On the first-order part of Ramsey's theorem for pairs

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Ramsey's theorem

We will argue in RCA_0.

Definition (Ramsey's theorem.)

- RT^n_k: for any $P : [\mathbb{N}]^n \to k$, there exists an infinite set $H \subseteq \mathbb{N}$ such that $|P([H]^n)| = 1$.
- $\mathrm{RT}^n := \forall k\, \mathrm{RT}^n_k$.
- $\mathrm{RT} := \forall n\, \mathrm{RT}^n$.

Proposition (RCA_0)

1. *If* $n' \leq n, k' \leq k$, *then* $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^{n'}_{k'}$.
2. $\mathrm{RT}^n_k \Rightarrow \mathrm{RT}^n_{k+1}$.

Ramsey's theorem

Proposition (RCA_0)

For any $n \in \omega$, $\mathrm{RT}^{n+1}_2 \Rightarrow \mathrm{RT}^n$.

Theorem (Jockusch/Simpson)

- ACA_0 *proves* RT^n_k *for any* $n, k \in \omega$.
- *Over* RCA_0, RT^3_2 *implies* ACA_0.

Thus,

$$\mathsf{RCA}_0 = \mathrm{RT}^1_2 \leq \mathrm{RT}^1 \leq \mathrm{RT}^2_2 \leq \mathrm{RT}^2 \leq \mathrm{RT}^3_2 = \mathrm{RT}^3 = \cdots = \mathsf{ACA}_0$$

Computability theoretic strength of RT^2_2

- $\mathrm{RCA}_0 \nvdash \mathrm{RT}^2_2$. (Specker 1971)
  $\Uparrow$ there exists a computable coloring for pairs
  which has no computable homogeneous set.
  Later, $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \vdash \mathrm{DNR}$ (HJKLS 2008).
- $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{RT}^3_2$. (Seetapun 1995)
  $\Uparrow$ Cone avoidance theorem.
  Later, low_2-basis theorem (CJS 2001).
- $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \nvdash \mathrm{WKL}_0$. (Liu 2011)
- (and many more works, see, 'Slicing the truth' by Hirschfeldt.)

We have the following separation on ω models,

$$\begin{array}{ccccc}
\mathrm{RT}^1_2(\leq \mathrm{RT}^1) & < & \mathrm{RT}^2_2(\leq \mathrm{RT}^2) & < & \mathrm{RT}^3_2 \\
\| & & \not\curlyvee \not\curlywedge & & \| \\
\mathrm{RCA}_0 & < & \mathrm{WKL}_0 & < & \mathrm{ACA}_0
\end{array}$$

How about first-order consequences?

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First-order part and ω-extensions

Let $\mathrm{RCA}_0 \subseteq T_0 \subseteq T_1$ be $\mathcal{L}_2$-theories.

Theorem (ω-extension property)

Assume that T_0 and T_1 satisfy the following condition:

- *for any countable model $(M, S) \models T_0$ and $A \in S$, there exists $\bar{S} \subseteq \mathcal{P}(M)$ such that $A \in \bar{S}$ and $(M, \bar{S}) \models T_1$.*

Then, T_1 is a Π^1_1-conservative extension of T_0.

Theorem (Conservation results by ω-extension property)

- RCA_0 *and* WKL_0 *are* Π^1_1*-conservative extensions of* $\mathrm{I}\Sigma^0_1$*. (Harrington, et al.)*
- $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ *and* $\mathrm{WKL}_0 + \mathrm{B}\Sigma^0_2$ *are* Π^1_1*-conservative extensions of* $\mathrm{B}\Sigma^0_2$*. (Hajek)*
- ACA_0 *is a* Π^1_1*-conservative extension of* PA ($\mathrm{I}\Sigma^0_{<\infty}$).

