Operational measures for squeezing

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joined work with Daniel Lercher and Michael M. Wolf

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What about the weird affiliation?

- Worked with Michael M. Wolf as a graduate student at TUM.
- Graduated recently and left academia.
- Work for TNG Technology Consulting:
  - Munich based German speaking IT development and consulting firm
  - > 50% PhD in Physics, Mathematics and Computer Science
  - special needs software consulting/development for various areas from telecommunications to autonomous driving
  - even a few quantum information people
We pave the way to investigate squeezing as a resource

If one specifies an error tolerance no larger than some error $\varepsilon > 0$ and allows for using $n$ instances of a given resource, what communication rates are achievable?

In this talk:

- “New” resource theory with the usual questions: Squeezing of formation, distillation of squeezing, etc.
- Interesting due to connections with entanglement theory, experimental difficulties, maybe even on its own.
- Providing new tools to study the question in continuous variable quantum information.
The electromagnetic field can be modeled as non-interacting harmonic oscillators (second quantisation).

Harmonic oscillator description: frequency $\omega_k$ and a set of position and momentum operators $Q_k, P_k$.

Usually finitely many $k$ are enough (e.g. in a cavity) ⇒ $R = (Q_1, P_1, \ldots, Q_n, P_n)$.

Photons are bosons ⇒ $R$ fulfil the CCR:

$$[R_k, R_l] = i\sigma_{kl}\mathbb{1}, \quad \sigma = \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Symplectic transformations $Sp(2n)$ leave the CCR invariant (corresponds to unitary transformations on the state).
For Gaussian states, the covariance matrix is your friend

- $Q, P$ must be unbounded $\Rightarrow$ use bounded representations $W_\xi = \exp(-i\xi \sigma R)$.
- Define the characteristic function $\chi(\xi) = \text{tr}(W_\xi \rho)$.
- The characteristic function of a state can often be described by its moments. Gaussian states are described by their first and second moments only:

$$d_k := \text{tr}(\rho R_k)$$
$$\gamma_{kl} := \text{tr}(\rho \{R_k - d_k \mathbb{I}, R_l - d_l \mathbb{I}\}_+)$$

- Operations on the state (such as time evolution) correspond to operations on the moments (such as symplectic transformations).
- Heisenberg’s uncertainty relation: $\gamma \geq i\sigma$.
- A squeezed state has an eigenvalue $\lambda$ of $\gamma$ with $\lambda < 1$. 
Squeezing is a resource

First noted by Braunstein: Squeezing remains invariant under linear optics [S.L. Braunstein. *PRA*, 71, 2005]

Free states: one-mode squeezed states: \( \text{diag}(s, s^{-1}) \)

Free operations:

1. Linear optics (symplectic orthogonal matrices \( S \) acting via \( \gamma \mapsto S^T \gamma S \)),
2. Free ancillary states (\( \gamma \mapsto \gamma \oplus \gamma_{\text{anc}} \)),
3. Add classical noise (\( \gamma_{\text{noise}} \geq 0 \) acting via \( \gamma \mapsto \gamma + \gamma_{\text{noise}} \)),
4. Weyl rotations (no change in covariance matrix),
5. Convex combinations (\( \lambda \gamma + (1 - \lambda)\tilde{\gamma} \)),
6. Homodyne detection:

\[
\gamma = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \quad \Rightarrow \quad \gamma \mapsto \lim_{d \to \infty} A - C(B + \text{diag}(d, 1/d))^{-1}C^T.
\]
If you want entanglement, you need squeezing

Theorem

Given a quantum state $\rho$ with covariance matrix $\gamma$, for an arbitrary two-mode subsystem of a quantum state we have

$$E_N \propto \max\{0, -\log_2(\lambda_1 \lambda_2)\},$$

where $\lambda_1, \lambda_2$ are the smallest eigenvalues of $\gamma$ and $E_N$ is the logarithmic negativity (an entanglement measure). [M.M. Wolf, J. Eisert, M. Plenio. PRL, 90, 2003]

Theorem (No super-activation without squeezing)

Let $T_1, T_2$ be passive Gaussian quantum channels. If each channel either has a symmetric extension or satisfies the PPT property, then $Q(T_1 \otimes T_2) = 0$. [D. Lercher, G. Giedke, M.M. Wolf. New J. Phys. 15, 2013]
Squeezing measures as entanglement witnesses

