Invariants of Kazhdan–Lusztig cells

Edmund Howse

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Let $S = \{s_1, \ldots, s_n\}$ be a finite non-empty set, and let $W$ be a group with presentation

$$W = \langle s_1, \ldots, s_n : s_i^2 = e, (s_is_j)^{m_{ij}} = e \rangle,$$

where $m_{ij} = m_{ji} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ if $i \neq j$.

Then we say that $W$ is a Coxeter group with generating set $S$, and the ordered pair $(W, S)$ is a Coxeter system.

Denote by $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ the corresponding length function.
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Denote by $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ the corresponding length function.

A weight function is any map $\mathcal{L} : W \rightarrow \mathbb{Z}$ such that:

$$\ell(yw) = \ell(y) + \ell(w) \Rightarrow \mathcal{L}(yw) = \mathcal{L}(y) + \mathcal{L}(w).$$

A weight function is determined by its values on $S$; we have

$$m_{ij} \text{ is odd} \Rightarrow \mathcal{L}(s_i) = \mathcal{L}(s_j).$$

Throughout this talk, we assume $\mathcal{L}(s) > 0$ for all $s \in S$. 
The *Iwahori–Hecke algebra* \( \mathcal{H} := \mathcal{H}(W, S, \mathcal{L}) \) associated to a weighted Coxeter system is a deformation of the group algebra of \( W \) over \( A = \mathbb{Z}[v, v^{-1}] \). It is an associative unital \( A \)-algebra, with:

- **basis:** \( \{ T_w : w \in W \} \),
- **identity:** \( T_e \),
- **generators:** \( \{ T_s : s \in S \} \),
- **parameters:** \( \{ v^\mathcal{L}(s) : s \in S \} \),
- **relations:**

\[
T_s T_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w), \\
T_{sw} + (v^\mathcal{L}(s) - v^{-\mathcal{L}(s)}) T_w & \text{if } \ell(sw) < \ell(w).
\end{cases}
\]

Let \( K \) be a the field of fractions of \( A \). If \( W \) is a finite Weyl group, then \( K \otimes_A \mathcal{H} \) is split semisimple, and isomorphic to \( K[W] \).
There exists a ‘new’ basis for $\mathcal{H}$ – the KL basis $\{C_w : w \in W\}$. Describing $C_w$ in terms of the standard basis defines the Kazhdan–Lusztig polynomials $P_{y,w} \in A$: 

$$C_w = \sum_{y \in W} P_{y,w} T_y.$$
There exists a ‘new’ basis for $\mathcal{H}$ – the KL basis $\{C_w : w \in W\}$. Describing $C_w$ in terms of the standard basis defines the Kazhdan–Lusztig polynomials $P_{y,w} \in \mathcal{A}$:

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The KL polynomials depend on $\mathcal{L}$. Suppose that $\mathcal{L} = \ell$. Then the coefficients of $P_{y,w}$ are all non-negative (Elias–Williamson, 2014).
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We have multiplication rules for the KL basis.

$$C_s C_w = \begin{cases} 
C_{sw} + \sum_{y : y < w, s y < y} M^s_{y,w} C_y & \text{if } \ell(sw) > \ell(w), \\
(v\mathcal{L}(s) + v-\mathcal{L}(s)) C_w & \text{if } \ell(sw) < \ell(w).
\end{cases}$$
Kazhdan–Lusztig cells

\[ C_s C_w = \begin{cases} 
C_{sw} + \sum_{y: y < w} \sum_{s y < y} M_{y,w}^s C_y & \text{if } \ell(sw) > \ell(w), \\
(vL(s) + v-L(s)) C_w & \text{if } \ell(sw) < \ell(w).
\end{cases} \]

The \textit{left elementary} relation \( \leq_{L,E} \) defined by

\[ y \leq_{L,E} w \quad \text{if} \quad \left\{ \begin{array}{l}
\text{there exists some } s \in S \text{ such that } \\
C_y \text{ occurs in } C_s C_w
\end{array} \right. \]

can be extended to its reflexive, transitive closure – the Kazhdan–Lusztig preorder \( \leq_L \).
Kazhdan–Lusztig cells