First-order part and cuts of nonstandard models

Theorem (cuts of nonstandard models)

Assume that T_0 and T_1 satisfy the following condition:

- *for any countable nonstandard model $(M, S) \models T_0$ and for any $\varphi(\bar{a}, \bar{A}) \in \Pi^0_n$ with $\bar{a} \in M$ and $\bar{A} \in S$, there exists a cut $I \subseteq_e M$ such that $\bar{a} \in I$ and $(I, \mathrm{Cod}(M/I)) \models T_1 + \varphi(\bar{a}, \bar{A} \cap I)$. (Here, $\mathrm{Cod}(M/I) = S \upharpoonright I := \{X \cap I : X \in S\}$.)*

Then, T_1 is a $\tilde{\Pi}^0_{n+1}$-conservative extension of T_0.
(Here $\tilde{\Pi}^0_n$-formula is of the form $\forall X \theta$ where θ is Π^0_n.)

- any cut preserves $\varphi \in \Pi^0_1$
- preserving Π^0_2-statement
  $\Leftrightarrow$ preserving the totality of a function
- preserving Π^0_3-statement
  $\Leftrightarrow$ preserving the divergence of the form $\lim_{n\to\infty} f(n) = \infty$
- ⋮

First-order part and cuts of nonstandard models

Theorem (cuts of nonstandard models)

Assume that T_0 and T_1 satisfy the following condition:

- *for any countable nonstandard model $(M, S) \models T_0$ and for any $\varphi(\bar{a}, \bar{A}) \in \Pi^0_n$ with $\bar{a} \in M$ and $\bar{A} \in S$, there exists a cut $I \subseteq_e M$ such that $\bar{a} \in I$ and $(I, \mathrm{Cod}(M/I)) \models T_1 + \varphi(\bar{a}, \bar{A} \cap I)$. (Here, $\mathrm{Cod}(M/I) = S \upharpoonright I := \{X \cap I : X \in S\}$.)*

Then, T_1 is a Π^0_{n+1}-conservative extension of T_0.

Theorem (Conservation results by cuts of nonstandard models)

- $\mathrm{B}\Sigma^0_2$ *is a Π^0_3-conservative extension of* $\mathrm{I}\Sigma^0_1$. *(Parsons/Paris/Friedman)*
- $\mathrm{I}\Sigma^0_1$ *is a Π^0_2-conservative extension of Primitive Recursive Arithmetic* (PRA). *(Parsons)*

Actually, one can prove the full Π^1_1-conservation by cuts of nonstandard models.

Proposition

For $n \in \omega$, WKL_0 is a $\tilde{\Pi}^0_{2n+1}$-conservative extension of $\mathrm{I}\Sigma^0_1$.

To show this, for given $M \models \mathrm{I}\Sigma^0_1$ and $\varphi \in \Pi^0_{2n}$, one needs to find a cut $I \subseteq_e M$ such that $(I, \mathrm{Cod}(M/I)) \models \mathsf{WKL}_0$ and I preserves φ.

- Consider a combinatorial condition to find a cut for WKL_0 preserving φ.

$\Rightarrow$ indicator argument

Indicators

Let T be a theory of second-order arithmetic.
A Σ_0-definable function $Y : [M]^2 \to M$ is said to be an *indicator* for $T \supseteq \mathrm{WKL}_0^*$ if

- $Y(x, y) \leq y$,
- if $x' \leq x < y \leq y'$, then $Y(x, y) \leq Y(x', y')$,
- $Y(x, y) > \omega$ if and only if there exists a cut $I \subseteq_e M$ such that $x \in I < y$ and $(I, \mathrm{Cod}(M/I)) \models T$.

  (Here, $Y(x, y) > \omega$ means that $Y(x, y) > n$ for any standard natural number n.)

Example

- $Y(x, y) = \max\{n : \exp^n(x) \leq y\}$ is an indicator for WKL_0^*.
- $$Y(x, y) = \max\{n : \text{any } f[[x, y]]^n \to 2 \text{ has a homogeneous set } Z \subseteq [x, y] \text{ such that } |Z| > \min Z\}$$

  is an indicator for ACA_0.

