Gödel's Programm and Ultimate L

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Outline of Topics

1. C.T.'s Problem
2. Gödel's Program
3. Ultimate L
4. Conclusion Remark.

C.T.'s Problem

Professor Chong Chi Tat asks the following question:

Would Gödel think "V=Ultimate L " is true?

Considering Gödel thought CH was false but CH will be true in the Ultimate L, this is a quite interesting question.

Why is C.T.'s Problem interesting?

Historically:

- What is exactly Gödel's attitude toward to CH?
- Is Gödel a platonist since 1925 as he himself claimed, or he actually once changed his mind?
- What is the relation between Gödel's Programe and Woodin's Ultimate L ?

Why is C.T.'s Problem interesting?

Philosophically:

- What is exactly the philosophical positions behind the Gödel's Program as well as the Ultimate L?
- Platonism or Anti-platonism, which one is more plausible in light of today's foundation study ?
- What can philosophy do for mathematics ?

Critiques on the philosophy of Gödel

......however one wishes to understand the statement about his earlier views, it seems clear that in 1933 Gödel's beliefs were quite different from those of his later years......
Much of the information that casts doubt on the uniformity of Gödel's metaphysical stance only came to light after his death through documents found in his Nachlass...... (Martin Davis, 2005.)

But I have suggested that—whatever his concept is—it does not fit well with his account of mathematical truth, especially with his contention that mathematical truths are analytic. (Donald Martin, 2005.)

In this talk, we will focus on the following two problems.

1. What is exactly Gödel's program?
2. Is the axiom of V=Ultimate L, if it proved to be 'true', a realization of Gödel's program ?

We try to show, through the discussions of these questions, that philosophy and mathematics have a more close relation than we have had supposed to.

Continuum Hypothesis

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- Cohen [1963]: CH cannot be proved from ZFC.

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- Gödel and Cohen's results show that CH is independent from ZFC: We can not determinate the truth of CH by our today's mathematical tools.
- But what does it mean?

Formalism

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- CH, perhaps the first significant question about uncountable sets which can be asked, has no intrinsic meaning. (Paul Cohen, 1964)
- My mental picture is that we have many possible set theories, all conforming to ZFC. I do not feel a universe of ZFC is like "the Sun", it is rather like " a human being" or "a human being of some fixed nationality". (Saharon Shelah, 2000)

Realism

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- As far as the most important open problem: CH, we believe that the process we described above leads in directions that will eventually refine our theory to the extent that we shall have a definite answer for the value of the Continuum as well as answers to many other independent problems. (Menachem Magidor, 2012)

Gödel's Program

What is it?

> *Decide mathematically interesting questions, especially CH, independent of ZFC in well-justified extensions of ZFC. (John Steel)*

Gödel's Program

What does "well-justified" mean?

- Intrinsic justifications. Axioms being intrinsically justified on the basis of the iterative concept of set. Examples: large cardinal axioms
- Extrinsic justifications. Axioms with fruitfulness in consequences. Examples: PD

Intrinsic justification

these axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only *unfold the content of the concept of set* as explained above.(Gödel, 1947)

Extrinsic justification

There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any established physical theory. (Gödel, 1947.)

Intrinsic v.s. Extrinsic

Is it possible that an axiom with intrinsic justifications but with rare consequences, or, vice versa, an axiom with abundant consequences but without of any intrinsic justifications?

The case of PD

Theorem

The following are equivalent.

1. *PD.*
2. *for each n, every consequence in the language of second order arithmetic of the theory ZFC plus there are n Woodin cardinals is true.*

Thus PD is precisely the instrumentalists trace of Woodin cardinals in the language of second-order arithmetic. (John Steel, 2007)

Remarks

- So PD (with extrinsic justifications) and there exist woodin cardinals (with intrinsic justifications) are proved to coincide.
- the fact that two apparently fruitful mathematical themes turn out to coincide makes it all the more likely that they are tracking a genuine strain of mathematical depth. (Pen Maddy, 2011)
- This sort of convergence of conceptually distinct domains is striking and unlikely to be an accident. (Koellner, 2006)

a belief of platonism.

Belief. Every axiom with intrinsic justifications will be proved to coincide with some axiom with extrinsic justifications.

Gödel's L.