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can be extended to its reflexive, transitive closure – the Kazhdan–Lusztig preorder \( \leq_L \). The associated equivalence relation on \( W \) is denoted \( \sim_L \), and is defined by

\[ y \sim_L w \iff y \leq_L w \text{ and } w \leq_L y. \]

The resulting equivalence classes are called left cells. As the \( M \)-polynomials depend on \( \mathcal{L} \), so does the partition of \( W \) into cells.
Kazhdan–Lusztig cells

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C_{sw} + \sum_{y: y < w, sy < y} M_{y,w}^s C_y & \text{if } \ell(sw) > \ell(w), \\
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An analogous preorder \( \leq_R \) and equivalence relation \( \sim_R \) exist, with equivalence classes called right cells.
Finally, the two-sided preorder $\leq_{LR}$ arising from the relation

$$y \leq_{LR,E} w \iff y \leq_{L,E} w \text{ or } y \leq_{R,E} w$$

leads to the relation $\sim_{LR}$ and equivalence classes called two-sided cells.
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- $y \sim_{L} w \iff y^{-1} \sim_{R} w^{-1}$.
- The relation $\sim_{LR}$ contains the relations $\sim_{L}$ and $\sim_{R}$, and so two-sided cells are unions of both left and right cells.
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Let $\Gamma \subseteq W$ be a left cell, and $w \in \Gamma$. Then

$$I^\Gamma_{\leq} := \langle C_z : z \leq_{L} w \rangle_A$$
$$I^\Gamma_{<} := \langle C_z : z \leq_{L} w, z \not\sim_{L} w \rangle_A$$

are two left ideals of $\mathcal{H}$, and $[\Gamma] := I^\Gamma_{\leq}/I^\Gamma_{<}$ is a $\mathcal{H}$-module.

So, $[\Gamma]_K := K \otimes_A [\Gamma]$ is a $K \otimes_A \mathcal{H}$-module.
Cells of the symmetric group

\[ S_n \cong W(A_{n-1}) \]

\[ s_1 \quad s_2 \quad \ldots \quad s_{n-1} \]

For each \( w \in S_n \), we may associate a pair of standard tableaux \((P(w), Q(w))\), both of shape \( \text{sh}(w) \) via the Robinson–Schensted correspondence.

\[ \Gamma \cong \Gamma' \iff \Gamma, \Gamma' \text{ lie in the same two-sided cell.} \]

\[ K \otimes A H \cong \bigoplus_{\Gamma \subseteq S_n} \Gamma \]

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Invariants of Kazhdan–Lusztig cells
Cells of the symmetric group

$\mathfrak{S}_n \cong W(A_{n-1})$

For each $w \in \mathfrak{S}_n$, we may associate a pair of standard tableaux $(P(w), Q(w))$, both of shape $\text{sh}(w)$ via the Robinson–Schensted correspondence.

**Theorem (Kazhdan–Lusztig, 1979)**

Let $\mathcal{H}$ be the Iwahori–Hecke algebra of $\mathfrak{S}_n$, and let $y, w \in \mathfrak{S}_n$.

- $y \sim_L w \iff Q(y) = Q(w) \iff y \approx_\tau w$.
- $y \sim_R w \iff P(y) = P(w) \iff y \leftrightarrow w$.
- $y \sim_{LR} w \iff \text{sh}(y) = \text{sh}(w)$.
- Every left cell module is irreducible.
- Every irreducible $\mathcal{H}$-module is isomorphic to a left cell module.
- $[\Gamma] \cong [\Gamma']$ if and only if $\Gamma, \Gamma'$ lie in the same two-sided cell.
- $\mathcal{K} \otimes_A \mathcal{H} \cong \bigoplus_{\Gamma \subseteq \mathfrak{S}_n} [\Gamma]_\mathcal{K}$. 
The Coxeter group of type $B_n$

$$W_n := W(B_n)$$

Let $L : W_n \rightarrow \mathbb{Z}$ be a weight function; it suffices to describe its values on $S$, so denote $b := L(t), a := L(s_i)$. 

