Set-theoretic geologies

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Abstract

Set-theoretic geology is a study of the structure of all ground models of the universe V. We will show that the ground models are downward directed, and observe some properties following from the downward directedness.
We also consider some variants of set-theoretic geology:

- Set-theoretic geology without the Axiom of Choice.
- Set-theoretic geology of pseudo-grounds.

What is “forcing”

- For a model M of set-theory, forcing is a procedure to construct an extension model $M[G]$ of M via some poset $\mathbb{P} \in M$ and a filter $G \subseteq \mathbb{P}$. $M[G]$ is called a generic extension or forcing extension of M. M is a ground model of $M[G]$.
- The structure of $M[G]$ strongly depend on the choice of $\mathbb{P}$, e.g., there is some poset $\mathbb{P}$ and $G \subseteq \mathbb{P}$ such that the Continuum Hypothesis holds in $M[G]$, and there is another $\mathbb{Q}$ and $H \subseteq \mathbb{Q}$ such that the Continuum Hypothesis fails in $M[H]$.
- By forcing method, set-theorists has constructed many various extensions, and it turns out that there are many statements which are independent from ZFC.
- On the other hand, generic extensions and ground models are second order objects, and it would be hard to treat it in first order theory directly...

Definability of ground models

A ground means a ground model,

Fact (Laver, Woodin)

In the forcing extension $V[G]$ of V, the universe V is a (first order) definable class in $V[G]$ with some parameters from V.

In other words:

If $M \subseteq V$ is a ground of V, then M is definable.

So every ground of V is definable by some first order formula.

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Uniform definability of grounds

Actually all grounds can be defined by some uniform way.

Fact (Fuchs-Hamkins-Reitz)

There is a first order formula $\varphi(x, y)$ such that:

1. *For each set r, the class $W_r = \{x : \varphi(x, r)\}$ is a ground of V ($W_r = V$ is possible).*
2. *For every inner model $M \subseteq V$ of ZFC, if M is a ground of V, then there is r with $M = W_r$.*

Remark

The statement "$M \subseteq V$ is a model of ZFC" is expressible by one first order sentence of language $\{\in, M\}$.

Set-theoretic geology

This result allow us to study the structure of the collection of grounds $\{W_r : r \in V\}$ in ZFC:e.g.,

- One can define (in ZFC) the intersection of two grounds.
- One can ask (in ZFC) whether $\forall r \exists s\,(W_s \subsetneq W_r)$?

This study is now called set-theoretic geology.

Remark

- "... is a ground of ..." is a transitive relation on models.
- Set-theoretic geology is a study of this partial ordered set (frame) as well.

The mantle

The "mantle" is a natural concept indicated by uniform definability.

Definition

The mantle $\mathbb{M}$ is the intersection of all grounds of V.

The mantle is a first order definable class $\{x : \forall r\, \varphi(x, r)\}$.

- There are many open questions about the mantle.

An important question about the mantle is:

Question (Fuchs-Hamkins-Reitz)

Is the mantle a model of ZF or ZFC?

If V is $L[X]$, HOD, K, class forcing extensions of these models, or other known models, then the mantle is a model of ZFC.

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DDG

Another interesting question is the downward directedness of the grounds: Does every two grounds W_0, W_1 have a common ground $W \subseteq W_0, W_1$?

Definition (Fuchs-Hamkins-Reitz)

The downward directed grounds hypothesis (DDG, for short) is the assertion that every two grounds have a common ground: $\forall r_0, r_1 \exists r \, (W_r \subseteq W_{r_0} \cap W_{r_1})$.

The strong downward directed grounds hypothesis (strong DDG, for short) is the assertion that for every set X, the collection $\{W_r : r \in X\}$ of grounds have a common ground: $\forall X \exists r \, \forall s \in X \, (W_r \subseteq W_s)$.

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$\forall X \exists r\, \forall s \in X\, (W_r \subseteq W_s)$.

Fact (Fuchs-Hamkins-Rietz)

1. *Many known models such as* $L[X]$, HOD, K, *class forcing extensions*, $\ldots$ *satisfy the strong DDG.*
2. *If the strong DDG holds, then the mantle is a model of ZFC.*

Question (Fuchs-Hamkins-Reitz)

Does the DDG always hold? How is the strong DDG?

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Question (Fuchs-Hamkins-Reitz)

Does the DDG always hold? How is the strong DDG?

DDG is true

We can prove the strong DDG as a theorem of ZFC:

Theorem

The strong DDG always holds. Consequently, the mantle is a model of ZFC.