Basic properties of indicators

Theorem

If Y is an indicator for a theory T, then for any $n \in \omega$,

$$T \vdash \forall x \exists y Y(x, y) \geq n.$$

Theorem

If Y is an indicator for a theory T, then, T is a Π^0_2-conservative extension of $\mathrm{EFA} + \{\forall x \exists y Y(x, y) \geq n \mid n \in \omega\}$.

Let $F^Y_n(x) = \min\{y \mid Y(x, y) \geq n\}$.

Theorem

If Y is an indicator for a theory T and $T \vdash \forall x \exists y \theta(x, y)$ for some Σ_1-formula θ, then, there exists $n \in \omega$ such that

$$T \vdash \forall x \exists y < F^Y_n(x) \theta(x, y).$$

To find an indicator for $\mathsf{WKL}_0 + \varphi$, we will define a relation $X \Vdash^{\mathsf{WKL}_0}_m \varphi$ inductively. We will argue within RCA_0.
We write $\mathrm{para}(\varphi)$ for the max of number parameters in φ.

Definition (generalized m-largeness notion for WKL_0)

Let $\varphi \in \Pi^0_{2n}$. Let $X \subseteq_{\mathrm{fin}} \mathbb{N}$, and $m \in \mathbb{N}$.

- $X \Vdash^{\mathsf{WKL}_0}_0 \varphi$ if φ is Π^0_0 and $\varphi \wedge |X| > 2 \wedge \mathrm{para}(\varphi) < \min X$.
- $X \Vdash^{\mathsf{WKL}_0}_{m+1} \varphi$ if $m + 1 \geq n$ and
  - if $m \geq n$, then for any partition $Z_0 \sqcup \cdots \sqcup Z_{\ell-1} = X$ such that $\ell \leq Z_0 < \cdots < Z_{\ell-1}$, there exists $i < \ell$ such that $Z_i \Vdash^{\mathsf{WKL}_0}_m \varphi$, and,
  - if $\varphi \equiv \forall x \exists y \theta(x, y)$, then, for any $a < \min X$, there exists $Z \subseteq X$ and $b < \min Z$ such that $Z \Vdash^{\mathsf{WKL}_0}_m \varphi$ if $m \geq n$ and $Z \Vdash^{\mathsf{WKL}_0}_m \theta(a, b)$.

Note that for each $\varphi \in \Pi^0_{2n}$ "$X \Vdash^{\mathsf{WKL}_0}_m \varphi$" can be expressed by a Π^0_0-formula uniformly.

Put $Y_\varphi^{\mathrm{WKL}_0}(a,b) := \max\{m \mid [a,b] \Vdash_m^{\mathrm{WKL}_0} \varphi\}$.

Theorem

$Y_\varphi^{\mathrm{WKL}_0}$ *is an indicator for* $\mathrm{WKL}_0 + \varphi$.

By an easy combinatorics, we have

Lemma

For any $m \in \omega$ *and* $\varphi \in \Pi^0_{2n}$ *such that* $m \geq n$,

$$\mathrm{RCA}_0 \vdash \forall x \exists y Y_\varphi^{\mathrm{WKL}_0}(x,y) \geq m.$$

Proposition

For $n \in \omega$, WKL_0 *is a* $\tilde{\Pi}^0_{2n+1}$*-conservative extension of* $\mathrm{I}\Sigma^0_1$.

This argument can be reformulated by "forcing for generic cuts".
(We will see this later.)

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The first-order strength of Ramsey's theorem

Theorem

Over RCA$_0$,

1. RT^1_2 *is provable,*
2. RT^1 *is equivalent to* $\mathrm{B}\Sigma^0_2$,
3. *if* $n \geq 3$, RT^n_2 *is equivalent to* ACA$_0$.

Corollary

1. RCA$_0$ + RT^1_2 *is a* Π^1_1*-conservative extension of* $\mathrm{I}\Sigma^0_1$.
2. RCA$_0$ + RT^1 *is a* Π^1_1*-conservative extension of* $\mathrm{B}\Sigma^0_2$.
3. *For* $n \geq 3$, RCA$_0$ + RT^n_2 *and* RCA$_0$ + RT^n *are* Π^1_1*-conservative extensions of* PA.

