Dickson's lemma and weak Ramsey theory

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Introduction

This talk consists of 5 sections:

0. Introduction
1. Definitions
2. Equivalence
3. Weak Paris–Harrington–Ramsey numbers
4. Phase transition and higher dimensions

We consider *the weak Paris–Harrington principle* (WPH), in connection with *miniaturized Dickson's lemma* (MDL).

- WPH:
  A weak version of PH;
  originally used by Erdös and Mills (1981).
- MDL:
  A Friedman-style miniaturization of Dickson's lemma

Results

Our main result is:

- *WPH and MDL are equivalent.* (§2)

This equivalence is shown based on the construction between *bad colorings* and *sequences*.

This construction has consequences:

- A sharp classification of *weak Paris–Harrington–Ramsey numbers*. (§3)
- Bounds for *weak Ramsey numbers*. (§3)
- A *phase transition* for WPH. (§4)

Base theory RCA_0^*

We will work in RCA_0^* (Recursive Comprehension Axiom $*$).

RCA_0^* consists of...

- basic axioms together with $\exp$
- Σ_0^0-induction
- Δ_1^0-comprehension

$$\mathsf{RCA}_0^* = \mathsf{RCA}_0 - \Sigma_1^0\text{-ind} + \Sigma_0^0\text{-ind} + \exp$$

RCA_0^* is...

- Π_2^0-conservative over EFA (Elementary Function Arithmetic)
- conservative over $\mathsf{B}\Sigma_1^0$ (Σ_1^0 Bounding) $+ \exp$

Section 1

Definitions

Notation for colorings

In this talk: a, R, D, c(color)$, d$(dimension) $\in \mathbb{N}$, $f : \mathbb{N} \to \mathbb{N}$
identify: $R = \{0, .., R-1\}$

X: a set

- $[X]^2 = \{(m,n) \in X^2 \mid m < n\}$
  $=$ the set of (unordered) pairs in X
- $[X]^d = \{(m_0, \ldots, m_{d-1}) \in X^d \mid m_0 < \cdots < m_{d-1}\}$

For a while, d is 2.

- coloring: $C\colon [R]^2 \to c$
- $H \subseteq R$ is C-*homogeneous* if $C|_{[H]^2}$ is constant, i.e.
  $C(h,h') = C(h'',h''')$ for all $h < h', h'' < h'''$ in H.

FRT and PH

Let c (color) be given.

- Finite Ramsey's theorem for pairs:

$$(\mathrm{FRT}_c) \quad \begin{array}{l}(\forall a)\ (\exists R) \text{ s.t. for every } C\colon [R]^2 \to c \text{ there exists} \\ H \subseteq R \text{ which is } C\text{-homogeneous and } |H| > a.\end{array}$$

- The Paris–Harrington principle for pairs:

$$(\mathrm{PH}_c) \quad \begin{array}{l}(\forall a)\ (\exists R) \text{ s.t. for every } C\colon [R]^2 \to c \text{ there exists} \\ H \subseteq R \text{ which is } C\text{-homogeneous and } |H| > a + \min H.\end{array}$$

Weak FRT and weak PH

Given $C\colon [R]^2 \to c$,

- $H \subseteq R$ is C-*homogeneous* if $C|_{[H]^2}$ is constant, i.e. $C(h,h') = C(h'',h''')$ for all $h < h', h'' < h'''$ in H.
- $H = \{h_0 < h_1 < h_2 < \cdots\} \subseteq R$ is C-*weakly homogeneous* if $C(h_i,h_{i+1}) = C(h_{i+1},h_{i+2})$ for all h_i, h_{i+1}, h_{i+2} in H.

- Weak finite Ramsey's theorem for pairs:

(WFRT$_c$) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > a$.

- The weak Paris–Harrington principle for pairs:

(WPH$_c$) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > a + \min H$.

Appending function f

(WPH_c) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > a + \min H$.

We parametrize the "id" in "$a + \min H$":

(WPH_c^f) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > f(a + \min H)$.

(Note again that we treat pairs only, for a while)

Notation for sequences and Dickson's lemma:

For $\overline{m}, \overline{n} \in \mathbb{N}^c$, define $\overline{m} \leq \overline{n}$ if $(\forall k)\ (\overline{m})_k \leq (\overline{n})_k$
e.g. $(1,2,3) \leq (2,3,4)$, $(1,2,3) \not\leq (2,3,1)$

$\overline{m}_0, \overline{m}_1, \ldots$ in $\mathbb{N}^c$ is *good* if there exist $i < j$ such that $\overline{m}_i \leq \overline{m}_j$

A sequence which is not good (i.e. $\forall i < j\ \exists k\ (\overline{m}_i)_k > (\overline{m}_j)_k$) is *bad* sequence.

