Why do we study identically prepared state compression
POPULATION CODING

- "States" of a neuron: probability distributions of reactions to different stimuli.

\[
\rho_{\text{bite}} = 0.9|1\rangle\langle 1| + 0.1|0\rangle\langle 0| \quad \rho_{\text{touch}} = 0.2|1\rangle\langle 1| + 0.8|0\rangle\langle 0|
\]

1/0: a spike/no spike.

- A group of \( n \gg 1 \) neurons → tensor-power form states \( \rho_{\theta}^\otimes n \) (\( \theta = \text{bite/touch} \)).

- Population coding: the state \( \rho_{\theta}^\otimes n \) of a large group of neurons is a coding for the stimulus \( \theta \).
To build a robot using quantum techniques: "neurons" releasing different quantum states for different stimuli $\theta$.

Population coding $\rightarrow$ A population of quantum states $\rho_{\theta} \otimes^n$ carrying $\theta$.

Compression: reduce the cost of population transmission.
COMPRESS A QUANTUM POPULATION CODING

- Input: a quantum population $\rho_\theta^\otimes n$ with $\theta$ unknown.
- Encoder/Decoder characterized by quantum channels.
- Minimize the memory cost $\mathcal{M}$ (focusing on the leading order of $n$).
- Output: a state that has vanishing trace distance to $\rho_\theta^\otimes n$ (faithfulness).
MINIMUM DESCRIPTION LENGTH

- [Rissanen 84'] The shortest length to describe a $d$-parametric probability distribution $P_\theta(x)$ given $n$ samples $x_1, x_2, \ldots, x_n$:
  $$-\mathbb{E}_\theta \log P_\theta(x) + \frac{d}{2} \log n$$

  Entropy (data) Distribution

- A compression task aimed at reconstructing the (unknown) distribution (not the data, and thus beyond i.i.d.) [Rissanen & followups; Hayashi, Tan 17'].

- Nontrivial to generalize to quantum.
RELATED WORKS

- Identically prepared pure qubit states [Plesch, Buzek; PRA 10']: $\log n$ qubits. Demonstrated (3 copies $\rightarrow$ 2 copies) by Rozema et al.

- Mixed qubit states [YY, GC, MH; PRL 16']: $\log n$ qubits + $1/2 \log n$ bits (necessary only when the state's mixedness is unknown).

- Clock states [YY, GC, MH arxiv:1703.05876]: $1/2 \log n$ qubits.

- General finite dimensional systems [YY, GC, Ebler; PRL 16']: a protocol requiring $O(\log n)$-size memory. Not optimal in general.

- A general compression protocol, requiring the minimum total memory and less quantum memory?
MEMORY COST OF COMPRESSION

How many bits and qubits do we need to encode $\rho^{\otimes n}$?
CLASICAL AND QUANTUM PARAMETERS

A non-degenerate state family \(\{\rho_\theta^{\otimes n}: \theta = (\mu, \xi) \in \Theta\}\) is characterized by two kinds of parameters:

\[
\rho_\theta = U_\xi \rho_\mu U_\xi^\dagger
\]

- **Classical** (free) parameters \(\mu\): determining the spectrum
- **Quantum** (free) parameters \(\xi\): determining the eigenbasis
EXAMPLES

 Full qudit state family: \( f_c = d - 1 \) and \( f_q = d^2 - d \).

 Phase-covariant state family: \( f_c = 0 \) and \( f_q = d - 1 \)

\[
\rho_\theta = U_\theta \rho_0 U_\theta^\dagger \quad U_\theta = \sum_k e^{i\theta_k} |k\rangle\langle k|.
\]

 Classical distribution family: \( f_c = d - 1 \) and \( f_q = 0 \).

 Displaced thermal states at known/unknown temperature: \( f_c = 0/1 \) and \( f_q = 2 \)

\[
\rho_{\alpha,\beta} = D_\alpha \rho_\beta D_\alpha^\dagger \\
\rho_\beta = \sum_k (1 - \beta) \beta^m |m\rangle\langle m| \\
D_\alpha = e^{\alpha \hat{a}^\dagger - \bar{\alpha} \hat{a}}
\]
MEMORY COST OF THE COMPRESSION

➤ [Recall] Pure qubits:
\[ \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle \]
has two quantum parameters \((\theta, \phi)\) and requires \(\log n\) qubits.

➤ [Main result.] For each free parameter \(t\), it takes:

1. \((1/2 + \delta) \log n\) bits for \(t\) classical
2. \(1/2 \log n\) bits + \(\delta \log n\) qubits for \(t\) quantum

to encode faithfully the \(n\)-copy state.

➤ \(\delta > 0\) is a parameter independent of \(n\) (also affects the error), which can be arbitrarily close to zero.
OPTIMALITY

- Construct a mesh on $\Theta$ containing $n^{f/2-\delta}$ mutually distinguishable states for any $\delta > 0$. $f = f_c + f_q$.
- Consider any faithful compression protocol $(\mathcal{E}, \mathcal{D})$:

  \[ \theta \xrightarrow{\rho^{\otimes n}} \mathcal{E} \xrightarrow{\mathcal{M}} \mathcal{D} \xrightarrow{\tilde{\mathcal{M}}_{\tilde{\theta}}} \tilde{\theta} \]

- Can faithfully communicate $(f/2 - \delta)\log n$ bits of messages.
- The communication cost $\log |\mathcal{M}|$ cannot be smaller than the amount of messages.

\[ O(n^{-1/2+\alpha}), \alpha > 0 \Rightarrow \rho^{\otimes n}_{\theta_1} \text{ distinguishable from } \rho^{\otimes n}_{\theta_2} \]
COMPRESSION PROTOCOL

How to achieve the minimal memory cost
PROTOCOL FOR DISPLACED THERMAL STATES

- Displaced thermal states $\rho_{\alpha,\beta} = D_\alpha \rho_\beta D_\alpha^\dagger$ $\alpha \in \mathbb{C}$, where $D_\alpha$ is the displace operator and $\rho_\beta$ is a fixed thermal state (state of a system in equilibrium).

