Tableau combinatorics

A partition of $n$ is a weakly decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ of non-negative integers that sum to $n$. Identify $\lambda$ with its Young diagram $[\lambda] = \{(r,c) | 1 \leq c \leq \lambda_r\}$, which is an array of boxes in the plane.

Let $P_n^\lambda$ be the set of partitions of $n$.

Example The diagram of $(3,2)$ is

```
+ + +
+ +
```

A $\lambda$-tableau with values in a set $X$ is a function $t : [\lambda] \rightarrow X$, which we think of as a labelled diagram. A $\lambda$-tableau is standard if $X = \{1, \ldots, n\}$ and the entries increase along rows and down columns.

Let $\text{Std}(\lambda)$ be the set of standard $\lambda$-tableaux and $\text{Std}(P_n^\lambda) = \bigcup_{\lambda \in P_n^\lambda} \text{Std}(\lambda)$.

Example The standard $(3,2)$-tableaux are:

- $t^\lambda = \begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 \\
3 & 4 \\
\end{array}$
- $\begin{array}{cccc}
1 & 2 & 5 \\
3 & 4 \\
2 & 5 \\
\end{array}$
- $\begin{array}{cccc}
1 & 3 & 5 \\
2 & 4 \\
\end{array}$

Remark We will also use standard $\lambda$-tableaux where $\lambda = (\lambda^{(1)} | \ldots | \lambda^{(\ell)})$ is an $\ell$-partition. These are $\ell$-tuples of tableaux such that the entries increase along rows and down columns in each component.

Content functions for symmetric groups

The content of a node $(r,c)$ is $c - r$. If $t$ is standard and $1 \leq m \leq n$ then the content of $m$ in $t$ is $c_m(t) = c - r$, if $t(r,c) = m$.

Example If $\lambda = (4,3,3,2)$ then the contents in $[\lambda]$ are:

```
0 1 2 3
-1 0 1
-2 -1 0
-3 -2
```

Contents increase along rows, decrease down columns and are constant on the diagonals of $\lambda$. The addable nodes of $\lambda$ have distinct contents.

Lemma

Let $s \in \text{Std}(\lambda)$ and $t \in \text{Std}(\mu)$. Then $s = t$ if and only if $c_m(s) = c_m(t)$ for $1 \leq m \leq n$. Consequently, if $1 \leq r < n$ then $c_m(t) = c_m(t)$ for $r \neq m, m + 1$ if and only if $s = t$ or $s = s_r t$. 

Proof Follows easily by induction because addable nodes have distinct contents.
Quiver Hecke algebras

The symmetric group $S_n$ acts on $I^n$ by place permutations: $wI = (i_{w(1)}, \ldots, i_{w(n)})$, for $w \in S_n$ and $I \in I^n$

For $\alpha \in Q^+$ let $I^n = \{ i \in I^n | \alpha = \alpha_i + \cdots + \alpha_n \}$, where $n = \text{ht}(\alpha)$

Definition (Khovanov-Lauda, Rouquier 2008)

The quiver Hecke algebra, or KL algebra, $R_n$ is the unital associative $k$-algebra generated by $\{ 1_i | i \in I^n \} \cup \{ \psi_r | 1 \leq r < n \} \cup \{ \psi_s | 1 \leq s \leq n \}$ subject to the relations:

- $1_i 1_i = \delta_{i,i} 1_i$, $\sum_{i \in I^n} 1_i = 1$, $\psi_r 1_i = 1_i \psi_r$,
- $y_r 1_i = 1_i y_r$, $y_r y_t = y_t y_r$, $\psi_r^2 1_i = Q_{i,i+1}(y_r, y_{r+1}) 1_i$
- $\psi_s y_r = y_r \psi_r$ if $s \neq r, r + 1$, $\psi_s y_t = y_t \psi_r$ if $|r - t| > 1$
- $(\psi_r y_{r-1} - \psi_{r-1} y_r) 1_i = \delta_{i,i} 1_i = (y_r \psi_{r-1} - y_{r-1} \psi_r) 1_i$
- $(\psi_s y_{r+1} y_{r+1} - \psi_{r+1} y_{r+1}) 1_i = \delta_{i,i+1} y_{r+1} (y_r, y_{r+1}, y_{r+1}) 1_i$

Let $R_n = \bigoplus_{\alpha \in Q^+} R_{\alpha}$, where $Q^+_n = \{ \alpha \in Q^+ | \text{ht}(\alpha) = n \}$

Importantly, $R_n$ is graded with the grading determined by $\deg 1_i = 0$, $\deg y_r 1_i = (\alpha_i, \alpha_i)$, and $\deg \psi_r 1_i = -(\alpha_i, \alpha_{i+1})$
Cyclotomic Hecke algebras of type A

Fix $\xi \in k$ such that $e$ is minimal with $1 + \xi^2 + \cdots + \xi^{2(e-1)} = 0$.

