Two instances where current single-letter schemes are sub-optimal

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This work is done jointly with my students

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Some "simple" open problems in Network Information Theory

Two fundamental unsolved problems in Network Information Theory

Question: Is the set of rate pairs \((R_1, R_2)\) (bits per channel use), that can be reliably communicated, computable?
Some "simple" open problems in Network Information Theory

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![Discrete memoryless interference channel (Ahlswede '75)](image)

**Figure 1:** Discrete memoryless interference channel (Ahlswede ’75)

**Question:** Is the set of rate pairs \((R_1, R_2)\) (bits per channel use), that can be reliably communicated, *computable.*
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These problems are "hard" open problems (perhaps two among the millennium problems of the IT Society)

CHAPTER I.
INTRODUCTION

Thomas M. Cover and B. Gopinath

The papers in this volume are the contributions to a special workshop on problems in communication and computation conducted in the summers of 1984 and 1985 in Morristown, New Jersey, and the summer of 1986 in Palo Alto, California. The structure of this workshop was unique: no recent results, no surveys. Instead, we asked for outstanding open problems in the field. There are many famous open problems, including the question

\[ P = NP? \]

the simplex conjecture in communication theory, the capacity region of the broadcast channel, and the two-helper problem in information theory.
Some "simple" open problems in Network Information Theory

Two \textit{fundamental} unsolved problems in Network Information Theory

These problems are "hard" open problems (perhaps two among the millennium problems of the IT Society)

So what are the "simple" open problems in these two settings
Some "simple" open problems in Network Information Theory

Two **fundamental** unsolved problems in Network Information Theory

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So what are the "simple" open problems in these two settings

- **Open problems:**
  8.1. What is the capacity region of the general 3-receiver DM-BC with one common message to all three receivers and one private message to one receiver?

  8.2. Is superposition coding optimal for the general 3-receiver DM-BC with one message to all three receivers and another message to two receivers?

  8.3. What is the sum-capacity of the binary skew-symmetric broadcast channel?

  8.4. Is Marton’s inner bound tight in general?
Some "simple" open problems in Network Information Theory

Two fundamental unsolved problems in Network Information Theory

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So what are the "simple" open problems in these two settings

Questions I consider "simple" (perhaps..):

1. Does Randomized-Time-Division strategy achieve the capacity region (sum-capacity) of the binary skew symmetric broadcast channel?

2. Is Marton’s inner bound tight in general?
Some "simple" open problems in Network Information Theory

Two **fundamental** unsolved problems in Network Information Theory

These problems are "hard" open problems (perhaps two among the millennium problems of the IT Society)

So what are the "simple" open problems in these two settings

- **Open problems:**
  6.1. What is the capacity region of the Gaussian IC with weak interference?
  6.2. What is the generalization of strong interference to three or more user pairs?
  6.3. What is the capacity region of the 3-user-pair injective deterministic IC?
  6.4. Is the Han–Kobayashi inner bound tight in general?
Some "simple" open problems in Network Information Theory

Two fundamental unsolved problems in Network Information Theory

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So what are the "simple" open problems in these two settings

Questions I consider "simple" (perhaps..):

1. Does Han-Kobayashi scheme with Gaussian auxiliaries (and power control) achieve the capacity region of the scalar Gaussian interference channel for
   - Z-interference channel
   - the mixed interference regime
   - the weak interference regime
Idea: Sub-optimality of current schemes

Han-Kobayashi (HK) achievable region (1981) á la Chong et. al. (2006)

A rate-pair \((R_1, R_2)\) is achievable for the interference channel if

\[
R_1 < I(X_1; Y_1|U_2, Q), \\
R_2 < I(X_2; Y_2|U_1, Q), \\
R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q), \\
R_1 + R_2 < I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q), \\
R_1 + R_2 < I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q), \\
2R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q), \\
R_1 + 2R_2 < I(X_2, U_1; Y_2|Q) + I(X_2; Y_2|U_1, U_2, Q) + I(X_1, U_2; Y_1|U_1, Q)
\]

for some pmf \(p(q)p(u_1, x_1|q)p(u_2, x_2|q)\), where \(|U_1| \leq |X_1| + 4, |U_2| \leq |X_2| + 4\), and \(|Q| \leq 7\).

