-1} \hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in I, j \in l^\beta, j_r \neq b + j_s \right\}.$$ 

Let $\tilde{H}_\beta$ be the generalised Ore localization of $\widehat{H}_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ (\hat{x}_r - \hat{x}_s - b)^{-1} \hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in I, j \in l^\beta, j_r \neq b + j_s \right\}.$$
Generalised Ore localisation

Let $\widetilde{\mathcal{H}}(q)$ be the generalised Ore localization of $\hat{\mathcal{H}}(q)$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{(\hat{X}_r - q^b \hat{X}_s)^{-1} \hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in I, j \in I^\beta, j_r \neq b + j_s \right\}.$$

Let $\widetilde{H}_\beta$ be the generalised Ore localization of $\hat{H}_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ (\hat{X}_r - \hat{X}_s - b)^{-1} \hat{e}(j) \mid 1 \leq r \neq s \leq n, b \in I, j \in I^\beta, j_r \neq b + j_s \right\}.$$
Generalised Ore localisation

In the non-degenerate setting, we define $\tilde{R}_\beta$ to be the generalised Ore localization of $R_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ ((1-y_r)-q^b(1-y_s))^{-1}e(j), (1-y_s)^{-1}e(j) \mid 1 \leq r \neq s \leq n, j \in I^\beta \right\}.$$ 

In the degenerate setting, we define $\tilde{R}'_\beta$ to be the generalised Ore localization of $R_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ (b+y_r-y_s)^{-1}e(j) \mid 1 \leq r \neq s \leq n, j \in I^\beta, 0 \neq b \in I \right\}.$$
Generalised Ore localisation

In the non-degenerate setting, we define $\tilde{R}_\beta$ to be the generalised Ore localization of $R_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ ((1-y_r)-q^b(1-y_s))^{-1} e(j), (1-y_s)^{-1} e(j) \mid 1 \leq r \neq s \leq n, j \in I^\beta \right\};$$

In the degenerate setting, we define $\tilde{R}'_\beta$ to be the generalised Ore localization of $R_\beta$ with respect to the multiplicatively closed subsets generated by the elements in the following set:

$$\left\{ (b + y_r - y_s)^{-1} e(j) \mid 1 \leq r \neq s \leq n, j \in I^\beta, 0 \neq b \in I \right\}.$$
Henceforth, if $e > 0$ then let $\Gamma$ be the cyclic quiver with $e$-vertices; if $e = 0$ then let $\Gamma$ be the linear quiver with vertex set $\mathbb{Z}$. In both cases, $I = \mathbb{Z}/e\mathbb{Z}$.

Let $i \in I^n$ and $r$ be an integer with $1 \leq r < n$. If $i_r = i_{r+1}$ then set $P_r(i) = 1$; if $i_r \neq i_{r+1}$ and in the non-degenerate setting, then set

$$P_r(i) := \frac{1 - q}{1 - q^{i_r - i_{r+1}}} \left\{ 1 + \sum_{k \geq 0} \frac{y_r - y_{r+1}}{1 - q^{i_r+1 - i_r}} \left( \frac{q^{i_r+1}y_{r+1} - q^{i_r}y_r}{q^{i+1} - q^{i_r}} \right)^k \right\};$$

while if $i_r \neq i_{r+1}$ and in the degenerate setting, then set

$$P_r(i) := \frac{1}{i_r - i_{r+1}} \left\{ 1 + \sum_{k \geq 1} \left( \frac{y_r - y_{r+1}}{i_{r+1} - i_r} \right)^k \right\}.$$
Henceforth, if $e > 0$ then let $\Gamma$ be the cyclic quiver with $e$-vertices; if $e = 0$ then let $\Gamma$ be the linear quiver with vertex set $\mathbb{Z}$. In both cases, $l = \mathbb{Z}/e\mathbb{Z}$.

Let $i \in l^n$ and $r$ be an integer with $1 \leq r < n$. If $i_r = i_{r+1}$ then set $P_r(i) = 1$; if $i_r \neq i_{r+1}$ and in the non-degenerate setting, then set

$$P_r(i) := \frac{1 - q}{1 - q^{i_r - i_{r+1}}} \left\{ 1 + \sum_{k \geq 0} \frac{y_r - y_{r+1}}{1 - q^{i_{r+1} - i_{r}}} \left( \frac{q^{i_{r+1} - i_r} y_{r+1} - q^{i_r} y_r}{q^{i_{r+1} - i_{r}} - q^{i_r}} \right)^k \right\};$$

while if $i_r \neq i_{r+1}$ and in the degenerate setting, then set

$$P_r(i) := \frac{1}{i_r - i_{r+1}} \left\{ 1 + \sum_{k \geq 1} \left( \frac{y_r - y_{r+1}}{i_{r+1} - i_r} \right)^k \right\}.$$
Brundan–Kleshchev's isomorphism

Henceforth, if $e > 0$ then let $\Gamma$ be the cyclic quiver with $e$-vertices; if $e = 0$ then let $\Gamma$ be the linear quiver with vertex set $\mathbb{Z}$. In both cases, $I = \mathbb{Z}/e\mathbb{Z}$.