Fix integers $r_1, \ldots, r_l$ with $\{1 \leq i \leq l \mid r_i \equiv i \pmod{e}\} = (h_i, \Lambda)$

For $m \in \mathbb{N}$ define the $\xi$-quantum integer $[m]_{\xi} = \frac{2^{m-1} - 1}{\xi - 1}$.

Definition (Ariki-Koike, Hu-M.)

The cyclotomic Hecke algebra of type $A$ is the unital associative $k$-algebra $\mathcal{H}_n^\Lambda = \mathcal{H}_n^\Lambda(\xi)$ with generators $T_1, \ldots, T_{n-1}, L_1, \ldots, L_n$ and relations

$$\prod_{i=1}^l (1 - \xi^{r_i}) - 1 = 0, \quad (T_r - \xi) (T_r + \xi^{-1}) - 1 = 0, \quad L_r L_r = L_r L_r,$$

$$T_s T_{s+1} T_s = T_{s+1} T_s T_{s+1}, \quad T_r T_s = T_s T_r \text{ if } |r-s| > 1,$$

$$T_r L_r = L_r T_r \text{ if } t \neq r, r+1, \quad L_{r+1} = T_r L_r + T_r.$$

When $\xi^2 \neq 1$, $\mathcal{H}_n^\Lambda$ is an Ariki-Koike algebra, which is a deformation of the group algebra of $\mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$. If $\xi^2 = 1$ then $\mathcal{H}_n^\Lambda$ is a degenerate Ariki-Koike algebra.

If $\ell = 1$ and $\xi^2 = 1$ then $\mathcal{H}_n^\Lambda \cong k\mathfrak{S}_n$.

Theorem (Ariki-Koike)

The algebra $\mathcal{H}_n^\Lambda$ is free as a $k$-module with basis

$$\{ L_1^{a_1} \cdots L_n^{a_n} T_w \mid 0 \leq a_k < \ell \text{ and } w \in \mathfrak{S}_n \}.$$ 

In particular, $\mathcal{H}_n^\Lambda$ is free of rank $\ell^n n! = \#(\mathbb{Z}/\ell \mathbb{Z} \wr \mathfrak{S}_n)$.

Seminormal forms for cyclotomic KLR algebras

If $t$ is a standard tableau the $e$-residue sequence of $t$ is the sequence $i^t = (i_1^t, \ldots, i_n^t)$, where $i_n^t = c_m(t) + e \mathbb{Z}$.

Brundan and Kleshchev obtained the following analogue of the seminormal form for $\mathcal{B}_n^\Lambda \cong k\mathfrak{S}_n$ ($e=p$) or, more generally, $\mathcal{B}_n^\Lambda \cong \mathcal{H}_n^\Lambda(\mathfrak{S}_n)$.

Proposition (Brundan-Kleshchev)

Suppose that $k$ is a field, $\ell = 1$ and $e > n$. Then for each partition $\lambda$ there is a (unique) irreducible graded $\mathcal{B}_n^\Lambda$-module, or $k\mathfrak{S}_n$-module, $S^\lambda$ with basis

$$\{ \nu_t \mid t \in \text{Std}(\lambda) \}$$ 

such that $\deg \nu_t = v_t = 0$ for all $t \in \text{Std}(\lambda)$ and

$$1_t \nu_t = \delta_{1_t \nu_t}, \quad y_r \nu_t = 0 \quad \text{and} \quad \psi_r \nu_t = \nu_{a_t r}.$$ 

Kleshchev and Ram generalised this result to give seminormal forms for irreducible graded $\mathcal{B}_n^\Lambda$-modules that are concentrated in one degree, for $C$ a Cartan matrix of finite type.