There are various consequences of this results:

1. The generic mantle, generic HOD,
2. Generic multiverse,
3. Modal logic of forcing, etc.

Generic Multiverse

Definition (Woodin)

A generic multivese is a collection $\mathcal{F}$ of (countable) models of ZFC such that:

1. If $M \in \mathcal{F}$ and N is a ground of M then $N \in \mathcal{F}$.
2. If $M \in \mathcal{F}$ and N is a generic extension of M then $N \in \mathcal{F}$.
3. For every $M, N \in \mathcal{F}$, there are finitely many $M_0, \dots, M_n$ such that $M_0 = M$, $M_n = N$, and each M_{i+1} is a forcing extension or a ground of M_i.

Consequences from DDG

Remark

Generic multivese is not upward directed.

Theorem

Let $\mathcal{F}$ be a generic multiverse.

1. For every $M, N \in \mathcal{F}$, there is a common ground $W \in \mathcal{F}$ of M and N, so $\mathcal{F}$ is downward directed.
2. For every $M, N \in \mathcal{F}$, $M \subseteq N$ if and only if M is a ground N.
3. The intersection $\mathcal{F}$ is a model of ZFC, and it is the mantle of some/any $M \in \mathcal{F}$.

Key tool for the proof

Definition

Let $M \subseteq V$ be a transitive model of ZFC. Let κ be a cardinal. M satisfies κ-uniform covering property for V if for every ordinal α and every function $f : \alpha \to ON$, there is $F \in M$ such that $F : \alpha \to [On]^{<\kappa}$ and $f(\beta) \in F(\beta)$ for $\beta < \alpha$.

Fact (Bukovsky)

Let $M \subseteq V$ be a transitive model of ZFC. Then the following are equivalent:

1. *M satisfies the κ-uniform covering property for V some κ.*
2. *M is a ground model of V.*

Using Bukovsk's theorem, we can constructed a common ground of given grounds.

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Bedrock and large cardinals

A bedrock is a minimal ground, a minimal element of the grounds of V.
Is there such element?

Fact (Fuchs-Hamkins-Reitz)

1. *It is consistent that V has a bedrock. Moreover there is a class forcing $\mathbb{P} \subseteq V$ such that if G is $(V, \mathbb{P})$-generic, then $\mathbb{M}^{V[G]} = V[G]$. This forcing notion preserves almost all large cardinals.*
2. *It is consistent that V has no bedrock. "No bedrock" is consistent with $\exists$ supercompact.*

However it is unknown whether "no bedrock exist" is consistent with large cardinals which are stronger than supercompact cardinals.
We will show that some large cardinal is inconsistent with "no bedrock".

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New large cardinal

Definition

An infinite cardinal κ is hyper huge if for every cardinal $\lambda > \kappa$, there is an inner model M of ZFC and an elementary embedding $j : V \to M$ such that:

1. The critical point of j is κ.
2. $\lambda < j(\kappa)$.
3. M is closed under $j(\lambda)$-sequences.

Super-2-huge $\Rightarrow$ hyper-huge $\Rightarrow$ superhuge $\Rightarrow$ supercompact limit of supercompact.

The mantle under very large cardinal

Theorem

Suppose hyper huge cardinal κ exists. Then V has only $< \kappa$ many grounds.

Consequently,

1. The mantle is a ground of V, hence V has a unique bedrock.
2. Moreover the mantle is a minimum universe of the generic mulitverse of V.
3. κ remains hyper-huge in the mantle.

- This means that if very large cardinal exists, then V must be very close to its "core".
- This also shows that there is some essential "gap" between supercompact cardinals and very large cardinals in the sense of forcing.

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FAQ

Someone had asked me the following:

If hyper-huge cardinal exists, is the mantle a model of $V =$ ultimate L?

- If V is a model of $V =$ ultimate L, then V satisfies $V = \mathrm{HOD}$ and CH.
- Moreover V is a minimum universe of the generic multiverse of V.
- If hyper-huge cardinal exists, then the mantle is a minimum universe of the generic multiverse of V.
- So, it is natural to ask if the mantle is a model of $V =$ ultimate L.

Answer is NO.

$V = \mathbb{M} + \exists$ hyper-huge $+ \neg$CH (or $V \neq \mathrm{HOD}$) is consistent.

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Geologies

We can consider various geologies of collections of models, and can ask the uniform definability, DDG, the mantle, etc.