How about RT^2_2 or RT^2?

The first-order strength of Ramsey's theorem for pairs

Theorem (Hirst)

Over RCA_0, RT^2_2 *implies* $\mathrm{B}\Sigma^0_2$ *and* RT^2 *implies* $\mathrm{B}\Sigma^0_3$.

Cholak/Jockusch/Slaman reformulated low_2-solution on nonstandard models, and obtained ω-extension property for RT^2_2 and RT^2.

Theorem (Cholak/Jockusch/Slaman)

1. $\mathsf{WKL}_0 + \mathrm{I}\Sigma^0_2 + \mathrm{RT}^2_2$ *is a* Π^1_1*-conservative extension of* $\mathrm{I}\Sigma^0_2$.
2. $\mathsf{WKL}_0 + \mathrm{I}\Sigma^0_3 + \mathrm{RT}^2$ *is a* Π^1_1*-conservative extension of* $\mathrm{I}\Sigma^0_3$.

$\mathrm{B}\Sigma^0_2 \leq (\mathsf{RCA}_0 + \mathrm{RT}^2_2)_{\Pi^1_1} \leq \mathrm{I}\Sigma^0_2$ and $\mathrm{B}\Sigma^0_3 \leq (\mathsf{RCA}_0 + \mathrm{RT}^2)_{\Pi^1_1} \leq \mathrm{I}\Sigma^0_3$.

The first-order strength of Ramsey's theorem for pairs

Here are the recent developments for RT^2_2 and RT^2.

Theorem (Chong/Slaman/Yang 2014)

$\mathsf{RCA}_0 + \mathrm{RT}^2_2$ *does not imply* $\mathrm{I}\Sigma^0_2$.

Theorem (Patey/Y)

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathrm{I}\Sigma^0_1$.

Theorem (Slaman/Y)

$\mathsf{WKL}_0 + \mathrm{RT}^2$ *is a* Π^1_1*-conservative extension of* $\mathrm{B}\Sigma^0_3$.

The first-order part of RT^2

Theorem (Slaman/Y)

$\mathrm{RCA}_0 + \mathrm{RT}^2$ *is a* Π^1_1*-conservative extension of* $\mathrm{B}\Sigma^0_3$.

This is an easy consequence of the following lemma.

Lemma

Let (M, S) *be a model of* $\mathrm{B}\Sigma^0_3$ *and let* $P : [M]^2 \to k$ *(*$k \in M$*) be a member of* S*. Then, there exists a set* $G \subseteq M$ *such that* $P \restriction [G]^2$ *is constant,* G *is unbounded in* M*, and* $(M, S \cup \{G\}) \models \mathrm{B}\Sigma^0_3$.

This is proved by showing that any coloring $P : [\mathbb{N}]^2 \to k$ has a low$_2$ homogeneous set (preserving $\mathrm{B}\Sigma^0_3$) and the construction refers to $\mathbf{0}''$ small number of times.

- Note that the proof provides feasible (canonical polynomial) proof-interpretation for Π^1_1-consequences.

Calibrating the first-order part of RT^2_2

Question

Is $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ a Π^1_1-conservative extension of $\mathrm{B}\Sigma^0_2$?

The answer is yes up to the level of Π^0_3.

Theorem (Patey/Y)

$\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathrm{I}\Sigma^0_1$.

This is proved by using cuts obtained by Paris's indicator argument.

Definition (RCA_0, Paris)

- A finite set $X \subseteq \mathbb{N}$ is said to be 0-*dense* if $|X| > \min X$.
- A finite set X is said to be $m+1$-*dense* if for any $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is m-dense and P-homogeneous.

Note that "X is m-dense" can be expressed by a Σ^0_0-formula.