All of these modules belong to semisimple blocks.

We want to find seminormal forms that we can use to understand non-semisimple blocks of $\mathcal{B}_n^\Lambda$. Our starting point was a deformation of Brundan and Kleshchev’s graded isomorphism theorem by Hu-M.
Examples of content systems

- If $\Gamma = A_\infty \sqcup \cdots \sqcup A_\infty$, so that $I = J$, then $r(k, a) = (k, a)$ and $c(k, a) = 0$
  - is a content system with coefficients in $\mathbb{Z}$

- If $\Gamma$ is a quiver of type $A_{e+1}$ then a content system is given by:
  \[
  r_0 1 2 \ldots e 0 1 \ldots \\
  c 0 x 2x \ldots ex (e+1)x (e+2)x \ldots 
  \]

- If $\Gamma$ is a quiver of type $C_{e+1}$ then
  \[
  r_0 1 \ldots e 1 \ldots 1 0 1 \ldots \\
  c 0 x (e-1)x (ex) (e+1)x \ldots (2e-1)x (2e+1)x \ldots 
  \]
  Generically, content systems are defined over $\mathbb{Z}[x, x_1, \ldots, x_d]$.

All of the content systems above are defined over $k[x]$ and there is a natural (homogeneous) specialisation map

\[ \mathcal{R}^\Lambda_n(Q_I, K_I) \rightarrow \mathcal{R}^\Lambda_n \]

given by tensoring with $\mathbb{Z}[x]/x\mathbb{Z}[x]$ — that is, specialising $x$ to 0.

Content systems corresponding to $\mathcal{R}^\Lambda_n$ under specialisation are not unique.

Semisimplicity and uniqueness

Theorem (Young's seminormal form, 1901)

For each partition $\lambda$ there is a (unique) absolutely irreducible $Q_\Sigma_n$-module $S^\lambda$ with basis $\{ v_t \mid t \in \operatorname{Std}(\lambda) \}$ such that

\[ s_k v_t = \frac{1}{\rho_k(t)} v_t + \frac{1+\rho_k(t)}{\rho_k} v_{s_k t} \quad \text{and} \quad L_k v_t = c_k(t) v_t \]

where $\rho_k(t) = c_{k+1}(t) - c_k(t)$ and $v_{s_k t} = 0$ if $s_k t \notin \operatorname{Std}(\lambda)$.

Theorem (Homogeneous seminormal form, 2017)

For each partition $\lambda$ there is a (unique) absolutely irreducible $Q(x)\Sigma_n$-module $S^\lambda_{Q(x)}$ with basis $\{ v_t \mid t \in \operatorname{Std}(\lambda) \}$ such that

\[ \psi_k v_t = \frac{1}{x \rho_k(t)} v_t + \frac{1+\rho_k(t)}{\rho_k} v_{s_k t} \]
\[ 1_i v_t = \delta_{i1} v_t \quad \text{and} \quad y_k v_t = x c_k(t) v_t \]

Seminormal representations

Fix a content system $(c, r)$ for $\mathcal{R}^\Lambda_n(Q_I, K_I)$.

If $t$ is a standard tableau and $m$ appears in row $a$, column $b$ and component $k$ of $t$ define $c_m(t) = c(k, b - a)$ and $r_m(t) = r(k, b - a)$.

Set $c(t) = (c_1(t), \ldots, c_n(t))$ and $r(t) = (r_1(t), \ldots, r_n(t))$.

Then $s = t$ if and only if $c(s) = c(t)$ and $r(s) = r(t)$, for $s, t \in \operatorname{Std}(P^\lambda_n)$.

Proposition

Let $K$ be the field of fractions of $k$ and suppose that $\lambda \in P^\Lambda_n$. Then there exists a (unique) irreducible graded $\mathcal{R}^\Lambda_n(Q_I, K_I)$-module $\mathcal{S}^\Lambda_n$ with basis $\{ v_t \mid t \in \operatorname{Std}(\lambda) \}$ such that

\[ 1_i v_t = \delta_{i1} v_t \]
\[ y_k v_t = c_k(t) v_t \]
\[ \psi_k v_t = \beta_k(t) v_{s_k t} + \frac{\delta_{c_k(t) - c_k(t)}^{c_k(t) - 1}}{1} v_t \]

where $\{ \beta_k(t) \}$ is a set of scalars that satisfy some natural conditions.