1. $\{M : M$ is a definable model of ZFC$\}$ with the relation "... is a definable in ...".
2. $\{M : V$ is a class forcing extension of $M\}$ with "... is a class forcing ground of ...".
3. $\{M : M$ is a model of ZF and a ground model of $V\}$ with "... is a ground of ...".
4. $\{M : M$ is a pseudo-ground of $V\}$ with "... is a pseudo-ground of ...".
5. Suppose $\mathcal{F} \subseteq \mathcal{P}(V)$ and $(V, \mathcal{F})$ is a model of NGB or MK. $\{M \in \mathcal{F} : M$ is a model of ZFC$\}$. with relation "... is a submodel of ..."(?).

Question

- Which collection is uniformly definable?
- Which frame satisfy DDG? Is the mantle definable? Is it a model of ZFC?

Fact (Folklore)

It is possible that V is a class forcing extension of $M \subseteq V$ but M is not definable in V.

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Set-theoretic geology without AC

- It is possible that V satisfies AC but V has a ground which does not satisfy AC.
- Forcing over model of ZF is also useful to construct various models: e.g., Woodin's $\mathbb{P}_{max}$ over $L(\mathbb{R})$.

Question

- $\{M \subseteq V \mid M$ is a ground of V and is a model of ZF$\}$ with "$\ldots$ is a ground of $\ldots$".
- Suppose V satisfies only ZF.

Can we develop the set-theoretic geology without the Axiom of Choice?

- A first problem is the definablity of grounds of V.

Covering and approximation properties

In ZFC, Hamkins's covering and approximation properties are important tools for proving the (uniform) definability of grounds.

Definition (Hamkins)

Let κ be a cardinal, and $M \subseteq V$ an inner model of ZF.

1. M satisfies the κ-covering property if for every $a \in [\mathrm{ON}]^{<\kappa}$ there exists $b \in M \cap [\mathrm{ON}]^{<\kappa}$ with $a \subseteq b$.
2. M satisfies the κ-approximation property if for every set A of ordinals, $\forall a \in [ON]^{<\kappa} \cap M (A \cap a \in M)$ then $A \in M$.

Fact (Hamkins)

Let M, N be inner models of ZF. If M and N satisfies the κ-covering and the κ-approximation properties, and $\mathcal{P}(\kappa^+) \cap M = \mathcal{P}(\kappa^+) \cap N$, then $M \cap \mathcal{P}(ON) = N \cap \mathcal{P}(ON)$. Furthermore $M = N$ if M and N satisfy AC.

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Definition

An inner model $M \subseteq V$ of ZFC is a pseudo-ground if M satisfies the κ-covering and the κ-approximation properties for some κ.

One can check that, in ZFC, every ground is a pseudo-ground.

Fact (Hamkins, Fuchs-Hamkins-Reitz)

All pseudo-grounds are uniformly definable: There is a first order formula $\varphi(x, y)$ such that:

1. *For each set r, the class $W_r = \{x : \varphi(x, r)\}$ is a pseudo-ground of V.*
2. *For every pseudo-ground $M \subseteq V$, there is r with $M = W_r$.*

1. In the proof of the uniform definability of pseudo-grounds, the property $M \cap \mathcal{P}(ON) = N \cap \mathcal{P}(ON) \Rightarrow M = N$ is essential. However this is not valid if AC fails.
2. Moreover, it is not clear that, in ZF, every ground satisfies the covering and approximation properties.

Question (in ZF, open)

1. For every poset $\mathbb{P}$ and generic G, is V definable in $V[G]$?
2. Are all grounds of V uniformly definable?

Fact (Gitman-Johnstone, ZF)

Suppose DC_κ holds. Then for every poset $\mathbb{P}$ with size $< \kappa$ (hence $\mathbb{P}$ is assumed to be well-orderable), V is definable in $V^{\mathbb{P}}$ with some parameters.

Their result does not need the full AC but a weak AC, and it is not known if $\mathbb{P}$ is not well-orderable.

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New class of cardinals in ZF

For the uniform definability of grounds, we give a partial answer: It is possible if there are many good cardinals.

Definition (ZF)

We say that a cardinal κ is Löwenheim-Skolem if for every $\alpha > \kappa$, $\gamma < \kappa$, and $p \in V_\alpha$, there is $\beta > \alpha$ and $X \prec V_\beta$ such that:

1. $p \in X$
2. ${}^{V_\gamma}(X \cap V_\alpha) \subseteq X$.
3. The transitive collapse of X belongs to V_κ.

- In ZFC, Löwenheim-Skolem cardinal is not a large cardinal: κ is Löwenheim-Skolem if and only if $\beth_\kappa = \kappa$, so there are always proper class many Löwenheim-Skolem cardinals.

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Definition (Woodin, ZF)

An uncountable cardinal κ is supercompact if for every $\alpha > \kappa$, there is $\beta > \alpha$, an transitive set X, and an elementary embedding $j : V_\beta \to X$ such that the critical point of j is κ, $\alpha < j(\kappa)$, and ${}^{V_\alpha}X \subseteq X$.