Let $i \in I^n$ and $r$ be an integer with $1 \leq r < n$. If $i_r = i_{r+1}$ then set $P_r(i) = 1$; if $i_r \neq i_{r+1}$ and in the non-degenerate setting, then set

$$P_r(i) := \frac{1 - q}{1 - q^{i_r-i_{r+1}}} \left\{ 1 + \sum_{k \geq 0} \frac{y_r - y_{r+1}}{1 - q^{i_{r+1}-i_r}} \left( \frac{q^{i_{r+1} y_{r+1} - q^{i_r y_r}}}{q^{i_{r+1}} - q^{i_r}} \right)^k \right\};$$

while if $i_r \neq i_{r+1}$ and in the degenerate setting, then set

$$P_r(i) := \frac{1}{i_r - i_{r+1}} \left\{ 1 + \sum_{k \geq 1} \left( \frac{y_r - y_{r+1}}{i_{r+1} - i_r} \right)^k \right\}.$$
Brundan–Kleshche's isomorphism

Henceforth, if \( e > 0 \) then let \( \Gamma \) be the cyclic quiver with \( e \)-vertices; if \( e = 0 \) then let \( \Gamma \) be the linear quiver with vertex set \( \mathbb{Z} \). In both cases, \( I = \mathbb{Z} / e\mathbb{Z} \).

Let \( i \in I^n \) and \( r \) be an integer with \( 1 \leq r < n \). If \( i_r = i_{r+1} \) then set \( P_r(i) = 1 \); if \( i_r \neq i_{r+1} \) and in the non-degenerate setting, then set

\[
P_r(i) := \frac{1 - q}{1 - q^{i_r - i_{r+1}}} \left\{ 1 + \sum_{k \geq 0} \frac{y_r - y_{r+1}}{1 - q^{i_{r+1} - i_r}} \left( \frac{q^{i_{r+1} y_{r+1} - q^{i_r} y_r}}{q^{i_{r+1} - i_r}} \right)^k \right\};
\]

while if \( i_r \neq i_{r+1} \) and in the degenerate setting, then set

\[
P_r(i) := \frac{1}{i_r - i_{r+1}} \left\{ 1 + \sum_{k \geq 1} \left( \frac{y_r - y_{r+1}}{i_{r+1} - i_r} \right)^k \right\}.
\]
Theorem (Brundan–Kleshche, Invent. Math. 2009)

Let $\mathcal{H}_{\beta}^{\Lambda} \in \{ \mathcal{H}_{\beta}^{\Lambda}(q), H_{\beta}^{\Lambda} \}$. Then there is an isomorphism of $K$-algebras $\theta_{\Lambda} : \mathcal{R}_{\beta}^{\Lambda} \cong \mathcal{H}_{\beta}^{\Lambda}$ that sends $e(i) \mapsto e(i)$, for all $i \in I_{\beta}$ and

$y_r e(i) \mapsto \begin{cases} 
(1 - q^{-i_r} L_r) e(i), & \text{if } \mathcal{H}_{\beta}^{\Lambda} = \mathcal{H}_{\beta}^{\Lambda}(q), \\
(L_r - i_r) e(i), & \text{if } \mathcal{H}_{\beta}^{\Lambda} = H_{\beta}^{\Lambda}.
\end{cases}$

$\psi_k e(i) \mapsto \begin{cases} 
(T_k + P_k(i)) Q_k(i)^{-1} e(i), & \text{if } \mathcal{H}_{\beta}^{\Lambda} = \mathcal{H}_{\beta}^{\Lambda}(q), \\
(s_k + P_k(i)) Q_k(i)^{-1} e(i), & \text{if } \mathcal{H}_{\beta}^{\Lambda} = H_{\beta}^{\Lambda},
\end{cases}$

where $1 \leq r \leq n$, $1 \leq k < n$, $P_k(i), Q_k(i) \in K[y_k, y_{k+1}]$ are certain polynomials in $y_k, y_{k+1}$. 
Brundan–Kleshchev’s isomorphism