Well-known idea: To show sub-optimality (or optimality): suffices to produce a channel where the 2-letter HK region beats 1-letter HK region (show for every channel the 2-letter HK region matches 1-letter HK region)
Idea: Sub-optimality of current schemes

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\]

\[
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\]

\[
R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q),
\]

\[
R_1 + R_2 < I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q),
\]

\[
R_1 + R_2 < I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q),
\]

\[
2R_1 + R_2 < I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q),
\]

\[
R_1 + 2R_2 < I(X_2, U_1; Y_2|Q) + I(X_2; Y_2|U_1, U_2, Q) + I(X_1, U_2; Y_1|U_1, Q)
\]

for some pmf \(p(q)p(u_1, x_1|q)p(u_2, x_2|q)\), where \(|U_1| \leq |X_1| + 4, |U_2| \leq |X_2| + 4, \) and \(|Q| \leq 7\).

Difficulty in implementing the idea:
Evaluating the HK region even for "random" channels with binary inputs is not easy
Evaluating the 2-letter HK region was rather out of the question
Idea: Sub-optimality of current schemes

To obtain counterexamples via this approach:

- One needs to come up with classes of channels where evaluation is rather easy
- On the other hand, the single-letter strategies must not be optimal for these "simple" classes

Classes of channels where evaluation is rather easy:

- **Analogy**: Gray-Wyner source coding region (only one auxiliary)
- Been shown (Beigi and Gohari ’15) that evaluation of this region is equivalent to evaluation of the *hypercontractivity region*
- Evaluation of hypercontractivity region is a hard problem in mathematics
  - Only been done for BSC with uniform inputs (Bonami-Beckner ’70s) or Gaussians (Gross ’75)
  - Same in network information theory: BSC (using Mrs. Gerber’s lemma), and Gaussians (using EPI)
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- **Message**: Even for simple channels, one would need to do **work** to evaluate regions
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  - Same in network information theory: BSC (using Mrs. Gerber’s lemma), and Gaussians (using EPI)
- **Message**: Even for simple channels, one would need to do work to evaluate regions
- **This talk**: Some of the ideas that were involved in this evaluation
Introduction: Three receiver BC with two degraded message sets

Question: Is superposition coding region optimal?
Introduction: Three receiver BC with two degraded message sets

Question: Is superposition coding region optimal?
Is the capacity region: the set of rate pairs \((R_0, R_1)\) satisfying:

\[
R_0 \leq I(U; Z)
\]

\[
R_0 + R_1 \leq I(U; Z) + \min\{I(X; Y|U), I(X; \hat{Y}|U)\}
\]

\[
R_0 + R_1 \leq \min\{I(X; Y), I(X; \hat{Y})\}
\]

from some \(U \indep X \indep (Y, \hat{Y}, Z)\).
Looking at this question from a different angle

Optimality of superposition coding for two receiver degraded message sets

- Established by Körner and Márton (1977)
- Relies on the image-size characterization over two channels
- If this image-size characterization problem can be solved over three channels, then we would have capacity region
Looking at this question from a different angle

Optimality of superposition coding for two receiver degraded message sets

- Established by Körner and Márton (1977)
- Relies on the image-size characterization over two channels
- If this image-size characterization problem can be solved over three channels, then we would have capacity region

3.1 SOME BASIC MATHEMATICAL PROBLEMS OF MULTIUSER SHANNON THEORY

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2. Image Size Characterization Problem.

The $\eta$-image size $g_W(A, \eta)$ of a set $A \subset X^n$ over a discrete memoryless channel (DMC) $W : X \rightarrow Y$ is the minimum cardinality of $B \subset Y^n$ such that $W^n(B \mid x) \geq \eta$ for each $x \in A$. The problem is to find, for a distribution $P$ on $X$ and DMCs $\{W_i : X \rightarrow Y_i\}$, $i = 1, \ldots, k$, a single-letter characterization of the limit of the sets of all $(k + 1)$-dimensional vectors

$$\left\{ \frac{1}{n} \log |A|, \frac{1}{n} \log g_{W_1}(A, \eta), \ldots, \frac{1}{n} \log g_{W_k}(A, \eta) \right\}.$$ 

Here $A \subset X^n$ is any set of $P$-typical sequences, and $0 < \eta < 1$ is fixed (the result is independent of $\eta$).
Körner’s region (1984)