Invariants of Kazhdan–Lusztig cells
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- The resulting partition of $W_n$ into Kazhdan–Lusztig cells depends only on the value $b/a$. 

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- The cells of $W_n$ are independent of the exact value of $b/a$ provided it is sufficiently large (with respect to $n$); this situation is known as the ‘asymptotic case’, and occurs precisely when $b/a > n - 1$. 
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- The cells of $W_n$ are independent of the exact value of $b/a$ provided it is sufficiently large (with respect to $n$); this situation is known as the ‘asymptotic case’, and occurs precisely when $b/a > n - 1$.
- We know the cells for $W_n$ with respect to $\mathcal{L}$ if $b/a$ is equal to...

Invariants of Kazhdan–Lusztig cells

Garfinkle (1993)


Lusztig (1983)
The Coxeter group of type $B_n$

For each $w \in W_n$, we may associate a pair of standard bitableaux $(A(w), B(w))$, both of shape $\text{sh}(w)$, via a generalised Robinson–Schensted correspondence.

Let $\mathcal{H}$ be the Iwahori–Hecke algebra of $(W_n, S, \mathcal{L})$, and let $y, w \in W_n$.

Theorem (Bonnafé–Iancu, Bonnafé, 2003)

$L$ is an asymptotic weight function if and only if $b/a > n - 1$.

Suppose that we are in the asymptotic case. Then:

$y \sim Lw \iff B(y) = B(w)$.

$y \sim Rw \iff A(y) = A(w)$.

$y \sim LRw \iff \text{sh}(y) = \text{sh}(w)$.

Every left cell module is irreducible.

Every irreducible $\mathcal{H}$-module is isomorphic to a left cell module.

$[\Gamma] \sim = [\Gamma']$ if and only if $\Gamma$, $\Gamma'$ lie in the same two-sided cell.

$K \otimes A \mathcal{H} \sim = \bigoplus_{\Gamma \subseteq W_n} [\Gamma] K$.
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- $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{H} \cong \bigoplus_{\Gamma \subseteq W_n} [\Gamma]_{\mathcal{K}}$.
“The problem seems to have two parts: the use of algebraic methods to reduce to questions about Weyl groups, and then combinatorics to study these questions.” – Vogan

Let \((W, S, \mathcal{L})\) be an arbitrary weighted Coxeter system, and let the right descent set of \(w \in W\) be \(\mathcal{R}(w) := \{ s \in S : \ell(ws) < \ell(w) \}\). Then \(y \sim_L w \Rightarrow \mathcal{R}(y) = \mathcal{R}(w)\).

\[
W = \bigsqcup_{I \subseteq S} \{ w \in W : \mathcal{R}(w) = I \}.
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\]

Bonnafé–Geck generalise this as follows.

Let \(S^\mathcal{L} := S \cup \{ sts : \mathcal{L}(t) > \mathcal{L}(s) \}\),

\[
\mathcal{R}^\mathcal{L}(w) := \{ \sigma \in S^\mathcal{L} : \ell(w\sigma) < \ell(w) \}.
\]

Then \(y \sim_L w \Rightarrow \mathcal{R}^\mathcal{L}(y) = \mathcal{R}^\mathcal{L}(w)\). If \(W\) is of type \(I_2(m)\), then the converse holds too.
Let \( S^L := S \cup \left\{ s_k \cdots s_1 t s_1 \cdots s_k : \mathcal{L}(t) > k \cdot \mathcal{L}(s_i) \text{ and } m_{i,i+1} = 3 \text{ for } 1 \leq i \leq k - 1 \right\} \),

\[ R^L(w) := \{ \sigma \in S^L : \ell(w\sigma) < \ell(w) \} \].