The Brundan–Kleshcheb’s isomorphism between $\mathcal{H}_\beta^\Lambda$ and $\mathcal{R}_\beta^\Lambda$ depends on the choice of certain polynomials $Q_r(i)$ for $1 \leq r < n$. In the degenerate setting, we set

$$Q_r(i) := \begin{cases} 1 + y_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\ 1 + \sum_{k \geq 1} (y_{r+1} - y_r)^k, & \text{if } i_r = i_{r+1} + 1; \\ P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1. \end{cases} \quad (4.2)$$

In the non-degenerate setting, following Stroppel–Webster, we set

$$Q_r(i) := \begin{cases} 1 - q + qy_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\ \frac{1}{1-q^{-1}} \left( 1 + \sum_{k \geq 1} \left( \frac{y_{r+1} - qy_r}{1-q} \right)^k \right), & \text{if } i_r = i_{r+1} + 1; \\ P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1. \end{cases} \quad (4.3)$$
Brundan–Kleshchev’s isomorphism

The Brundan–Kleshchev’s isomorphism between $H^\lambda_\beta$ and $R^\lambda_\beta$ depends on the choice of certain polynomials $Q_r(i)$ for $1 \leq r < n$. In the degenerate setting, we set

$$Q_r(i) := \begin{cases} 
1 + y_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\
1 + \sum_{k \geq 1} (y_{r+1} - y_r)^k, & \text{if } i_r = i_{r+1} + 1; \\
P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1.
\end{cases} \quad (4.2)$$

In the non-degenerate setting, following Stroppel–Webster, we set

$$Q_r(i) := \begin{cases} 
1 - q + q y_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\
\frac{1}{1-q^{-1}} \left(1 + \sum_{k \geq 1} \left(\frac{y_{r+1}-q y_r}{1-q}\right)^k\right), & \text{if } i_r = i_{r+1} + 1; \\
P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1.
\end{cases} \quad (4.3)$$
Brundan–Kleshchev’s isomorphism

The Brundan–Kleshchev’s isomorphism between $\mathcal{H}^\Lambda$ and $\mathcal{R}^\Lambda$ depends on the choice of certain polynomials $Q_r(i)$ for $1 \leq r < n$. In the degenerate setting, we set

$$Q_r(i) := \begin{cases} 
1 + y_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\
1 + \sum_{k \geq 1} (y_{r+1} - y_r)^k, & \text{if } i_r = i_{r+1} + 1; \\
P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1.
\end{cases} \quad (4.2)$$

In the non-degenerate setting, following Stroppel–Webster, we set

$$Q_r(i) := \begin{cases} 
1 - q + qy_{r+1} - y_r, & \text{if } i_{r+1} = i_r; \\
\frac{1}{1-q} \left(1 + \sum_{k \geq 1} \left(\frac{y_{r+1} - qy_r}{1-q}\right)^k\right), & \text{if } i_r = i_{r+1} + 1; \\
P_r(i) - 1, & \text{if } i_r \neq i_{r+1}, i_{r+1} + 1.
\end{cases} \quad (4.3)$$
Brundan–Kleshchev’s isomorphism endows both $H_\beta^\Lambda(q)$ and $H_\beta^\Lambda$ a nontrivial $\mathbb{Z}$-grading.

Next we are going to lift Brundan-Kleshchev’s isomorphism to the setting of modified versions of affine Hecke algebras and quiver Hecke algebras?
Brundan–Kleshchev’s isomorphism endows both $\mathcal{H}_\beta^\Lambda(q)$ and $H^\Lambda_\beta$ a nontrivial $\mathbb{Z}$-grading.

Next we are going to lift Brundan-Kleshchev’s isomorphism to the setting of modified versions of affine Hecke algebras and quiver Hecke algebras?
Expressing \( e(i) \) as polynomials in Jucys-Murphy operators

**Lemma**

Let \( \beta \in Q_n^+ \). For each \( i \in I^\beta \), we can associate with a polynomial \( f_i(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n] \) which depends only on \( i \), such that for any \( i \in I^\beta \),

- \( e(i) = f_i(L'_1, \cdots, L'_n) \) holds in \( \mathcal{H}_n^\Lambda \) if \( q \neq 1 \); or
- \( e(i) = f_i(L_1, \cdots, L_n) \) holds in \( \mathcal{H}_n^\Lambda \) if \( q = 1 \); and
- \( f_{s_r}(t_1, \cdots, t_n) = s_r(f_i(t_1, \cdots, t_n)) \) for any \( 1 \leq r < n \).