Körner had proposed a region for the image sizes over three channels

\[\text{Theorem: } \text{For every RV’s } T, U, \text{ and } V \text{ such that } \]
\[TUV \to S \to XYZ,\]

nonnegative numbers \(t, t',\) and \(t''\), the point \((r_x, r_y, r_z)\) with coordinates

\[r_x \triangleq \min \left[ H(X), H(X|T) + t, H(X|TU) + t', H(X|TUV) + t'' \right],\]

\[r_y \triangleq \min \left[ H(Y), H(Y|T) + t, H(Y|TU) + t', H(Y|TUV) + t'' \right],\]

\[r_z \triangleq \min \left[ H(Z), H(Z|T) + t, H(Z|TU) + t', H(Z|TUV) + t'' \right].\]

is an element of \(\mathcal{H}(X; Y; Z|S).\) \(\square\)
Körner’s region (1984)

Körner had proposed a region for the image sizes over three channels

\emph{Theorem:} For every RV’s \( T, U, \) and \( V \) such that \( TUV \to S \to XYZ \), nonnegative numbers \( t, t', \) and \( t'' \), the point \( (r_x, r_y, r_z) \) with coordinates

\[
\begin{align*}
    r_x &\triangleq \min \left[ H(X), H(X|T) + t, H(X|TU) + t', H(X|TUV) + t'' \right], \\
    r_y &\triangleq \min \left[ H(Y), H(Y|T) + t, H(Y|TU) + t', H(Y|TUV) + t'' \right], \\
    r_z &\triangleq \min \left[ H(Z), H(Z|T) + t, H(Z|TU) + t', H(Z|TUV) + t'' \right]
\end{align*}
\]

is an element of \( \mathcal{H}(X; Y; Z|S) \).

As a consequence of our work, we know that such points do no exhaust \( \mathcal{H}(X; Y; Z|S) \).
Counter-example: Multilevel product broadcast erasure channel

\[ X_a \rightarrow Y_a : B E C(e_a), \quad X_b \rightarrow Y_b : B E C(e_b) \]
\[ X_a \rightarrow \hat{Y}_a : B E C(\hat{e}_a), \quad X_b \rightarrow \hat{Y}_b : B E C(\hat{e}_b) \]
\[ X_a \rightarrow Z_a : B E C(f_a), \quad X_b \rightarrow Z_b : B E C(f_b) \]

\[ \hat{e}_a \geq f_a \geq e_a \quad \& \quad e_b \geq f_b \geq \hat{e}_b \]

\[ C_Z = (1 - f_a) + (1 - f_b) \]
Counter-example: Multilevel product broadcast erasure channel

\[ X_a \to Y_a : BEC(e_a), \; X_b \to Y_b : BEC(e_b) \]
\[ X_a \to \hat{Y}_a : BEC(\hat{e}_a), \; X_b \to \hat{Y}_b : BEC(\hat{e}_b) \]
\[ X_a \to Z_a : BEC(f_a), \; X_b \to Z_b : BEC(f_b) \]

\[ \hat{e}_a \geq f_a \geq e_a \; \& \; e_b \geq f_b \geq \hat{e}_b \]

\[ C_Z = (1 - f_a) + (1 - f_b) \]

**Theorem**

For
\[ e_a = 1/2 \]
\[ e_b = 1/2 \]
\[ \hat{e}_a = 1 \]
\[ \hat{e}_b = 0 \]
\[ f_a = 17/22 \]
\[ f_b = 9/34 \]

1-letter \( SC \):
\[ R_0 + R_1 \leq 1 \]
\[ \frac{11}{10} R_0 + R_1 \leq \frac{18}{17} \]
\[ \frac{11}{10} C_Z \]

2-letter \( SC \):
\[ R_0 + R_1 \leq 1 \]
\[ \frac{484}{435} R_0 + R_1 \leq \frac{528}{493} \]
\[ \frac{484}{435} C_Z \]
Plot

2-letter & 1-letter SC regions
Plot

$R_0$

$\frac{75}{119}$

$\frac{10}{17}$

$\frac{44}{119}$

$\frac{7}{17}$

$R_1$

2-letter & 1-letter $SC$ regions

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Main: Computation of $SC$-region(s)

**Idea 1:** Computing the supporting hyperplane of the superposition coding region(s)

**Lemma**

\[
\max_{\lambda \geq 1} (\lambda R_0 + R_1) \text{ for } \lambda \geq 1 \text{ is equal to }
\]

\[
\min_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \left( \max_{p(u,x)} \left( (\lambda - 1)I(U;Z) + \alpha_1 (I(U;Z) + I(X;Y|U)) + \alpha_2 I(X;Y) + \alpha_3 (I(U;Z) + I(X;\hat{Y}|U)) + \alpha_4 I(X;\hat{Y}) \right) \right)
\]

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$, $\alpha_i > 0$ and $|U| \leq |X|$