**Proposition (H., 2017)**

- Let \((W, S, L)\) be a finite weighted Coxeter system. Then \( y \sim_L w \Rightarrow R^L(y) = R^L(w) \).
- Consider \((W_n, S, L)\) with \( b/a \in (k, k + 1] \subseteq (1, n] \), with \( k \in \mathbb{Z} \). Then \( R^L \) partitions \( W_n \) into \( 2^{n-k} \cdot 3^k \) non-empty subsets.
Let $\overline{S^L} := S \cup \left\{ s_k \cdots s_1 s t s_1 \cdots s_k : \begin{array}{c} \mathcal{L}(t) > k \cdot \mathcal{L}(s_i) \\
\text{and} \\
m_{i,i+1} = 3 \text{ for } 1 \leq i \leq k - 1 \end{array} \right\}$,

\[
\overline{R^L}(w) := \{ \sigma \in \overline{S^L} : \ell(w\sigma) < \ell(w) \}.
\]

**Proposition (H., 2017)**

- Let $(W, S, \mathcal{L})$ be a finite weighted Coxeter system. Then $y \sim_L w \Rightarrow \overline{R^L}(y) = \overline{R^L}(w)$.
- Consider $(W_n, S, \mathcal{L})$ with $b/a \in (k, k + 1] \subseteq (1, n]$, with $k \in \mathbb{Z}$. Then $\overline{R^L}$ partitions $W_n$ into $2^{n-k} \cdot 3^k$ non-empty subsets.

Suppose that we are in the asymptotic case; that is, $b/a > n - 1$.

<table>
<thead>
<tr>
<th></th>
<th>$W_3$</th>
<th>$W_4$</th>
<th>$W_5$</th>
<th>$W_6$</th>
<th>$W_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}$</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
</tr>
<tr>
<td>$\overline{R^L}$</td>
<td>18</td>
<td>54</td>
<td>162</td>
<td>486</td>
<td>1456</td>
</tr>
<tr>
<td>cells</td>
<td>20</td>
<td>76</td>
<td>312</td>
<td>1384</td>
<td>6512</td>
</tr>
</tbody>
</table>
Consider \((W, S, \mathcal{L})\) and let \(I \subseteq S\) be non-empty. Then \(W_I := \langle I \rangle\) is a subgroup called a parabolic subgroup. Aspects of its structure can be realised in the group \(W\), for instance, the Kazhdan–Lusztig cells of \(W_I\) (with respect to the weight function \(\mathcal{L}|_{W_I}\)).

If \(\varphi : W_I \to W_I\) is any map, then \(\varphi\) (left) extends to a map \(\varphi^L : W \to W\) in a natural way.
Consider some $(W, S, \mathcal{L})$ and let $I = \{s_i, s_j\} \subseteq S$ with $m_{ij} = 3$. The parabolic subgroup $W_I \cong S_3$ has six elements.
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\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}
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\end{array} \hspace{1cm} \begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{array}\]

The map \(\sigma: W_I \to W_I\) has the following properties:

- \(\Gamma\) is a left cell of \(W_I\) \(\iff\) \(\sigma(\Gamma)\) is a left cell of \(W_I\).
- \(w \sim R, I \sigma(w) \forall w \in W_I\).

The map \(\sigma_L\) is a \(*\)-operation for \(W\) (Vogan, Kazhdan–Lusztig).
Consider some \((W, S, \mathcal{L})\) and let \(I = \{s_i, s_j\} \subseteq S\) with \(m_{ij} = 3\). The parabolic subgroup \(W_I \cong \mathfrak{S}_3\) has six elements.
Consider some \( (W, S, \mathcal{L}) \) and let \( I = \{s_i, s_j\} \subseteq S \) with \( m_{ij} = 3 \). The parabolic subgroup \( W_I \cong \mathfrak{S}_3 \) has six elements. Define \( \sigma : W_I \longrightarrow W_I \) by:

\[
\begin{align*}
\sigma &: \text{left cell of } W_I \quad \Rightarrow \quad \text{left cell of } W_I \\
\sigma &: \text{orbit of } u \quad \Rightarrow \quad \text{orbit of } \sigma(u)
\end{align*}
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Note that \(\sigma : W_I \rightarrow W_I\) has the following properties:

- \(\Gamma\) is a left cell of \(W_I\) \iff \(\sigma(\Gamma)\) is a left cell of \(W_I\).
- \(u \sim_{R,I} \sigma(u)\) \(\forall u \in W_I\).
Consider some \((W, S, \mathcal{L})\) and let \(I = \{s_i, s_j\} \subseteq S\) with \(m_{ij} = 3\). The parabolic subgroup \(W_I \cong S_3\) has six elements. Define \(\sigma : W_I \to W_I\) by:

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It can be seen that the map \(\sigma^L : W \to W\) is such that:

- \(\Gamma\) is a left cell of \(W\) \iff \(\sigma^L(\Gamma)\) is a left cell of \(W\).
- \(w \sim_R \sigma^L(w)\) \ \forall w \in W\)

The map \(\sigma^L\) is a \(*\)-operation for \(W\) (Vogan, Kazhdan–Lusztig).
For \((W, S, \mathcal{L})\), these involutive maps \(\sigma^L\) permute the elements of \(W\), so \(\mathcal{P}(\ast) := \langle \sigma^L : \sigma^L \text{ is a } \ast\text{-operation for } W \rangle\) is a permutation group.

The \(\ast\)-operations, together with \(\mathcal{R}\), are used to define the generalised \(\tau\)-invariant (Vogan, 1979).
Generalised $\tau$-invariant

For $(W, S, \mathcal{L})$, these involutive maps $\sigma^L$ permute the elements of $W$, so $\mathcal{P}(\ast) := \langle \sigma^L : \sigma^L \text{ is a } \ast\text{-operation for } W \rangle$ is a permutation group. The $\ast$-operations, together with $\mathcal{R}$, are used to define the generalised $\tau$-invariant (Vogan, 1979).

- Let $y, w \in W$. If $\mathcal{R}(y) = \mathcal{R}(w)$, continue.
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- Let $y, w \in W$. If $\mathcal{R}(y) = \mathcal{R}(w)$, continue.
- Apply a $\ast$-operation $\sigma^L$ to $y$ and $w$. If $\mathcal{R}(\sigma^L(y)) = \mathcal{R}(\sigma^L(w))$, continue.
For $(W, S, \mathcal{L})$, these involutive maps $\sigma^L$ permute the elements of $W$, so $\mathcal{P}(\ast) := \langle \sigma^L : \sigma^L \text{ is a } \ast\text{-operation for } W \rangle$ is a permutation group.

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- Apply a $\ast$-operation $\sigma^L$ to $y$ and $w$. If $\mathcal{R}(\sigma^L(y)) = \mathcal{R}(\sigma^L(w))$, continue.
- If $\mathcal{R}(\nu(y)) = \mathcal{R}(\nu(w))$ for all $\nu \in \mathcal{P}(\ast)$, then we write $y \approx_{\tau} w$, and say that $y$ and $w$ have the same generalised $\tau$-invariant.
For \((W, S, \mathcal{L})\), these involutive maps \(\sigma^L\) permute the elements of \(W\), so \(\mathcal{P}(\ast) := \langle \sigma^L : \sigma^L \text{ is a } \ast\text{-operation for } W \rangle\) is a permutation group.

The \(\ast\)-operations, together with \(\mathcal{R}\), are used to define the generalised \(\tau\)-invariant (Vogan, 1979).

- Let \(y, w \in W\). If \(\mathcal{R}(y) = \mathcal{R}(w)\), continue.
- Apply a \(\ast\)-operation \(\sigma^L\) to \(y\) and \(w\). If \(\mathcal{R}(\sigma^L(y)) = \mathcal{R}(\sigma^L(w))\), continue.
- If \(\mathcal{R}(\nu(y)) = \mathcal{R}(\nu(w))\) for all \(\nu \in \mathcal{P}(\ast)\), then we write \(y \approx_\tau w\), and say that \(y\) and \(w\) have the same generalised \(\tau\)-invariant.
- \(y \sim_L w \Rightarrow y \approx_\tau w\).
Let $I \subseteq S$ and $\delta : W_I \to W_I$ be such that the following are satisfied:

(A1) If $\Gamma \subseteq W_I$ is a left cell, then so is $\delta(\Gamma)$.

(A2) The map $\delta$ induces a $\mathcal{H}_I$-module isomorphism $[\Gamma] \cong [\delta(\Gamma)]$.

(A3) We have $u \sim_{R,I} \delta(u)$ for all $u \in W_I$.

Then the pair $(I, \delta)$ is called a strongly KL-admissible pair.