In particular, the block idempotent \( e(\beta) \) in \( \mathcal{H}_n^\Lambda \) is a symmetric polynomial in \( L_1, \cdots, L_n \).
Expressing $e(i)$ as polynomials in Jucys-Murphy operators

Lemma

Let $\beta \in Q_n^+$. For each $i \in I^\beta$, we can associate with a polynomial $f_i(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n]$ which depends only on $i$, such that for any $i \in I^\beta$, 

- $e(i) = f_i(L'_1, \cdots, L'_n)$ holds in $\mathcal{H}^\wedge_n$ if $q \neq 1$; or
- $e(i) = f_i(L_1, \cdots, L_n)$ holds in $\mathcal{H}^\wedge_n$ if $q = 1$; and
- $f_{s_r}(t_1, \cdots, t_n) = s_r(f_i(t_1, \cdots, t_n))$ for any $1 \leq r < n$.

In particular, the block idempotent $e(\beta)$ in $\mathcal{H}^\wedge_n$ is a symmetric polynomial in $L_1, \cdots, L_n$. 
Expressing $e(i)$ as polynomials in Jucys-Murphy operators

**Lemma**

Let $\beta \in Q_n^+$. For each $i \in I^\beta$, we can associate with a polynomial $f_i(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n]$ which depends only on $i$, such that for any $i \in I^\beta$,

1. $e(i) = f_i(L'_1, \cdots, L'_n)$ holds in $\mathcal{H}_n^\wedge$ if $q \neq 1$; or
2. $e(i) = f_i(L_1, \cdots, L_n)$ holds in $\mathcal{H}_n^\wedge$ if $q = 1$; and
3. $f_{s_r}(t_1, \cdots, t_n) = s_r(f_i(t_1, \cdots, t_n))$ for any $1 \leq r < n$.

In particular, the block idempotent $e(\beta)$ in $\mathcal{H}_n^\wedge$ is a symmetric polynomial in $L_1, \cdots, L_n$. 

Hu Jun (Beijing Institute of Technology)
Expressing $e(i)$ as polynomials in Jucys-Murphy operators

**Lemma**

Let $\beta \in Q_n^+$. For each $i \in I^\beta$, we can associate with a polynomial $f_i(t_1, \ldots, t_n) \in K[t_1, \ldots, t_n]$ which depends only on $i$, such that for any $i \in I^\beta$,

1. $e(i) = f_i(L'_1, \ldots, L'_n)$ holds in $\mathcal{H}_n^\wedge$ if $q \neq 1$; or
2. $e(i) = f_i(L_1, \ldots, L_n)$ holds in $\mathcal{H}_n^\wedge$ if $q = 1$; and

$\circ$ $f_{s_r}(t_1, \ldots, t_n) = s_r(f_i(t_1, \ldots, t_n))$ for any $1 \leq r < n$.

In particular, the block idempotent $e(\beta)$ in $\mathcal{H}_n^\wedge$ is a symmetric polynomial in $L_1, \ldots, L_n$. 

Hu Jun (Beijing Institute of Technology)
Expressing $e(i)$ as polynomials in Jucys-Murphy operators

Lemma

Let $\beta \in \mathbb{Q}_+^n$. For each $i \in I^\beta$, we can associate with a polynomial $f_i(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n]$ which depends only on $i$, such that for any $i \in I^\beta$,

1. $e(i) = f_i(L'_1, \cdots, L'_n)$ holds in $H_n^\Lambda$ if $q \neq 1$; or $e(i) = f_i(L_1, \cdots, L_n)$ holds in $H_n^\Lambda$ if $q = 1$; and
2. $f_{s_r i}(t_1, \cdots, t_n) = s_r(f_i(t_1, \cdots, t_n))$ for any $1 \leq r < n$.

In particular, the block idempotent $e(\beta)$ in $H_n^\Lambda$ is a symmetric polynomial in $L_1, \cdots, L_n$. 
Expressing $e(i)$ as polynomials in Jucys-Murphy operators

**Lemma**

Let $\beta \in Q_n^+$. For each $i \in I^\beta$, we can associate with a polynomial $f_i(t_1, \cdots, t_n) \in K[t_1, \cdots, t_n]$ which depends only on $i$, such that for any $i \in I^\beta$,

1. $e(i) = f_i(L'_1, \cdots, L'_n)$ holds in $\mathcal{H}_n^\Lambda$ if $q \neq 1$; or $e(i) = f_i(L_1, \cdots, L_n)$ holds in $\mathcal{H}_n^\Lambda$ if $q = 1$; and
2. $f_{s_r i}(t_1, \cdots, t_n) = s_r(f_i(t_1, \cdots, t_n))$ for any $1 \leq r < n$.