- Variances (in position and momentum) can be measured very well in the lab.
- “Spin squeezing” measures have been used as entanglement measures for years.
- Similar “squeezing measures” have been proposed recently [M. Gessner, et al. Quantum, 2017-07-10]:

\[
\xi^2(\gamma) = \min_{g \in \mathbb{R}^{2n}, \|g\|_2 = 1} (g^T \sigma^T \gamma \prod(\rho) \sigma g)(g^T \gamma \rho g)
\]

where \( \prod(\rho) = \prod_{i=1}^{N} \rho_i \) with the reduced density matrices \( \rho_i \). The separability criterion reads:

\[
\xi^2(\gamma_{\text{sep}}) \geq 1
\]

- Different goal: find entanglement, not study squeezing as is
Current squeezing measures work well for one-mode states

Currently: $G_{\text{squeeze}} = \lambda_{\text{min}}(\gamma)$. [B. Kraus et al. PRA 67:0402314, 2003]

Problems:

- Consider multimode states

$$\begin{pmatrix} s & \frac{1}{s} \\ \frac{1}{s} & 1 \end{pmatrix}, \quad \begin{pmatrix} s & \frac{1}{s} \\ \frac{1}{s} & s \end{pmatrix}, \quad \begin{pmatrix} s^2 & \frac{1}{s^2} \\ \frac{1}{s^2} & 1 \end{pmatrix}.$$ 

The first and second should not have the same squeezing.

- Operational measures?
Defining measures of squeezing which also work for multiple modes

**Cost:** A one-mode squeezer $S = \text{diag}(s, s^{-1})$ has costs $\log(s)$.

**Idea:** For a symplectic matrix $S$ minimise one-mode squeezing for decompositions $S = S_1 \ldots, S_m$ with $S_i$ passive ($K(n)$) or a one-mode squeezer ($Z(n)$).

Then we have the measure:

$$F(S) = \inf \left\{ \sum_{i=1}^{m} \log s_i(S_i) \mid S = S_1 \ldots S_m, S_i \in K(n) \cup Z(n) \right\}.$$
Minimal squeezing for symplectic matrices is given by the Euler decomposition

Proposition (M.I., D. Lercher, M.M. Wolf (2016))

*For any symplectic matrix $S$ we have*

$$F(S) = \sum_{i=1}^{n} \log(s_i^\dagger(S)).$$

*Equality is achieved by the Euler decomposition*

$$S = K_1 \text{diag}(s_1, s_1^{-1}, \ldots, s_n, s_n^{-1})K_2$$

*with passive $K_1, K_2$.***
Proof

See Whiteboard
This already is an operational measure for pure states

- If \( \text{diag}(s, s^{-1}) \) are resource states, the optimal way to prepare a pure state with covariance matrix \( \gamma \) is given by the Euler decomposition.
- Preparation costs can be read from the Euler decomposition.
- For general states: Take a pure state and add noise.

**Idea:** Suggestion for a measure for general states:

\[
G(\gamma) := \inf \{ F(S) | \gamma \geq S^T S, S \in Sp(2n) \}
\]
An operational measure

Minimal squeezing for states is given by a simple optimisation problem

Minimise costs over all sequences

\[ \gamma_0 \rightarrow \gamma_1 \rightarrow \ldots \rightarrow \gamma_N = \gamma \]

with \( \gamma_0 \) resource states and each operation being an allowed operation.
An operational measure

Minimal squeezing for states is given by a simple optimisation problem

Minimise costs over all sequences

$$\gamma_0 \rightarrow \gamma_1 \rightarrow \ldots \rightarrow \gamma_N = \gamma$$

with $\gamma_0$ resource states and each operation being an allowed operation.

Theorem (M.I., D. Lercher, M.M. Wolf (2016))

*Given a quantum state $\rho$ with covariance matrix $\gamma$, the minimal amount of one-mode squeezing needed for its creation is given by*

$$G(\gamma) = \min \{F(S)|\gamma \geq S^T S, S \in Sp(2n)\}$$
Proof

See Whiteboard
Detour: Cayley transformation

Regular Cayley transform: Transform upper complex half plane to unit disk. Matrix Cayley transform:

- Transformation of positive half-plane $Z > 0$ to unit disc $\|H\| < 1$ [D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, 1998].

- Transformation of skew-Hermition matrix to unitary matrices.