Theorem (Bonnafé–Geck, 2015)

Let $(I, \delta)$ be a strongly KL-admissible pair. Then $(S, \delta^L)$ is a strongly KL-admissible pair.
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Let $(I, \delta)$ be a strongly KL-admissible pair. Then $(S, \delta^L)$ is a strongly KL-admissible pair.
Consider a weighted Coxeter system \((W, S, \mathcal{L})\), and let:
- \(\Delta\) be a collection of strongly KL-admissible pairs for \((W, S, \mathcal{L})\),
- \(\rho\) be an invariant of the left cells of \(W\),
- \(\mathcal{P}(\Delta) := \langle \delta^L : (I, \delta) \in \Delta \rangle\).

We say that \(y, w \in W\) are in the same left Vogan \((\Delta, \rho)\)-class if:
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\rho(\nu(y)) = \rho(\nu(w)) \quad \forall \nu \in \mathcal{P}(\Delta).
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**Theorem (Bonnafé–Geck, 2015)**

If \(y \sim_L w\) then \(y \approx^{\Delta, \rho} w\).

Write \(y \xleftarrow{\Delta} w\) if \(w = \nu(y)\) for some \(\nu \in \mathcal{P}(\Delta)\), and say that \(y\) and \(w\) lie in the same \(\Delta\)-orbit. Then we have

- \(y \xleftarrow{\Delta} w \Rightarrow y \sim_R w\) and \(y \sim_L w \iff y^{-1} \sim_R w^{-1}\), so:
- \(y^{-1} \xleftarrow{\Delta} w^{-1} \Rightarrow y \sim_L w \Rightarrow y \approx^{\Delta, \rho} w\).
We have $W_K = W_{n-1}$ and $W_J \cong \mathfrak{S}_n$. The cells of $\mathfrak{S}_n$ are understood. If $b/a > n - 2$, then we are in the asymptotic case for $W_K$, and its cells are known as well. **From now on, we assume** $b/a > n - 2$, putting us in one of the following cases:
Let \((I, \delta)\) be a strongly KL-admissible pair. We say that \((I, \delta)\) is \textit{maximally KL-admissible} if it additionally satisfies the condition:

(A4) If \(u \sim_{R,I} v\), then \(\exists k \in \mathbb{Z}_{\geq 0}\) such that \(u = \delta^k(v)\).
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There exist maps \(\varepsilon : W_J \to W_J\) and \(\psi : W_K \to W_K\) such that both \((J, \varepsilon)\) and \((K, \psi)\) are maximally KL-admissible pairs.

Set \(\Xi := \{(J, \varepsilon), (K, \psi)\}\) and \(\mathcal{P}(\Xi) := \langle \varepsilon^L, \psi^L \rangle\).
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Set \(\Xi := \{(J, \varepsilon), (K, \psi)\}\) and \(\mathcal{P}(\Xi) := \langle \varepsilon^L, \psi^L \rangle\).

\textbf{Proposition (H., 2017)}

- The group \(\mathcal{P}(\Xi)\) is independent of the choices made during the construction of the maps \(\varepsilon\) and \(\psi\).
- Therefore, the left Vogan \((\Xi, \rho)\)-classes are well-defined.
- We have \(\mathcal{P}(\Delta) \leq \mathcal{P}(\Xi)\). Therefore, for all \(y, w \in W_n\), we have \(y \approx^{\Xi, \rho} w \implies y \approx^{\Delta, \rho} w\).
Orbits of $\mathcal{P}(\Xi)$ partition $W_n$; right cells of $W_n$ are unions of $\Xi$-orbits.
Left Vogan \((\Xi, \overline{R}^\ell)\)-classes

Orbits of \(P(\Xi)\) partition \(W_n\); right cells of \(W_n\) are unions of \(\Xi\)-orbits.

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<tr>
<th>cells (b/a = n - 1)</th>
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Theorem (H., 2017)

Consider \( (W_n, S, L) \) with \( b/a \geq n - 1 \). Then:

\[
y \sim_L w \iff y \approx \Xi, \overline{R}^L w.
\]
Thank you for your attention!