- Every supercompact cardinal is Löwenheim-Skolem, and a limit of Löwenheim-Skolem cardinals.

Theorem (ZF)

Suppose there are proper class many Löwenheim-Skolem cardinals. Then all grounds are uniformly definable: There is a first order formula $\varphi(x, y)$ such that:

1. For each set r, the class $W_r = \{x : \varphi(x, r)\}$ is a ground of V.
2. For every ground $M \subseteq V$, there is r with $M = W_r$.

In particular, if there are proper class many supercompact cardinals, then all grounds are uniformly definable

Unfortunately, the existence of a Löwenheim-Skolem cardinal is not provable from ZF:

Lemma (ZF)

If κ is a singular limit of Löwenheim-Skolem cardinals, then κ^+ is regular, and the non-stationary ideal over κ^+ is κ^+-complete.

In Gitik's model with no regular uncountable cardinals, Löwenheim-Skolem cardinal does not exist.

Note that:

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If AC is forceable

Lemma (ZF)

The statement that "there are proper class many Löwenheim-Skolem cardinals" is forcing absolute.

This statement is provable from ZFC. Hence we have:

Corollary (ZF)

Suppose there is a poset which forces AC. Then there are proper class many Löwenheim-Skolem cardinals, and all grounds are uniformly definable.

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If $V = L(X)$ for some set X, then all grounds are uniformly definable.

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When AC is forceable

Fact (Blass, in ZF)

The following are equivalent:

1. *There is a poset which forces AC.*
2. *There is a set X such that for every set Y, there is an ordinal α and a sujection $f : X \times \alpha \to Y$.*

Theorem (ZF)

The following are equivalent:

1. There is a poset which forces AC.
2. There is a definable model M of ZFC and a set X such that $M(X) = V$, where $M(X)$ be the minimal model of ZF containing $M \cup \{X\}$.

(2) $\Rightarrow$ (1) is trivial. For the converse, we use the DDG.

When AC is forceable

Fact (Blass, in ZF)

The following are equivalent:

1. *There is a poset which forces AC.*
2. *There is a set X such that for every set Y, there is an ordinal α and a sujection $f : X \times \alpha \to Y$.*

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If AC is forceable...

Corollary (ZF)

Suppose AC is forceable.

1. Every ground of V is of the form $M(X)$ for some definable inner model M of ZFC and set X.
2. A weak form of the strong DDG holds: For every grounds $\{W_r : r \in R\}$, there is a forcing extension $V[G]$ of V and a ground W of $V[G]$ such that W satisfies AC, $W \subseteq \bigcap_{r \in R} W_r$ and $V = W(X)$ for some set X. (W is definable in V but would not be a ground of V).
3. The mantle is a model of ZF.

Remark

Woodin conjectured: In ZF, if there is a sufficiently large cardinal, then AC is forceable.

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Set-theoretic geology of pseudo-grounds

We return to ZFC-world.

All pseudo-grounds are uniformly definable, and we can consider the geology of all pseudo-grounds.

- There are some connections between pseudo-grounds, Woodin's weak extender models, and HOD-conjecture.

At this moment, however, we know few things about this geology, or even non-trivial models.

Question (open)

1. Does the DDG for pseudo-grounds hold? How is the strong DDG?
2. Let $p\mathbb{M}$, the pseudo-mantle, be the intersection of all pseudo-grounds. Is $p\mathbb{M}$ a model of ZFC?
3. Is it possible that $V = p\mathbb{M}$ (in non-trivial sense)?

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Weak extender model and pseudo-ground

Definition (Woodin)

Let κ be a supercompact. An innder model M of ZFC is a weak extender model for κ supercompact if for every cardinal λ, there is a normal measure U over $\mathcal{P}_\kappa\lambda$ such that $M \cap \mathcal{P}_\kappa\lambda \in U$ and $U \cap M \in M$.

Fact (Hamkins, Woodin)

Let M be a pseudo-ground (weak extender model). If there are proper class many supercompact cardinals, then so does in M. Supercompact cardinals can be replaced by extendible, huge, etc.

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Pseudo-ground = weak extender model

Lemma

1. *If κ is a supercompact and M is a weak extender model for κ supercompact, then M is a pseudo-ground.*
2. *If there are proper class many supercompact cardinals, then every pseudo-ground is a weak extender model for κ supercompact for some large κ.*

Under large cardinal assumption, set-theoretic geology of pseudo-grounds is a study of Wooding's weak extender models.