- Symplectic Cayley transform transforms symplectic matrices into symmetric ones [M. deGosson, *Symplectic Geometry and Quantum Mechanics*, 2006].
Detour: Cayley transformation

\[ C : \{ H \in \mathbb{R}^{n \times n} | \text{spec}(H) \cap \{+1\} = \emptyset \} \mapsto \mathbb{R}^{n \times n}, \quad H \mapsto \frac{1 + H}{1 - H}. \]

\[ C^{-1} : \{ S \in \mathbb{R}^{n \times n} | \text{spec}(H) \cap \{-1\} = \emptyset \} \mapsto \mathbb{R}^{n \times n}, \quad S \mapsto \frac{S - 1}{S + 1}. \]

- \( C \) and \( C^{-1} \) are diffeomorphisms onto their respective images.
- \( C \) is operator monotone and operator convex on matrices \( A \) with \( \text{spec}(A) \subset (-1, 1) \).
- \( C^{-1} \) is operator monotone and operator concave on matrices \( A \) with \( \text{spec}(A) \subset (-1, \infty) \).
- \( C : \mathbb{R} \to \mathbb{R} \) is log-convex on \([0, 1)\).
- \( C(H) = \text{Sp}(2n) \cap \{ \gamma \geq i\sigma \} \), where

\[ \mathcal{H} := \left\{ H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} | A \in \mathbb{R}^{2n \times 2n}, A^T = A, B^T = B, -1 < H < 1 \right\}. \]
We can rewrite $G$ using the Cayley transform

$$G(\gamma) = \inf\{F(S)|\gamma \geq S^T S, S \in Sp(2n)\}$$

$$= \inf\{F(\gamma^{1/2})|\gamma \geq \gamma_0 \geq iJ\}$$

$$= \inf \left\{ \frac{1}{2} \sum_{i=1}^{n} \log \left( \frac{1 + s_i(A + iB)}{1 - s_i(A + iB)} \right) \left| C^{-1}(\gamma) \geq H, H \in \mathcal{H} \right\}$$

with (again)

$$\mathcal{H} := \left\{ H = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \left| A \in \mathbb{R}^{2n \times 2n}, A^T = A, B^T = B, -1 < H < 1 \right\}.$$
Proof

See Whiteboard
The measure improves preparation procedures

We provide numerical calculations for examples (two-parameter family of states, code can be found at https://github.com/Martin-Idel/operationalsqueezing):

[L. Mišta, N. Korolkova. PRA 77:050302, 2008]
$G$ is superadditive and probably subadditive

We have:

$$\frac{1}{2}(G(\gamma_A) + G(\gamma_B)) \leq G(\gamma_A \oplus \gamma_B) \leq G(\gamma_A) + G(\gamma_B).$$

**Conjecture:** $G$ is subadditive

- Supported by numerical data.
- True at least if $\gamma_A$ is pure.
We have bounds for $G$, but the upper bound is bad

Best bounds:

$$-\frac{1}{2} \sum_{\lambda_i(\gamma) < 1} \log(\lambda_i(\gamma)) \leq G(\gamma) \leq F(S)$$

where $S$ is the symplectic matrix in Williamson’s theorem $\gamma = S^T DS$.

- lower bound achieved, if the eigenvectors to eigenvalues $< 1$ can be extended to an orthonormal symplectic basis.
- upper bound can be arbitrarily bad: Thermal state with $\gamma = nI$ and $S = \text{diag}(\sqrt{N-1}, 1/\sqrt{N-1}, \ldots) \in Sp(2n)$. 
Detour: Set-valued analysis

- Work with functions with sets and not just points as values.
- Define continuity, norms, etc.

Definition

Let $X, Y \subseteq \mathbb{R}^{n \times m}$ and $f : X \rightarrow 2^Y$ be a set-valued function. It is upper semicontinuous (upper hemicontinuous) at $x_0 \in X$ if:

for all open neighbourhoods $Q$ of $f(x_0)$ there exists an open neighbourhood $W$ of $x_0$ such that $W \subseteq \{x \in X | f(x) \subset Q\}$.

Likewise, we call it lower semicontinuous (often called lower hemicontinuous) at a point $x_0$ if for any open set $V$ intersecting $f(x_0)$, we can find a neighbourhood $U$ of $x_0$ such that $f(x) \cap V \neq \emptyset$ for all $x \in U$. 
Detour: Set-valued analysis

Just for fun:

**Theorem**

Let $S$ be a non-empty, compact and convex subset of some Euclidean space $\mathbb{R}^n$. Let $f : S \rightarrow 2^S$ be a set-valued function on $S$ with a closed graph and the property that $f(x)$ is non-empty and convex for all $x \in S$. Then $f$ has a fixed point.