Cuts for RT^2_2

Theorem (Bovykin/Weiermann)

If $(M, S) \models \mathrm{RCA}_0$ is countable nonstandard and $[a, b] \subseteq M$ is m-dense for any $m \in \omega$, then there exists a cut $a \in I \subseteq_e M$ such that $(I, \mathrm{Cod}(M/I)) \models \mathrm{WKL}_0 + \mathrm{RT}^2_2$.

Theorem (Patey/Y)

For any $m \in \omega$, RCA_0 proves the following:

$m\mathrm{PH}^2_2$: any infinite set contains m-dense set.

In fact, if X is ω^{300^m}-large then X is m-dense within RCA_0, which is shown only by finite cominatorics (Kołodziejczyk/Y).

Corollary

$\mathrm{WKL}_0 + \mathrm{RT}^2_2$ is a Π^0_2-conservative extension of RCA_0.

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Indicator plus forcing for generic cuts

To analyze Π^0_n-consequences for $n \geq 3$, we will sharpen the indicator argument.
(joint work with Kołodziejczyk, Wong, et al.)

- Let $M = (\mathbb{N}^M, S; U^M)$ be a countable model of $\mathrm{RCA}_0 +$"$U \subseteq \mathbb{N}$ is a proper cut"$+(\forall m \in U)(m\mathrm{PH}^2_2)$.

  (Any nonstandard model has an expansion for such U by putting $U^M = \omega$.)
- Within M, consider a poset $(\mathbb{P}, \trianglelefteq)$:

  $\mathbb{P} = \{Y \subseteq_{M\text{-fin}} M : Y$ is a-dense for some $a \notin U\}$,
  $Y \trianglelefteq X \Leftrightarrow Y \subseteq X$ (inclusion order, smaller set is strong).
- For a given generic filter G on $\mathbb{P}$, put

  $$I_G := \sup\{\min Y : Y \in G\} \subseteq_e M,$$

  then $M[G] := (I_G, \mathrm{Cod}(M/I_G))$ is a model of $\mathrm{WKL}_0 + \mathrm{RT}^2_2$.

Indicator plus forcing for generic cuts

Syntactical part is defined as follows: let $X \in \mathbb{P}$,

- if $\bar{a} \in \mathbb{N}$ and $\bar{A} \in [\mathbb{N}]^{<\mathbb{N}}$,
  $X \Vdash \psi(\bar{a}, \bar{A}) \Leftrightarrow \psi(\bar{a}, \bar{A} \cap [0, \max X]) \wedge \bar{a} < \min X$,
- $\wedge, \vee, \neg$ defined as usual,
- $X \Vdash \exists x \psi(x) \Leftrightarrow \forall Y \trianglelefteq X \, \exists Z \trianglelefteq Y \, \exists a < \min Z \; Z \Vdash \psi(a)$,
- $X \Vdash \exists X \psi(X) \Leftrightarrow \forall Y \trianglelefteq X \, \exists Z \trianglelefteq Y \, \exists A \subseteq [0, \max Z] \; Z \Vdash \psi(A)$.

For a given $\mathcal{L}_2$-formula ψ, "$X \Vdash \psi$" is $\Sigma^{0,U}_0$.

Theorem

$\mathrm{WKL}_0 + \mathrm{RT}^2_2$ is a Π^0_{n+1}-conservative extension of RCA_0+"U is a cut"+$\{\psi \to \exists X(X \Vdash \psi) : \psi \in \Pi^0_n\}$.

Eliminating "U is a cut"

Combine "density for RT^2_2" and generalized indicator for WKL_0.

Definition (generalized m-density notion for RT^2_2)

Let $\varphi \in \Pi^0_{2n}$. Let $X \subseteq_{\mathrm{fin}} \mathbb{N}$, and $m \in \mathbb{N}$.