The pseudo-mantle would not be a pseudo-ground

Theorem (Recall)

If κ is hyper-huge, then $\mathbb{M}$ is a minimal ground of V.

For pseudo-grounds, an analog result does not hold:

Fact (Woodin, Sakai)

Let κ be a measurable cardinal, M a ultrapower of V by some normal measure over κ. If there is a supercompact cardinal $> \kappa$, then M is a pseudo-ground of V.

Corollary

If there are proper class many supercompact cardinals, then every pseudo-ground has a proper pseudo-ground. In particular $p\mathbb{M}$ is not a pseudo-ground.

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HOD

A set x is ordinal definable if there are ordinals $\alpha_0, \ldots, \alpha_n$ and a formula φ of set-theory such that

$$x = \{y : \varphi(y, \alpha_0, \ldots, \alpha_n)\}.$$

OD is the class of all ordinal definable sets, and HOD is the class of all hereditary ordinal definable sets.

Fact (Gödel(?))

HOD *is a first order definable model of ZFC.*

Unlike the constructible universe L, HOD is not absolute model;
It is consistent that $\mathrm{HOD}^{\mathrm{HOD}}$, the HOD defined in HOD, is strictly smaller than HOD.
However, recently Woodin conjectured HOD is very close to the universe V.

Jensen's Dichotomy

Fact (Jensen)

Let L be the constructible universe. Then exactly one of the following holds:

1. *For every singular cardinal λ, λ is singular in L and L computes λ^+ correctly (L is close to V), or*
2. *Every uncountable cardinal is inaccessible in L (L is far from V).*

Woodin's Dichotomy

An uncountable cardinal κ is extendible if for every ordinal $\alpha > \kappa$, there is $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ with critical point κ.
hyper-huge $\Rightarrow$ extendible $\Rightarrow$ supercompact.

Fact (Woodin)

Suppose κ is extendible. Then exactly one of the following holds:

1. *For every singular cardinal $\lambda \geq \kappa$, λ is singular in HOD and HOD computes λ^+ correctly (HOD is close to V), or*
2. *Every regular cardinal $\geq \kappa$ is measurable in HOD (HOD is far from V).*

Woodin's HOD-conjecture is the assertion that if there exists an extendible cardinal, then HOD is close to V, that is, (2) alway holds.

HOD-Conjecture

Fact (Woodin)

HOD*-conjecture is equivalent to the statement that if* κ *is extendible, then* HOD *is a weak extender model for* κ *supercompact.*

HOD-conjecture is "equivalent" to the statement that HOD is a pseudo-ground.

Lemma

Suppose HOD-Conjecture is true. If there are proper class many extendible cardinals, and the DDG for pseudo-grounds holds, then the strong DDG for pseudo-grounds holds as well. In particular the pseudo-mantle is a model of ZFC.

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Iterated HOD-construction

If HOD-conjecture is true and there are large cardinals, then HOD is a pseudo-ground, and HOD of HOD is a pseudo-ground as well. Can we iterate this procedure?

Definition

Define HOD^α by:

- $\mathrm{HOD}^0 = V$.
- $\mathrm{HOD}^{\alpha+1} = (\mathrm{HOD})^{\mathrm{HOD}^\alpha}$.
- $\mathrm{HOD}^\alpha = \bigcap_{\beta<\alpha} \mathrm{HOD}^\beta$ if α is limit.

Fact

1. *(McAloon) It is consistent that AC fails in* HOD^ω.
2. *(McAloon, Harrington) It is consistent that the class* $\{\mathrm{HOD}^n : n < \omega\}$ *is not defiable, and* HOD^ω *is not a model of ZF.*

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Theorem

Suppose HOD-conjecture is "true".

1. If there are proper class many extendible cardinals, then $\{\mathrm{HOD}^\alpha : \alpha \in ON\}$ is definable in V, and HOD^α is a model of ZF for every limit α.
2. Furthermore if κ is a hyper-huge cardinal, then there is $\alpha < \kappa$ such that $\mathrm{HOD}^{\alpha+1} = \mathrm{HOD}^\alpha$, so HOD^α is a model of $V = \mathrm{HOD}$. Moreover, the mantle of HOD^α is a model of $V = \mathrm{HOD} = \mathbb{M} + \kappa$ is hyper-huge.

- The mantle of HOD^α is a highly canonical model of $V = \mathrm{HOD} = \mathbb{M}$,
- Every pseudo-ground has a pseudo-ground satisfying $V = \mathrm{HOD} = \mathbb{M}$.
- Again, however, the mantle of HOD^α would not be a model of $V =$ ultimate L.

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Thank you for your attention!