Proved in 1941 by Shizuo Kakutani and used in the one-page paper “Equilibrium points in $N$-person games” by John F. Nash.
G is probably continuous

Recall:
\[ G(\gamma) = \inf \{ F(\tilde{\gamma}^{1/2}) | \gamma \geq \tilde{\gamma} \geq i\sigma \} \]

For continuity, this means we have the intersection of two convex, non-empty and set-valued functions:
- \( f(A) : A \geq i\sigma \).
- \( g(B) : B \leq \gamma \) (this one varies continuously).

Heuristic: This should be continuous.
We should be able to prove continuity using set-valued analysis

**Current state:** $G$ is lower semicontinuous on the set of covariance matrices and continuous on its interior.

**Conjecture:** $G$ is continuous, since any compact intersection of set-valued functions consisting of matrix cones is continuous.

- Any intersection of non-empty compact sets with non-empty interior is continuous.
- Non-polynomial matrix cones make it somewhat difficult.
Potential other applications

Would directly prove continuity of all functions consisting of optimisation of continuous functions over convex cones.

**Example:** Gaussian entanglement of formation: e.g. [M.M. Wolf., PRA 69, 2004]

\[
E_{\text{form}}(\gamma_{AB}) = \min \{ H(\gamma_p) | \gamma_{AB} \geq S^T S, S \in Sp(2n) \}
\]

where \(\gamma_p\) is the reduced state of \(S^T S\) and \(H\) is entanglement entropy.
On the road to more realistic measures

- $\log(s)$ could be interpreted as “interaction time” of the squeezing Hamiltonian.
- In experiments: linear difficulty until cutoff at about 15 $dB$ squeezing e.g. [H. Vahlbruch et al., PRL 117, 2016].
- Maybe exponential difficulty without explicit cutoff.

Problematic parts:
- Convexity of $G$.
- Submultiplicativity of $F$ (irrelevant for resource theory).
Can we do better using Lie algebras?

Idea: Maybe simple products of the form $S = S_1 \cdots S_n$ are not optimal. How about general paths on the symplectic group?

Proposition (M.I., D. Lercher, M.M. Wolf (2016))

*General paths on $\text{Sp}(2n)$ cannot decrease squeezing costs.*

We define the measure as follows:

$$
\tilde{F}(S) := \inf \left\{ \int_0^1 \| \tilde{c}^a_{\alpha}(t) \|_1 \, dt \left| \alpha \in C^r(S), \dot{\alpha}(t) = (\tilde{c}^p_{\alpha}(t)g^p(\alpha(t)), \tilde{c}^a_{\alpha}(t)g^a(\alpha(t)))^T \right. \right\}
$$

with $c_{\alpha} \in L^\infty([0,1], \mathfrak{sp}(2n))$. 

$\tilde{c}^a_{\alpha} \in \mathfrak{sp}(2n)$. 

$\tilde{c}^p_{\alpha} \in \mathbb{C}$. 

$g^a$ and $g^p$ are functions from $\mathbb{C}$ to $\mathfrak{sp}(2n)$. 

$C^r(S)$ denotes the space of $r$-times continuously differentiable functions on $S$. 

$\| \cdot \|_1$ denotes the $L^1$-norm.
See Whiteboard
Can we distill squeezing and use it in channels?


**Partial Answer:** Distillation of squeezing is prohibited for Gaussian states [B. Kraus, et al. *PRA* 67, 2003], hence not for the usual measure. For our measure, there are clear upper bounds ⇒ not so interesting.
Can we distill squeezing and use it in channels?


Partial Answer: Distillation of squeezing is prohibited for Gaussian states [B. Kraus, et al. *PRA* 67, 2003], hence not for the usual measure. For our measure, there are clear upper bounds ⇒ not so interesting.


Problem: You cannot work with the covariance matrices only.
I still have many open questions

- Fermionic quantum systems?
- Squeezing is related to the spectrum of the covariance matrix, while entanglement is related to the symplectic spectrum of submatrices. Can we have more explicit direct bounds?
- State interconvertibility is more complicated. Can we have even “better” measures?
- Can we have trade-off functions between squeezing and (e.g.) superactivation?