- $X \Vdash_0 \varphi$ if φ is Π^0_0 and $\varphi \wedge |X| > 2 \wedge \mathrm{para}(\varphi) < \min X$.
- $X \Vdash_{m+1} \varphi$ if $m+1 \geq n$ and
  - if $m \geq n$, then for any partition $Z_0 \sqcup \cdots \sqcup Z_{\ell-1} = X$ such that $\ell \leq Z_0 < \cdots < Z_{\ell-1}$, there exists $i < \ell$ such that $Z_i \Vdash^{\mathsf{WKL}_0}_m \varphi$,
  - if $m \geq n$, then for any $P : [X]^2 \to 2$, there exists a P homogeneous set $Z \subseteq X$ such that $Z \Vdash_m \varphi$, and,
  - if $\varphi \equiv \forall x \exists y \theta(x, y)$, then, for any $a < \min X$, there exists $Z \subseteq X$ and $b < \min Z$ such that $Z \Vdash_m \varphi$ if $m \geq n$ and $Z \Vdash_m \theta(a, b)$.

Eliminating "U is a cut"

Proposition

If $\psi \in \Pi^0_{2n}$, $m \in \omega$ and $m \geq n$, then

$$\mathsf{WKL}_0 + \mathrm{RT}^2_2 \vdash \psi \to \exists X (X \Vdash_m \psi).$$

Given a cut U, put $\mathbb{P} = \{X : X \Vdash_a \psi \text{ for some } a \notin U\}$, then we have

- $X \Vdash_m \psi$ for any $m \in U \Rightarrow X \Vdash \psi$.

Thus, if $M \models \mathsf{RCA}_0 + \{\psi \to \exists X (X \Vdash_m \psi) : m \in \omega\}$ and M is nonstandard, then one can obtain a cut to be a model of $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ with forcing ψ.
(Put $U^M = \omega$.)

Theorem

$\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *is a* Π^0_{2n+1}*-conservative extension of* $\mathsf{RCA}_0 + \{\psi \to \exists X (X \Vdash_m \psi) : m \in \omega, \psi \in \Pi^0_{2n}, m \geq n\}$.

What is the first-order part of RT^2_2?

Question

Is $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ a Π^1_1-conservative extension of $\mathrm{B}\Sigma^0_2$?

The answer is yes if

- $\mathrm{RCA}_0 + \mathrm{B}\Sigma^0_2$ proves $\psi \to \exists X (X_m \Vdash \psi)$ for any $\psi \in \Sigma^1_0$ and $m \in \omega$.

This is true for the case $\psi \in \Pi^0_2$, thus we have Π^0_3-conservation:

- to force the totality of f defined by $\psi \in \Pi^0_2$ with $\mathrm{para}(\psi) < a$: if for any $x, y \in X$, $x < y \to f(x) < y$ and X is m-dense, then $X \Vdash_m f$ is total,
- one can find an m-dense set $X \subseteq \{a, f(a), f(f(a)), \dots\}$ in $\mathrm{I}\Sigma^0_1$.

Theorem (Patey/Y)

$\mathrm{RCA}_0 + \mathrm{RT}^2_2$ *is a* $\tilde{\Pi}^0_3$*-conservative extension of* $\mathrm{I}\Sigma^0_1$.

Feasible Π^0_3-conservation?

The previous argument may provide canonical proof-transformation.

Conjecture (Kołodziejczyk/Wong/Y)

There is a canonical polynomial proof transformation between $\mathsf{WKL}_0 + \mathrm{RT}^2_2$ *and* $\mathrm{I}\Sigma^0_1$ *for* $\tilde{\Pi}^0_3$*-formulas.*

For example, if a Π^0_2-formula $\forall x \exists y \theta(x, y)$ is provable from $\mathsf{WKL}_0 + \mathrm{RT}^2_2$, then one may feasibly extract a primitive recursive function $f : \omega \to \omega$ from the proof so that $\omega \models \forall x \exists y < f(x) \theta(x, y)$.

Thank you!

- Andrey Bovykin and Andreas Weiermann. The strength of infinitary Ramseyan principles can be accessed by their densities. to appear.
- Ludovic Patey and Y, The proof-theoretic strength of Ramsey's theorem for pairs and two colors, draft, available at `http://arxiv.org/abs/1601.00050`
- Theodore A. Slaman and Y, The strength of Ramsey's theorem for pairs and arbitrary many colors, draft.

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