- $\min \max$ interchange [Geng, Gohari, Nair, Yu ’14]
Main: Computation of \( SC\)-region(s)

**Idea 1:** Computing the supporting hyperplane of the superposition coding region(s)

**Lemma**

\[
\max_{SC} (\lambda R_0 + R_1) \text{ for } \lambda \geq 1 \text{ is equal to }
\]

\[
\min_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \left( \max_{p(u,x)} ((\lambda - 1)I(U; Z) + \alpha_1(I(U; Z) + I(X; Y|U)) + \alpha_2I(X; Y) + \alpha_3(I(U; Z) + I(X; \hat{Y}|U)) + \alpha_4I(X; \hat{Y})) \right)
\]

where \( \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1, \alpha_i > 0 \) and \(|\mathcal{U}| \leq |\mathcal{X}|\)

- **min max** interchange [Geng, Gohari, Nair, Yu ’14]

- **Difficulty:** 15 dimensional non-convex optimization in the 2-letter region computation for binary input channels
Upperbounding 1-letter $SC$

Lemma

For the product binary erasure broadcast channel with

\[
\begin{align*}
\max_{1\text{-letter } SC} \left( \frac{11}{10} R_0 + R_1 \right) &\leq \frac{11}{10} C_Z = \frac{18}{17} \\
\end{align*}
\]

where $C_Z = \frac{18 \times 10}{17 \times 11}$ is the point to point capacity of $W(z|x)$

\[
\begin{align*}
e_a &= \frac{1}{2}, \quad \hat{e}_a = 1, \quad f_a = \frac{17}{22} \\
e_b &= \frac{1}{2}, \quad \hat{e}_b = 0, \quad f_b = \frac{9}{34}
\end{align*}
\]
Upperbounding 1-letter $SC$

Lemma

For the product binary erasure broadcast channel with

$$\max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right) \leq \frac{11}{10} C_Z = \frac{18}{17}$$

where $C_Z = \frac{18 \times 10}{17 \times 11}$ is the point to point capacity of $W(z|x)$

Proof. In the equation for $\max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right)$, set $\alpha_1 = \alpha_3 = \frac{1}{2}$ and $\alpha_2 = \alpha_4 = 0$

$$\max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right) \leq \max_{p(u,x)} \left( \frac{11}{10} I(U; Z) + \frac{1}{2} (I(X; Y|U) + I(X; \hat{Y}|U)) \right)$$

Idea 2: Using concave envelope characterization, the above maximum equals to

$$\max_{p(x)} \left( \frac{11}{10} I(X; Z) + \epsilon \left[ \frac{1}{2} (I(X; Y) + I(X; \hat{Y})) - \frac{11}{10} I(X; Z) \right] \right)$$
Upperbounding 1-letter SC

Lemma

For the product binary erasure broadcast channel with

\[ \max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right) \leq \frac{11}{10} C_Z = \frac{18}{17} \]

where \( C_Z = \frac{18 \times 10}{17 \times 11} \) is the point to point capacity of \( W(z|x) \)

Proof. In the equation for \( \max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right) \), set \( \alpha_1 = \alpha_3 = \frac{1}{2} \) and \( \alpha_2 = \alpha_4 = 0 \)

\[ \max_{1\text{-letter} \ SC} \left( \frac{11}{10} R_0 + R_1 \right) \leq \max_{p(u,x)} \left( \frac{11}{10} I(U; Z) + \frac{1}{2} (I(X; Y|U) + I(X; \hat{Y}|U)) \right) \]

Idea 3: Applying a symmetrization trick, it equals

\[ \frac{11}{10} C_Z + \max_{p(x)} \left[ \frac{1}{2} (I(X; Y) + I(X; \hat{Y})) - \frac{11}{10} I(X; Z) \right] \]
Upperbounding 1-letter SC: continued

Lemma

Consider a product erasure channel mapping $X_1, .., X_k$ to $Y_1, .., Y_k$ with erasure probabilities $\epsilon_1, ..., \epsilon_k$. Then

$$I(X_1, ..., X_k; Y_1, ..., Y_k) = \sum_{S \subseteq [1:k]} \left( \prod_{i \in S}(1 - \epsilon_i) \prod_{j \notin S} \epsilon_j \right) H(X_S)$$

where $X_S = (X_i : i \in S)$

Expanding the function inside the envelope using this lemma

$$\frac{1}{2}(I(X; Y) + I(X; \hat{Y})) - \frac{11}{10} I(X; Z) = -\frac{1}{17} H(X_b | X_a)$$

Hence the two lines

$$\begin{cases} R_0 + R_1 = 1 \\ \frac{11}{10} R_0 + R_1 = \frac{18}{17} \end{cases}$$

upperbound 1-letter SC
Characterization of 1-letter $SC$

Lemma

The intersection point $(R_0, R_1) = \left(\frac{10}{17}, \frac{7}{17}\right)$ of the two lines lies within the 1-letter superposition coding region.