Idea of proof

Check the relations.

Semisimplicity and uniqueness

Theorem (Evseev-M.)

Suppose that $\mathcal{R}^\Lambda_n(Q_I, K_I)$ has a content system over $k$ and let $K$ be the field of fractions of $k$. Then $\mathcal{R}^\Lambda_n(Q_I, K_I)$ is a split semisimple graded $K$-algebra that is canonically isomorphic to a cyclotomic quiver Hecke algebra for the quiver $A_\infty \sqcup \cdots \sqcup A_\infty$ with vertex set $J$.

In particular, different content systems determine the same algebra over $K$.

The proof follows by splitting the idempotents $1_i = \sum_{t \in \operatorname{Std}(I)} F_t$ and using this to construct an isomorphism.
Jantzen filtrations of Specht modules

The modules $S_\lambda^\Lambda$ and $S_\Lambda^\lambda$ come equipped with non-degenerate symmetric bilinear forms that are homogeneous of degree zero.

Let $A = \mathbb{Z}[x](x)$ be the localisation of $\mathbb{Z}[x]$ at the prime ideal $(x)$.

The Jantzen filtration of an $A$-module $M$ with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ is

$$J_k(M) = \{ m \in M \mid \langle m, a \rangle \in xA \text{ for all } a \in M \}$$

Let $M$ be the $R$-module obtained by specialising $x$ to 0.

The Jantzen filtration of $M$ is given by

$$J_k(M) = (J_k(M) + xM)/xM, \text{ for } k \geq 0$$

By construction, $J_k(M)/J_{k+1}(M)$ has a non-degenerate homogeneous bilinear form of degree $-2k$.

The aim is to find a Jantzen sum formula that explicitly describes $\sum_{k>0} J_k(M)$ in the Grothendieck group $\text{Rep}(R_n^\Lambda)$.

In particular, we have Jantzen filtrations of the graded Specht modules $S_\Lambda^\lambda$ and of the dual graded Specht modules $S_\Lambda^\lambda$.

Cellular bases

Theorem (Evseev-M.)

Suppose that $R_n^\Lambda(Q_I, K_I)$ has a content system over $k$. Then $R_n^\Lambda(Q_I, K_I)$ is a split semisimple graded cellular algebra.

Following Hu-M., there exist “integral elements” $\psi_{at}, \psi_{bt} \in R_n^\Lambda(Q_I, K_I)_{\mathfrak{K}}$.

Theorem (Evseev-M.)

Suppose that $R_n^\Lambda(Q_I, K_I)$ has a content system over $k$. Then $R_n^\Lambda(Q_I, K_I)_{\mathfrak{K}}$ is a graded cellular $k$-algebra with “dual” cellular bases $\{\psi_{at}\}$ and $\{\psi_{bt}\}$.

The proof of this theorem is quite hard: the problem is in showing that these elements span $R_n^\Lambda(Q_I, K_I)_{\mathfrak{K}}$ — for this we need Webster algebras.

Corollary (Evseev-M.)

Let $R_n^\Lambda$ be a quiver Hecke algebra of type $C_\infty^{(1)}$. Then $R_n^\Lambda$ is a graded cellular algebra.
Examples of Webster diagrams

Example Let $\ell = 1$, $\theta = (0)$ and $\lambda = (4, 2, 1)$. Then $N = 15 = L$ and $\SSS\theta(\lambda, \mu)$ contains the tableaux:

The corresponding Webster diagram $1_\lambda$ is:

![Webster Diagram Example](image)

The elements of $\mathcal{R}_n$ can be described diagrammatically:

We want similar, but more complicated diagrams, to define an algebra $\mathcal{W}_n^{\theta, \Lambda}$

Webster diagrams have three types of strings:

- Thick red vertical strings with $x$-coordinates $N\theta_1, \ldots, N\theta_\ell$
- Solid strings of residues $n_1, \ldots, n_L$, for some $i \in I^n$
- Dashed grey ghost strings that are translates, $L$-units to the left of the solid strings. A ghost string has the same residue as the corresponding solid string

Diagrams are defined up to isotopy and solid strings can have dots

The following crossings are not allowed for red, solid or ghost strings:

![Diagram of Allowed Crossings](image)

Examples of Webster diagrams II

Now let $\lambda = (2^2 | 2, 1)$, so that $N = 15$ and $L = 30$.