- $L_0 = \emptyset$.
- $L_{\alpha+1} = \mathscr{P}_{def}(L_\alpha)$.
- $L_\gamma = \bigcup_{\beta<\gamma} L_\beta$

$$L = \bigcup_{\alpha \in On} L_\alpha.$$

Gödel's Axiom A

$$V = L$$

Gödel's L.

- L does provide a clear conception of the Universe of Sets. It has condensation, GCH is true in L, etc.
- But unfortunately, we have the following theorem.

Theorem (Danna Soctt, 1961)

In ZFC we can proof: If there exists a measurable cardinal, then $V \neq L$.

- So $V = L$ is false, since it limits the large cardinal axioms which are intrinsic justified.

Did Gödel once believe V=L

This question is discussed extensively, the following are the mostly often-quoted passage:

> *The proposition A [i.e., $V = L$] added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way...... (Gödel, 1938.)*

Did Gödel once believe V=L

Acceptance of V=L as an axiom of set theory would not be incompatible with the philosophical realism Gödel expressed later, although it would be with the mathematical views he expressed in concern with the continuum problem. But regarding the concept of an arbitrary infinite "vague notion" certainly does not square with Gödel's view in 1964 continuum problem has a definite answer. (Parsons, 1995)

Did Gödel once believe V=L ?

In this connection it is important that the consistency proof for A does not break down if stronger axioms of infinity (e.g. the existency of inaccessible numbers) are adjoined to T. Hence the consistency of A seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase. (Gödel, 1938.)

- Even in 1938, Gödel does not believe V=L unconditionally.
- He was quite aware of the importance of that it must be compatible with stronger axioms of infinity.
- At that time, inaccessible cardinal and Mahlo cardinal are known to be compatible with Axiom A and Scott's theorem would be proved 23 years later.
- So it could not be regarded as an evidence of he hold different philosophical positions at that time.

Gödel's Program

Summery.

1. CH must be either true or false.
2. ZFC is by far not complete.
3. New axioms which extend ZFC must be compatible with all the stronger axioms of infinity.
4. New axioms might be A-like.
5. In the enlargement of ZFC, CH might be false.

In the following we try to show: Woodin's Axiom of V=Ultimate L is compatible with all the aboves except the last one, so it could be regarded as a perfect realization of Gödel's Program.

Strengthen Axiom A.

Two ways to strengthen the Axiom A:

1. A more general concept of definability, e.g. Gödel's HOD.
2. The notion of relative definability, e.g. Kunen's L[U].

Gödel's HOD

According to Gödel 1946,

- We have an absolute concept of computability, that is, not depending on the formalism chosen.
- We can expect the same thing in the concept of demonstrability and definability.
- The concept of definability in terms of ordinal is absolute in a sense.

Gödel's HOD

Definition

A set x is ordinal-definable if x is definable in V from ordinal parameters. x is hereditary ordinal-definable if its transitive closure is ordinal definable.

HOD is a model of ZFC.

Kunen's L[U]

Theorem (Kunen)

Suppose U is a κ-complete normal non-principal ultrafilter on κ, then in $L[U]$, κ is the only measurable cardinal.

This is the start of inner model program.

Inner Model Program

- L is an inner model, that is, a model of ZFC with all ordinals in it.
- Try to find an inner model which could accommodate certain large cardinals.

The difficulties of Inner Model Program

- Each new construction of an enlargement of L meeting the challenge of a specific large cardinal axiom comes with a theorem that no stronger large cardinal axiom can hold in that enlargement.
- Since it seems very unlikely that there could ever be a strongest large cardinal axiom, this methodology seems unable by its very nature to ever succeed in providing the requisite axiom for clarifying the conception of the Universe of Sets. (Woodin, 2010)

Universality

The situation has now changed dramatically and there is for the first time a genuine prospect for the construction of an ultimate enlargement of L. This arises ...... from the unexpected discovery that at a specific critical stage in the hierarchy of large cardinal axioms, the construction of an enlargement of L compatible with this large cardinal axiom must yield......an enlargement which is compatible with all stronger large cardinal axioms.

Weak Extender Model

Definition

Suppose N is a transitive class and $N \models \text{ZFC}$. Then N is a *weak extender model for δ is supercompact*, if for all $\lambda > \delta$, there is a δ-complete normal fine ultrafilter U on $\mathscr{P}_\delta(\lambda)$ such that

1. $\mathscr{P}_\delta(\lambda) \cap N \in U$.
2. $U \cap N \in N$.