- Let $U$ be a ternary random variable

$$
\begin{align*}
P(U = 0) &= \frac{13}{34} \quad (X_a, X_b)|U = 0 = (0, 0) \\
P(U = 1) &= \frac{7}{34} \quad (X_a, X_b)|U = 1 = (M, 0) \\
P(U = 2) &= \frac{21}{34} \quad (X_a, X_b)|U = 2 = (M, M)
\end{align*}
$$

where $M$ is an unbiased binary random variable.

- Let $Q$ be the random variable that symmetrizes the distribution of $X$
- Let $\tilde{U} = (U, Q)$ and substitute $(\tilde{U}, X)$ into $SC$

$$
R_0 \leq I(\tilde{U}; Z) = \frac{10}{17}
$$

$$
R_0 + R_1 \leq I(U; Z) + I(X; Y|U) = 1
$$

$$
R_0 + R_1 \leq I(U; Z) + I(X; \hat{Y}|U) = 1
$$

$$
R_0 + R_1 \leq I(X; Y) = 1
$$

$$
R_0 + R_1 \leq I(X; \hat{Y}) = 1
$$
Upperbounding 2-letter $SC$

**Lemma**

For the product binary erasure broadcast channel with

$$\max_{2\text{-letter } SC} (\frac{484}{435} R_0 + R_1) \leq \frac{484}{435} C_Z = \frac{528}{493}$$

where $C_Z = \frac{528 \times 435}{493 \times 484}$ is the point to point capacity of $W(z|x)$

**Proof.**

- In the equation for $\max_{2\text{-letter } SC} (\frac{484}{435} R_0 + R_1)$, set $\alpha_1 = \frac{88}{174}, \alpha_3 = \frac{86}{174}$ and $\alpha_2 = \alpha_4 = 0$

$$\max_{2\text{-letter } SC} (\frac{484}{435} R_0 + R_1) \leq \max_{p(u,x)} (\frac{484}{435} I(U; Z) + \frac{88}{174} I(X; Y|U) + \frac{86}{174} I(X; \hat{Y}|U))$$

- Following the same line of argument for 2-letter $SC$, the above equals

$$\frac{484}{435} C_Z + \max_{p(x)} \left[ \frac{88}{174} I(X; Y) + \frac{86}{174} I(X; \hat{Y}) - \frac{484}{435} I(X; Z) \right]$$
Characterization of 2-letter $SC$

- The expression inside the envelope $\frac{88}{174} I(X; Y) + \frac{86}{174} I(X; \hat{Y}) - \frac{484}{435} I(X; Z)$
Introduction

Characterization of 2-letter $SC$

- The expression inside the envelope $\frac{88}{174} I(X; Y) + \frac{86}{174} I(X; \hat{Y}) - \frac{484}{435} I(X; Z)$

$$= - \frac{17}{174} I(X_{b1}; X_{b2}) - \frac{19}{2958} (I(X_{b1}; X_{b2}|X_{a1}) + I(X_{b1}; X_{b2}|X_{a2}))$$
$$- \frac{2}{29} I(X_{a1}; X_{a2}|X_{b1}X_{b2}) - \frac{2543}{50286} I(X_{b1}; X_{b2}|X_{a1}X_{a2})$$
$$- \frac{35}{493} (H(X_{b1}|X_{a1}X_{a2}X_{b2}) + H(X_{b2}|X_{a1}X_{b1}X_{a2}))$$
$$- \frac{1}{174} (I(X_{a1}; X_{b1}|X_{a2}) + I(X_{a1}; X_{b2}|X_{a2})$$
$$+ I(X_{a2}; X_{b1}|X_{a1}) + I(X_{a2}; X_{b2}|X_{a1}))$$
$$- \frac{1}{174} (I(X_{a1}; X_{b1}|X_{a2}X_{b2}) + I(X_{a1}; X_{b2}|X_{a2}X_{b1})$$
$$+ I(X_{a2}; X_{b1}|X_{a1}X_{b2}) + I(X_{a2}; X_{b2}|X_{a1}X_{b1})),$$
Characterization of 2-letter SC