In particular, the block idempotent $e(\beta)$ in $\mathcal{H}_n^\Lambda$ is a symmetric polynomial in $L_1, \cdots, L_n$. 
Expressing \( e(i) \) as polynomials in Jucys-Murphy operators

**Lemma**

Let \( \beta \in Q_n^+ \). For each \( i \in I^\beta \), we can associate with a polynomial \( f_i(t_1, \ldots, t_n) \in K[t_1, \ldots, t_n] \) which depends only on \( i \), such that for any \( i \in I^\beta \),

1. \( e(i) = f_i(L'_1, \ldots, L'_n) \) holds in \( \mathcal{H}_n^\wedge \) if \( q \neq 1 \); or
2. \( e(i) = f_i(L_1, \ldots, L_n) \) holds in \( \mathcal{H}_n^\wedge \) if \( q = 1 \); and
3. \( f_{s_r,i}(t_1, \ldots, t_n) = s_r(f_i(t_1, \ldots, t_n)) \) for any \( 1 \leq r < n \).

In particular, the block idempotent \( e(\beta) \) in \( \mathcal{H}_n^\wedge \) is a symmetric polynomial in \( L_1, \ldots, L_n \).
Brundan–Kleshche’s isomorphism for modified versions

Theorem (Hu-Li)

In the non-degenerate case, there is a $K$-algebra isomorphism $\theta : \widetilde{R}_\beta \cong \widetilde{H}_\beta(q)$, such that $e(i) \mapsto \hat{e}(i)$, $y_s e(i) \mapsto \hat{e}(i)(1 - q^{-i_s} \hat{X}_s) \hat{e}(i)$ and

$$
\psi_r e(i) \mapsto \begin{cases}
q^{i_r}(\hat{T}_r + 1)(\hat{X}_r - q\hat{X}_{r+1})^{-1} \hat{e}(i) & \text{if } i_r = i_{r+1}; \\
q^{-i_r} \left( \hat{T}_r(\hat{X}_r - \hat{X}_{r+1}) + (q - 1)\hat{X}_{r+1} \right) \hat{e}(i) & \text{if } i_{r+1} \to i_r; \\
\left( \hat{T}_r(\hat{X}_{r+1} - \hat{X}_r) + (1 - q)\hat{X}_{r+1} \right) \\
\times (\hat{X}_r - q\hat{X}_{r+1})^{-1} \hat{e}(i) & \text{otherwise.}
\end{cases}
$$

for any $i \in I_\beta$, $1 \leq s \leq n$ and $1 \leq r < n$. 
The inverse map $\eta$ is given by:

$$
\eta(\hat{e}(i)) = e(i), \quad \eta(\hat{X}_s^{\pm 1} e(i)) = q^{\pm i_s} (1 - y_s)^{\pm 1} e(i),
$$

and $\eta(\hat{T}_r \hat{e}(i))$ is equal to $\psi_r (1 - q + q y_{r+1} - y_r) e(i) - e(i)$ if $i_r = i_{r+1}$; or

$$
\left( q^r e(i) - (q - 1)(1 - y_{r+1}) e(i) \right) \left( q(1 - y_r) - (1 - y_{r+1}) \right)^{-1} e(i),
$$

if $i_r = i_{r+1} + 1$; or otherwise

$$
\psi_r (q^i_r - q^{i_{r+1}+1} - q^i_r y_r + q^{i_{r+1}+1} y_{r+1}) (q^{i_{r+1}} - q^i_r + q^i_r y_r - q^{i_{r+1}} y_{r+1})^{-1}.
$$
The center of non-degenerate cyclotomic Hecke algebras

We have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{R}_\beta & \xrightarrow{\sim} & \tilde{H}_\beta \\
\downarrow_{p_1(\Lambda)} & & \downarrow_{\pi_1(\Lambda)} \\
\mathbb{R}^\Lambda & \xrightarrow{\sim} & \mathbb{H}^\Lambda \\
\end{array}
\]

Theorem (Hu)