- Dickson's lemma:

(DL_c) Every infinite sequence $\overline{m}_0, \overline{m}_1, \ldots$ in $\mathbb{N}^c$ is good.

Miniaturizing Dickson's lemma

We consider the following *Friedman-style miniaturization* of Dickson's lemma:

- Miniaturized Dickson's lemma:

(MDL_c^f) $(\forall a)$ $(\exists D)$ s.t. every sequence $\overline{m}_0, \ldots, \overline{m}_D$ in $\mathbb{N}^c$ with $(\forall i)$ $|\overline{m}_i|_\infty < f(a+i)$ is good

where $|\overline{m}|_\infty = \max_{k<c} (\overline{m})_k$ (max norm).

(We have function parameter f again)

Section 2

Equivalence

Main theorem

Theorem 1

For every c and f, WPH^f_c and MDL^f_c are equivalent.

(proof)

- Say $C\colon [R]^2 \to c$ is (a,f)-*bad* if for every C-weakly homogeneous set $H \subseteq R$, $|H| \leq f(a + \min H)$.
- $\overline{m}_0, \dots, \overline{m}_D$ in $\mathbb{N}^c$ is (a,f)-*bounded* if $(\forall i)\ |\overline{m}_i|_\infty < f(a+i)$. $|\overline{m}_i|_\infty < f(a+i)$ for all i. Call (a,f)-bounded bad sequences (a,f)-*bad*.

A bad coloring/sequence is a counter-example for $\mathrm{WPH}^f_c/\mathrm{MDL}^f_c$. The theorem is a direct consequence of the next lemma:

Construction of bad colorings/sequences

Lemma

1. Existence of an (a, f)-bad coloring $C\colon [R]^2 \to c$ implies existence of an (a, f)-bad sequence $\overline{m}_0, \ldots, \overline{m}_R$.
2. Existence of an (a, f)-bad sequence $\overline{m}_0, \ldots, \overline{m}_D$ implies existence of an (a, f)-bad coloring $C\colon [D]^2 \to c$.

(Sketch of the proof)

1. $\overline{m}_0, \ldots, \overline{m}_R$ is defined as: $(\overline{m}_i)_k > (\overline{m}_j)_k$ whenever $C(i, j) = k$.
2. $C\colon [D]^2 \to c$ is defined as: $C(i, j) = k$ where $(\overline{m}_i)_k > (\overline{m}_j)_k$.

Corollary: relativized WPH and DL

Theorem 1

For every c and f, WPH_c^f and MDL_c^f are equivalent.

Corollary 2

For every c, $\forall f \mathrm{WPH}_c^f$ and DL_c are equivalent.

Note:

- $\mathrm{DL}_c \leftrightarrow \mathrm{WO}(\omega^c)$
- $\mathrm{WO}(\omega^{c+4}) \rightarrow \forall f \mathrm{PH}_c^f$ and the converse is not known

Section 3

Weak Paris–Harrington–Ramsey numbers

Weak Paris–Harrington–Ramsey numbers for pairs

(WPH_c^f) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > f(a + \min H)$.

Define $R_c^f(a) =$ the least R such that this holds.

(MDL_c^f) $(\forall a)$ $(\exists D)$ s.t. every sequence $\overline{m}_0, \dots, \overline{m}_D$ in $\mathbb{N}^c$ with $(\forall i)$ $|\overline{m}_i|_\infty < f(a+i)$ is good.

Define $D_c^f(a) =$ the least D such that this holds.

Our construction shows:

Corollary 3

$$R_c^f(a) = D_c^f(a)$$

Classification for R_c^f

This gives a classification for R_c^f in *the fast growing hierarchy*, derived from those for D_c^f (Schnoebelen et al. 2011):

Corollary 4

Let $\gamma \geq 1$.
If f is nondecreasing and a proper member of $\mathfrak{F}_\gamma$, then R_c^f is a proper member of $\mathfrak{F}_{\gamma+c-1}$.
Where $\mathfrak{F}_\gamma$ is the class of γ-th level in the fast growing hierarchy.

$\mathfrak{F}_\gamma$ is the smallest class (containing some basic functions and the γ-th fast growing function F_γ) which is closed under composition and bounded primitive recursion.