- Concentration of the displacement

$$\otimes^n_{Beam\ Splitters} \rho_{\alpha,\beta} \rightarrow \rho_{\sqrt{n}\alpha,\beta} \otimes \rho_\beta^{\otimes(n-1)}$$

- Photon number distribution of $\rho_{\sqrt{n}\alpha,\beta}$: concentrated in an-$O(n)$ window.

- Photon number truncation $\rho_{\sqrt{n}\alpha,\beta} \rightarrow (1 + \delta) \log n$ qubits for any $\delta > 0$.

- Two free parameters ($\alpha \in \mathbb{C}$), each around $1/2 \log n$ qubits.
PROTOCOL FOR QUDIT STATES

- Localization.

- Local asymptotic equivalence of $n$-tensor power qudit states and Gaussian (displaced thermal $\otimes$ classical Gaussian) states.

- Compression of Gaussian states (Solved already!).
LOCALIZATION

- Take out a negligible portion of $n^{1-\delta/2}$ copies and use them for tomography.

- Tomography pins $\theta$ to a neighborhood $\Theta_L$ of size $O(n^{-\frac{1}{2}+\frac{\delta}{3}})$ with exponentially vanishing error.

- Encode the tomography outcome into a classical memory, so that the overall quantum memory cost can be reduced.

- The same strategy can be used in the displaced thermal state case.

- Left with $n - n^{1-\delta/2}$ copies (the lost copies can be retrieved later by amplification).
QUANTUM LOCAL ASYMPTOTIC NORMALITY (Q-LAN)

- Q-LAN [Kahn, Guta; CMP 09']

In the neighborhood $\Theta_L$, $\rho_\theta \otimes^n$ is asymptotically equivalent to a classical-quantum Gaussian state:

$$\rho_\theta \otimes^n \overset{Q-LAN}{\leftrightarrow} \gamma_\theta = \gamma_{\mu}^{\text{class}} \otimes \gamma_\xi^{\text{quant}}$$

- Classical mode $\gamma_{\mu}^{\text{class}}$: a Gaussian distribution with $f_c$ variates;

- Quantum mode $\gamma_\xi^{\text{quant}}$: a multimode (number of modes depending on $f_q$) displaced thermal state.

- Problem reduced to compression of Gaussian distributions and displaced thermal states.
The compression error is upper bounded as

\[ \epsilon_n = O(n^{-\delta/2}) + O\left(n^{-\kappa(\delta)}\right), \]

where the latter is the error of Q-LAN. Especially, \( \kappa(\delta) > 0 \) for \( \delta \in (0, 2/9) \).

Faithfulness \( \lim_{n \to \infty} \epsilon_n = 0 \) is guaranteed as long as \( \delta > 0 \).

The error vanishes slower when less quantum memory is used.
QUANTUM MEMORY IS ESSENTIAL

Why fully classical memory doesn’t work
Is it possible to reversibly convert $\rho_\theta^\otimes n$ into classical bits with an error vanishing in $n$?

Fact: a state family can be perfectly compressed into classical memory if and only if it is classical, i.e. $[\rho_1, \rho_2] = 0$ for any $\rho_1, \rho_2$ from the family.

Compression is only approximately perfect. Cannot directly apply the fact.
FULLY CLASSICAL MEMORY DOESN’T WORK

- Consider $\rho_{\theta_0}^\otimes n$ and $\rho_{\theta_0 + t/\sqrt{n}}^\otimes n$; $t > 0$ is a vector of quantum parameters.
  - Approximation by Gaussian states:
    $$\rho_{\theta_0}^\otimes n \overset{Q-LAN}{\leftrightarrow} \gamma_0$$
    $$\rho_{\theta_0 + t/\sqrt{n}}^\otimes n \overset{Q-LAN}{\leftrightarrow} \gamma_t := D_t \gamma_0 D_t^+$$

  - $\| [\gamma_0, \gamma_t] \| > 0$ (independent of $n$). $\| \cdot \|$: operator norm.
  - A compression protocol for $\rho_{\theta_0}^\otimes n, \rho_{\theta_0 + t/\sqrt{n}}^\otimes n \rightarrow$ a protocol for $\gamma_0, \gamma_t$.
  - State families containing both $\rho_{\theta_0}^\otimes n$ and $\rho_{\theta_0 + t/\sqrt{n}}^\otimes n$ cannot be faithfully encoded in a classical memory.

- For state families with quantum parameters, compression with only classical memory cannot have vanishing error.
SUMMARY AND FUTURE WORKS

- Compression of $\rho_\theta \otimes^n$:
  1. minimal memory cost: approximately $1/2 \log n$ for each degree of freedom;
  2. the required memory is mainly classical;
  3. a fully classical memory is not OK

- Extension to non-product states with symmetry; e.g. states of bosonic systems.

- A second-order theory?
AUTHORS OF THIS WORK

Ge Bai

Yuxiang Yang

Giulio Chiribella

Masahito Hayashi
THANKS FOR YOUR ATTENTION!

See arXiv 1701.03372 for more details

And enjoy the workshop and Singapore ~