If $\theta = (0, 1)$ then $1_\lambda^1$ is the diagram

If $\theta = (0, 5)$ then $1_\lambda^1$ looks like:

Strings in diagrams from $\ell$-partitions “cluster” according to the diagonals:

![Diagram of Clustering](image)
Composing Webster diagrams

We compose Webster diagrams in the usual way: if \( D \) and \( E \) are Webster diagrams then the diagram \( D \circ E \) is \( 0 \) if their residues are different and when their residues are the same we put \( D \) on top of \( E \) and apply isotopy.

For example if \( D \) is the diagram

\[
\begin{array}{c}
\text{Diagram A} \\
\end{array}
\]

Let \( E \) be the diagram obtained by reflecting \( D \) in the line \( y = 0 \). Then \( D \circ E \) is the diagram

\[
\begin{array}{c}
\text{Diagram B} \\
\end{array}
\]

Relations for Webster algebras

...continued

- Strings can be pulled through crossings except for:
  \[
  \begin{aligned}
  \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle &= P_{i,s}(y_r, y_s) \quad \text{and} \quad \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle = P_{i,s}(y_r, y_s) \\
  \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle &= \delta_{ijr} \quad \text{and} \quad \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle = \delta_{ijr} \\
  \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle &= \delta_{ijk} \quad \text{and} \quad \langle \begin{array}{c}
  i & j \\
  \end{array} \rangle = \delta_{ijk}
  \end{aligned}
  \]

A diagrammatic cellular basis

Inside \( \mathcal{W}^\theta_\lambda \), for \( T \in T_{\text{Std}}(\theta, \mu) \) define the diagram \( C_T \) to be a Webster diagram with a minimal number of crossings such that for each node \((l, r, c) \in \lambda\) there is a solid string of residue \( \kappa_l + c - r + e \mathbb{Z} \) that starts with \( x \)-coordinate \( T(l, r, c) \in L_\theta(\mu) \) at the top of the diagram and that finishes with \( x \)-coordinate \( \theta(l, r, c) \in L_\theta(\lambda) \) at the bottom of the diagram.

The diagram \( C_T \) is not unique, in general.

Let \( C_T^* \) be the diagram obtained from \( C_T \) by reflecting it in the line \( y = 0 \).

Define \( C_{ST}^\theta = C_{ST} C_{ST}^* \).

Theorem (cf. Bowman, Webster)

The algebra \( \mathcal{W}^\theta_\lambda \) is spanned by the diagrams

\( \{ C_{ST}^\theta | S, T \in T_{\text{Std}}(\lambda) \} \)

Idea of proof First push all strings to the left so that they are concave, turning at the equator. This shows that if \( D \) is a Webster diagram then \( D \in \mathcal{W}^\theta_\lambda 1_\lambda \mathcal{W}^\theta_\lambda 1_\lambda \), for some \( \lambda \in \mathcal{P}\).

By resolving crossings it now follows that \( \mathcal{W}^\theta_\lambda \) is spanned by the \( \{ C_{ST}^\theta \} \).
Let $\omega_n = (0|\ldots|0|1^n)$ and that $\text{Std}_0(\Lambda) = S\text{Std}_0(\Lambda, \omega_n)$

**Theorem (cf. Bowman, Webster)**

There is an isomorphism of graded algebras $R_n^\Lambda \xrightarrow{\sim} 1_{\omega_n} \mathbb{W}^\Lambda \omega_n$

**Idea of proof** The isomorphism is given by:

$1_i \mapsto \begin{array}{c|c|c|c|c} \cdots & & & \cdots \\ \hline \rho_1 & \rho_\ell & i_1 & i_r & i_n \end{array}$

$y_r 1_i \mapsto \begin{array}{c|c|c|c|c} \cdots & & & \cdots \\ \hline \rho_1 & \rho_\ell & i_1 & i_r & i_n \end{array}$

$\psi_r 1_i \mapsto \begin{array}{c|c|c|c|c} \cdots & & & \cdots \\ \hline \rho_1 & \rho_\ell & i_1 & i_r & i_{r+1} \end{array}$

Pull all strings to the right and check the relations.