- The expression inside the envelope \( \frac{88}{174} I(X; Y) + \frac{86}{174} I(X; \hat{Y}) - \frac{484}{435} I(X; Z) \) evaluates to sum of non-positive terms
Characterization of 2-letter SC

- The expression inside the envelope \( \frac{88}{174} I(X; Y) + \frac{86}{174} I(X; \hat{Y}) - \frac{484}{435} I(X; Z) \) evaluates to sum of non-positive terms

Lemma

The intersection \((R_0, R_1) = (\frac{75}{119}, \frac{44}{119})\) of the two lines lies within the 2-letter superposition coding region

\[
\begin{align*}
M_1 & \& M_2 \\
\text{two independent unbiased binary r.v.} & & \left\{ \begin{array}{l}
P(U = 0) = \frac{20}{119} \\
P(U = 1) = \frac{88}{119} \\
P(U = 2) = \frac{11}{119}
\end{array} \right.
\end{align*}
\]

\(X_{a1}, X_{b1}, X_{a2}, X_{b2})|U = 0 = (0, 0, 0, 0)

\(X_{a1}, X_{b1}, X_{a2}, X_{b2})|U = 1 = (M_1, M_1, M_1, 0)

\(X_{a1}, X_{b1}, X_{a2}, X_{b2})|U = 2 = (M_1, 0, M_2, 0)

- Let \(Q\) be the random variable that symmetrizes the distribution of \(X\)
- Let \(\tilde{U} = (U, Q)\) and substitute \((\tilde{U}, X)\) into \(SC\) \(\Rightarrow (R_0, R_1) = (\frac{75}{119}, \frac{44}{119})\)
Computing limiting $n$-letter:

**Theorem (Concentration of mutual information over memoryless product erasure channel)**

Consider a product erasure channel, $W_a(y_a|x_a) \otimes W_b(y_b|x_b)$, mapping $X_a, X_b$ to $Y_a, Y_b$ with erasure probabilities $\epsilon_a, \epsilon_b$, respectively. Then

$$I(X_a^n, X_b^n; Y_a^n, Y_b^n) = \mathcal{H}([n(1 - \epsilon_a)], [n(1 - \epsilon_b)]) + O\left(\sqrt{n \log n}\right),$$

where

$$\mathcal{H}_n(k, l) = \frac{1}{\binom{n}{k}\binom{n}{l}} \sum_{S,T \subseteq [n]: |S|=k, |T|=l} H(X_aS, X_bT).$$
n-letter

Computing limiting $n$-letter:

Theorem (Concentration of mutual information over memoryless product erasure channel)

Consider a product erasure channel, $W_a(y_a|x_a) \otimes W_b(y_b|x_b)$, mapping $X_a, X_b$ to $Y_a, Y_b$ with erasure probabilities $\epsilon_a, \epsilon_b$, respectively. Then

$$I(X^n_a, X^n_b; Y^n_a, Y^n_b) = \mathcal{H}([n(1-\epsilon_a)], [n(1-\epsilon_b)]) + O\left(\sqrt{n \log n}\right),$$

where

$$\mathcal{H}_n(k, l) = \frac{1}{\binom{n}{k}\binom{n}{l}} \sum_{S,T \subseteq [n]: |S|=k, |T|=l} H(X_{aS}, X_{bT}).$$

Using **Shannon-type** information inequalities, we can establish that

$$\limsup_n \max_{p(x^n_a, x^n_b)} \frac{1}{n} \left( \frac{85}{160} \mathcal{H}(\frac{n}{2}, \frac{n}{2}) + \frac{75}{160} \mathcal{H}(0, n) - \frac{187}{160} \mathcal{H}(\frac{5n}{22}, \frac{25n}{34}) \right) \leq 0.$$
n-letter continued

Theorem (Outer bound)

Any achievable rate pair \((R_0, R_1)\) must satisfy the constraints.

\[
R_0 + R_1 \leq 1 \quad \text{and} \quad \frac{187}{160} R_0 + R_1 \leq \frac{18}{16}.
\]
n-letter continued