Conjecture C holds for \( \mathbb{H}^\Lambda \) if and only if Conjecture A holds for the cyclic quiver (when \( e > 0 \)) and linear quiver cases (when \( e = 0 \)).
**The main results: the degenerate case**

**Theorem (Hu-Li)**

*In the degenerate case, there is a $K$-algebra isomorphism $\theta' : \tilde{R}_\beta' \cong \tilde{H}_\beta$, such that $e(i) \mapsto \hat{e}(i)$, $y_s e(i) \mapsto \hat{e}(i)(\hat{x}_s - i_s)\hat{e}(i)$ and*

\[
\psi_r e(i) \mapsto \begin{cases} 
(\hat{s}_r + 1)(1 + \hat{x}_{r+1} - \hat{x}_r)^{-1}\hat{e}(i) & \text{if } i_r = i_{r+1}; \\
(\hat{s}_r(\hat{x}_r - \hat{x}_{r+1}) + 1)\hat{e}(i) & \text{if } i_r = i_{r+1} + 1; \\
(\hat{s}_r(\hat{x}_r - \hat{x}_{r+1}) + 1) \times (1 + \hat{x}_{r+1} - \hat{x}_r)^{-1}\hat{e}(i) & \text{otherwise.}
\end{cases}
\]

*for any $i \in I_\beta$, $1 \leq s \leq n$ and $1 \leq r < n$.**
The main results: the degenerate case

Theorem (Continued)

The inverse map $\eta$ is given by:

$$\eta(e(i)) = e(i), \quad \eta(xs e(i)) = (y_s + i_s)e(i),$$

and $\eta(s_r e(i))$ is equal to $\psi_r(1 + y_{r+1} - y_r)e(i) - e(i)$ if $i_r = i_{r+1}$; or

$$\left(\psi_r e(i) - e(i)\right) \left(1 - y_{r+1} + y_r\right)^{-1} e(i),$$

if $i_r = i_{r+1} + 1$; or otherwise

$$\psi_r \left(1 - i_r + i_{r+1} + y_{r+1} - y_r\right) \left(i_r - i_{r+1} - y_{r+1} + y_r\right)^{-1} e(i) - (i_r - i_{r+1} - y_{r+1} + y_r)^{-1} e(i).$$
We have the following commutative diagram:

\[
\begin{array}{ccc}
\mathcal{R}'_\beta & \xrightarrow{\sim} & \tilde{H}_\beta \\
\downarrow p_2(\Lambda) & & \downarrow \pi_2(\Lambda) \\
\mathcal{R}^\Lambda_\beta & \xrightarrow{\sim} & H^\Lambda_\beta
\end{array}
\]

**Theorem (Hu)**

Conjecture C holds for \( H^\Lambda_\beta \) if and only Conjecture A holds for the cyclic quiver \( \Gamma_p \) (when \( \text{char } K = p > 0 \)) or the linear quiver (when \( p = 0 \)).
Consequences

Applying the above two theorems and using previous results of Brundan, McGerty, we can now deduce that

**Corollary**

Conjecture A and B hold in any one of the following three cases:

1. $\Gamma$ is the linear quiver;
2. $\Gamma$ is the cyclic quiver with $p$ vertices and $\text{char } K = p > 0$;
3. $\Gamma$ is the cyclic quiver and $n = 2$. 
Consequences

Applying the above two theorems and using previous results of Brundan, McGerty, we can now deduce that

**Corollary**

Conjecture A and B hold in any one of the following three cases:

1. \( \Gamma \) is the linear quiver;
2. \( \Gamma \) is the cyclic quiver with \( p \) vertices and \( \text{char } K = p > 0 \);
3. \( \Gamma \) is the cyclic quiver and \( n = 2 \).
Consequences

Applying the above two theorems and using previous results of Brundan, Mcgerty, we can now deduce that

**Corollary**

Conjecture A and B hold in any one of the following three cases:

1. $\Gamma$ is the linear quiver;
2. $\Gamma$ is the cyclic quiver with $p$ vertices and $\text{char } K = p > 0$;
3. $\Gamma$ is the cyclic quiver and $n = 2$. 
Consequences

Applying the above two theorems and using previous results of Brundan, McGerty, we can now deduce that

**Corollary**

*Conjecture A and B hold in any one of the following three cases:*

1. $\Gamma$ is the linear quiver;
2. $\Gamma$ is the cyclic quiver with $p$ vertices and $\text{char } K = p > 0$;
3. $\Gamma$ is the cyclic quiver and $n = 2$. 
Thank You For Your Attention!