Weak Extender Model

If there is a generalization of L at the level of a supercompact cardinal then it should exist in a version which is a weak extender model for the supercompactness of some δ .

Universality Theorem

The importance of the concept of weak extender model lies in the following theorem:

Theorem (Univerality Theorem)

Suppose N is a weak extender model for δ is supercompact. Suppose $\kappa > \delta$ is a regular cardinal of N and

$$\pi : (H(\kappa^+))^N \to (H(\pi(\kappa)^+))^N$$

is an elementary embedding with $\mathrm{crt}(\pi) > \delta$, then $\pi \in N$.

That means: Large cardinal notions which is in V above δ must hold in N above δ. So, if we have such an N, the axiom V=N would be absolute in some sense.

But, is there a weak extender model for δ is supercompact? Let's recall Gödel's HOD.

HOD Dichotomy

Theorem

Assume that there is an extendible cardinals. Then exactly one of the following holds:

1. *For all singular cardinals γ, γ is singular in HOD, and moreover, $(\gamma^+)^V = (\gamma^+)^{HOD}$.*
2. *All cardinals greater than γ are ω-strongly measurable in HOD.*

That means HOD is either 'close' to V or very far from V.

But which side of the dichotomy are we on?

HOD Conjecture

HOD-Conjecture

In ZFC we can prove that HOD is close to V, i.e., the class

$$\{\kappa\colon \kappa \text{ is a regular cardinal but not an } \omega\text{-strongly measurable}\}$$

is a proper class in HOD.

HOD Conjecture and weak extender model

Theorem

Suppose κ is an extendible cardinal, then the followings are equivalent:

1. *HOD Conjecture is true.*
2. *HOD is a weak extender model for δ is supercompact.*

Ultimate L

Though a level by level definition of the Ultimate L (a weak extednder model for δ is supercomact) is still open, but there is a method for formulate the 'Axiom of V=Ultimate L' without knowing how to construct the model.

Definition

$\Theta = \sup\{\alpha \in Ord : \exists \pi : \mathbb{R} \to \alpha.\}$

Note. If AC holds, then $\Theta = \mathfrak{c}^+$ where $\mathfrak{c} = |\mathbb{R}|$.

Axiom of V=Ultimate L

Axiom. V=Ultimate L

1. *There is strong cardinal* κ *which is a limit of Woodin cardinals.*
2. *Suppose* φ *is a* Σ_3*-sentence such that* φ *holds in* V*. Then* $\exists A \subseteq \mathbb{R}$ *such that* A *is universally Baire and* $HOD^{L(A,\mathbb{R})} \cap V_\delta \models \varphi$*, where* $\delta \in \Theta^{L(A,\mathbb{R})}$.

Ultimate L Conjecture

Ultimate L Conjeture

Suppose δ is an extendible cardinal, then there exists a transitive class $N \subseteq V$ such that:

1. *N is a weak extender model for δ supercompact.*
2. *$N \models$ "$V =$ Ultimate L".*
3. *$N \subseteq HOD$.*

CH is true in Ultimate L

Theorem

Suppose Ultimate L Conjecture holds, then

1. *CH is true.*
2. *V=HOD.*

Conclusion Remark.

V=Ultimate L, if it is true, has the following consequences:

1. implies the Continuum Hypothesis.
2. reduces all questions of set theory to axioms of strong infinity (which are with intrinsic justification).
3. provides an axiomatic foundation for set theory which is immune to independence by Cohens method.

So it would be a realization of Gödel's Program.

Moreover, if V=Ultimate L is true, there would be a model of ZFC is the special one, in a sense, it is THE UNIVERSE of ZFC. This fact itself would be an evidence shows that the universe of sets is a well determined reality, could be regarded as a support to the philosophical position of Platonism.

A good question is: what would happen if the Ultimate L Conjecture is false?

Though Professor Woodin try to convince us that it would not be the case, we have many evidences toward a positive answer of the Ultimate L Conjecture, it is still possible that things turn into the other direction. So set theory come at a point with very different future: on the one hand,

- HOD Conjecture.
- Ultimate L Conjecture.
- Inner model theory for supercompact.

and on the other hand:

- Negation of HOD Conjecture.
- Negation of Ultimate L Conjecture.
- No Inner model theory for supercompact.

No matter which one would come to be true, philosophy will engage in more deeply than before.

Thank You !