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Achievability

If there is a non-trivial collection \((X^n_a, X^n_b)\) such that

\[
\begin{align*}
\mathcal{H}_n\left(\frac{n}{2}, \frac{n}{2}\right) & \equiv \mathcal{H}_n\left(\frac{n}{2}, \frac{25n}{34}\right), \\
\frac{5}{11} \mathcal{H}_n\left(\frac{n}{2}, \frac{25n}{34}\right) + \frac{6}{11} \mathcal{H}_n\left(0, \frac{25n}{34}\right) & \equiv \mathcal{H}\left(\frac{5n}{22}, \frac{25n}{34}\right), \\
\frac{8}{17} \mathcal{H}_n\left(0, n\right) + \frac{9}{17} \mathcal{H}_n\left(0, \frac{n}{2}\right) & \equiv \mathcal{H}_n\left(0, \frac{25n}{34}\right), \\
\frac{17}{25} \mathcal{H}_n\left(0, \frac{25n}{34}\right) & \equiv \mathcal{H}_n\left(0, \frac{n}{2}\right),
\end{align*}
\]

then there are non-trivial points of the outer bound that are achievable.
Any achievable rate pair \((R_0, R_1)\) must satisfy the constraints.

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\begin{align*}
\mathcal{H}_n\left( \frac{n}{2}, \frac{n}{2} \right) &\equiv \mathcal{H}_n\left( \frac{n}{2}, \frac{25n}{34} \right), \\
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\frac{8}{17} \mathcal{H}_n(0, n) + \frac{9}{17} \mathcal{H}_n(0, \frac{n}{2}) &\equiv \mathcal{H}_n(0, \frac{25n}{34}), \\
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\end{align*}
\]

then there are non-trivial points of the outer bound that are achievable.

Suggests: "MDS-like" code-construction for \(X^n_a\)

Structured codes needed
Recap

Considered a scenario where optimality of the single-letter scheme was not known
- Found a channel where 2-letter region strictly improves on single-letter region
- Developed tools to evaluate regions (for this and other settings)
  - min-max exchange
  - Recognized extremal auxiliaries essentially computed upper concave envelopes
  - Symmetrization argument
  - Information inequalities
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    - It was a multi-year (sometimes painful but ultimately rewarding) process

- Developed ideas to estimate capacity regions (limiting $n$-letter regions)
  - Structured codes seem necessary

Short term future directions:

- We relied heavily on erasure channel model for concentration results on limiting $n$-letter
  - Can one appropriately generalize this to arbitrary memoryless channels
  - My guess: yes

- More importantly: Can we use such ideas to compute capacity regions directly from limiting $n$-letter expressions.
Setting two: Interference Channel (Ahlswede ’75)

Figure 1: Discrete memoryless interference channel
Setting two: Interference Channel (Ahlswede ’75)

Figure 1: Discrete memoryless interference channel

- **Open problems:**
  6.1. What is the capacity region of the Gaussian IC with weak interference?
  6.2. What is the generalization of strong interference to three or more user pairs?
  6.3. What is the capacity region of the 3-user-pair injective deterministic IC?
  6.4. Is the Han–Kobayashi inner bound tight in general?
Han-Kobayashi achievable region (1981) à la Chong et. al.

A rate-pair \((R_1, R_2)\) is achievable for the interference channel if

\[
R_1 < I(X_1; Y_1 | U_2, Q),
\]
\[
R_2 < I(X_2; Y_2 | U_1, Q),
\]
\[
R_1 + R_2 < I(X_1, U_2; Y_1 | Q) + I(X_2; Y_2 | U_1, U_2, Q),
\]
\[
R_1 + R_2 < I(X_2, U_1; Y_2 | Q) + I(X_1; Y_1 | U_1, U_2, Q),
\]
\[
R_1 + R_2 < I(X_1, U_2; Y_1 | U_1, Q) + I(X_2, U_1, Y_2 | U_2, Q),
\]
\[
2R_1 + R_2 < I(X_1, U_2; Y_1 | Q) + I(X_1; Y_1 | U_1, U_2, Q) + I(X_2, U_1; Y_2 | U_2, Q),
\]
\[
R_1 + 2R_2 < I(X_2, U_1; Y_2 | Q) + I(X_2; Y_2 | U_1, U_2, Q) + I(X_1, U_2; Y_1 | U_1, Q)
\]

for some pmf \(p(q)p(u_1, x_1 | q)p(u_2, x_2 | q)\), where \(|U_1| \leq |X_1| + 4\), \(|U_2| \leq |X_2| + 4\), and \(|Q| \leq 7\).
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A rate-pair \((R_1, R_2)\) is achievable for the interference channel if