Weak Ramsey numbers for pairs

Define $wr_c(a)$ = the least R which witnesses

(WFRT$_c$) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^2 \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > a$.

(WFRT$_c$ is $\mathrm{WPH}_c^{f_a}$ where f_a is the constant function $x \mapsto a$)

$R_c^f(a) = D_c^f(a)$ implies the following simple formula:

Theorem 5

$$wr_c(a) = a^c$$

(Note: For (normal, not weak) Ramsey number $r_c(a)$, $r_2(5)$ is not known.)

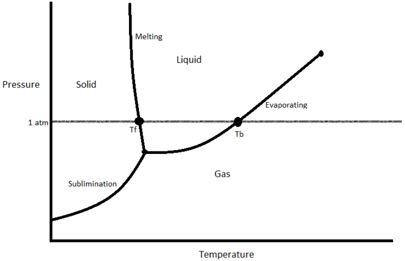
Section 4

Phase Transition and higher dimensions

Phase transition

φ_f: a statement which has a parameter $f\colon \mathbb{N} \to \mathbb{N}$.
Phase Transition for φ_f (over T): Find functions
$f_0 < f_1 < f_2 < \cdots \quad < f$ such that

$$(\forall n)\ T \vdash \varphi_{f_n} \qquad \text{and} \qquad T \nvdash \varphi_f$$



E.g. $f_n = n$-th fast growing function, $f =$ Ackermann function then

$$(\forall n)\ \mathrm{I}\Sigma_1 \vdash \mathrm{Tot}(f_n) \qquad \text{and} \qquad \mathrm{I}\Sigma_1 \nvdash \mathrm{Tot}(f)$$

Phase transition for WPH (for pairs)

We drop colors: $\mathrm{WPH}^f \equiv \forall c \mathrm{WPH}^f_c$

$wr_c(a) = a^c$ gives the following:

Theorem 6

1. RCA_0^* (or EFA) proves WPH^f for $f(x) = \log(x)$.
2. For all n, $\mathsf{RCA}_0^* + \mathrm{I}\Sigma^0_1$ (or PRA) does not prove WPH^{f_n} where $f_n(x) = \sqrt[n]{x}$.

For higher dimension

We will extend WPH for higher dimensions.
Firstly define weak Ramsey number for dimension d:

Given d, c and a, $wr_c^d(a)$ is the least R such that

(WFRT_c^d) $(\forall a)$ $(\exists R)$ s.t. for every $C\colon [R]^d \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > a$.

Lemma

For $a \geq d \geq 1$ and $c \geq 1$,

1. $wr_c^d(a) \leq M \Rightarrow wr_c^{d+1}(a) \leq 2^{M^{d+1}}$,
2. $wr_c^d(a) \geq M \Rightarrow wr_{5c}^{d+1}(a) \geq 2^M$.

Bounds for wr_c^d

By induction on d:

Theorem 7

1. For each (standard) $d \geq 2$,

$$wr_c^d(a) \leq 2^{\cdot^{\cdot^{\cdot^{2^{a^{k_0 c}}}}}} \Big\} (d-2) \text{ 2's}$$

where $k_0 = (d+1)!$.

2. For each (standard) $d \geq 2$, $c \geq 1$ and $a \geq d$,

$$wr_{k_1 c}^d(a) \geq 2^{\cdot^{\cdot^{\cdot^{2^{a^{c}}}}}} \Big\} (d-2) \text{ 2's}$$

where $k_1 = 5^{d-2}$.

Phase transition for WPH in higher dimensions

We use superscript to denote dimension:

$(\mathrm{WPH}^{d,f})$ $(\forall c)$ $(\forall a)$ $(\exists R)$ s.t. for every C: $[R]^d \to c$ there exists $H \subseteq R$ which is C-weakly homogeneous and $|H| > f(a + \min H)$.

Bounds for $wr_c^d(a)$ give us:

Theorem

Let $d \geq 2$ standard.

1. RCA_0^* (EFA) proves $\mathrm{WPH}^{d,f}$ for $f(x) = \log^{(d-1)}(x)$.
2. For all n, $\mathsf{RCA}_0^*(\mathsf{EFA}) + \mathsf{I}\Sigma^0_{d-1}$ does not prove WPH^{d,f_n} where $f_n(x) = \sqrt[n]{\log^{(d-2)}(x)}$.

Thank you very much!

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