\[
\begin{align*}
R_1 &< I(X_1; Y_1|U_2, Q), \\
R_2 &< I(X_2; Y_2|U_1, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1|Q) + I(X_2; Y_2|U_1, U_2, Q), \\
R_1 + R_2 &< I(X_2, U_1; Y_2|Q) + I(X_1; Y_1|U_1, U_2, Q), \\
R_1 + R_2 &< I(X_1, U_2; Y_1|U_1, Q) + I(X_2, U_1; Y_2|U_2, Q), \\
2R_1 + R_2 &< I(X_1, U_2; Y_1|Q) + I(X_1; Y_1|U_1, U_2, Q) + I(X_2, U_1; Y_2|U_2, Q), \\
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\end{align*}
\]

for some pmf \(p(q)p(u_1, x_1|q)p(u_2, x_2|q)\), where \(|U_1| \leq |X_1| + 4\), \(|U_2| \leq |X_2| + 4\), and \(|Q| \leq 7\).

First step:
- Find a channel class where HK region simplifies AND current tools for computing capacity region fails
- Defined a class: **very weak interference channels** (approx. 2010)
Very weak interference channels

Definition: Analogous and antipodal to very strong interference channels

An interference channel is said to have very weak interference if

\[
I(U_1; Y_1) \geq I(U_1; Y_2|X_2), \quad \forall p_1(u_1, x_1)p_2(x_2)W(y_1, y_2|x_1, x_2)
\]

\[
I(U_2; Y_2) \geq I(U_2; Y_1|X_1), \quad \forall p_1(x_1)p_2(u_2, x_2)W(y_1, y_2|x_1, x_2)
\]
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\[ I(U_2; Y_2) \geq I(U_2; Y_1|X_1), \quad \forall p_1(x_1)p_2(u_2, x_2)W(y_1, y_2|x_1, x_2) \]

If the interference channel has very-weak interference then any pair \((R_1, R_2)\) in Han-Kobayashi region satisfies

\[ R_1 + R_2 \leq I(X_1; Y_1) + I(X_2; Y_2). \]
Very weak interference channels

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\[ R_1 + R_2 \leq I(X_1; Y_1) + I(X_2; Y_2). \]

Unfortunately, we could not find an interference channel with very weak interference where sum-rate is strictly larger than that above, i.e. by treating interference as noise.
Very weak interference channels

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An interference channel is said to have very weak interference if

\[ I(U_1; Y_1) \geq I(U_1; Y_2|X_2), \quad \forall p_1(u_1, x_1)p_2(x_2)W(y_1, y_2|x_1, x_2) \]
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If the interference channel has very-weak interference then any pair \((R_1, R_2)\) in Han-Kobayashi region satisfies

\[ R_1 + R_2 \leq I(X_1; Y_1) + I(X_2; Y_2). \]

"Simple" open problem

Prove or disprove: The sum-capacity of an interference channel with very weak interference is given by

\[ \max_{p_1(x_1)p_2(x_2)} I(X_1; Y_1) + I(X_2; Y_2). \]
Very weak interference channels

Definition: Analogous and antipodal to very strong interference channels

An interference channel is said to have very weak interference if

\[
I(U_1; Y_1) \geq I(U_1; Y_2 | X_2), \quad \forall p_1(u_1, x_1)p_2(x_2)W(y_1, y_2 | x_1, x_2)
\]
\[
I(U_2; Y_2) \geq I(U_2; Y_1 | X_1), \quad \forall p_1(x_1)p_2(u_2, x_2)W(y_1, y_2 | x_1, x_2)
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If the interference channel has very-weak interference then any pair \((R_1, R_2)\) in Han-Kobayashi region satisfies

\[
R_1 + R_2 \leq I(X_1; Y_1) + I(X_2; Y_2).
\]

There are some non-trivial examples (BSZIC) where we could prove optimality. (2013)

An extremal BSZIC (not very weak): \(Y_1 = X_1 \land X_2, \quad Y_2 = X_1 \lor X_2\)

Conjecture (Ahlswede): maximum sum-rate is 1 (still open).
Beyond sum-rate: Clean Z Interference Channel (CZIC) Model

A rate-pair \((R_1, R_2)\) belongs to Han-Kobayashi region if and only if

\[
R_1 < I(X_1; Y_1|U_2, Q),
\]

\[
R_2 < I(X_2; Y_2|Q),
\]

\[
R_1 + R_2 < I(X_1, U_2; Y_1|Q) + H(X_2|U_2, Q),
\]

for some pmf \(p(q)p(x_1|q)p(u_2, x_2|q)\), where \(|U_2| \leq |X_2|\) and \(|Q| \leq 2\).
Beyond sum-rate: Clean Z Interference Channel (CZIC) Model

![Clean Z-interference channel diagram](image)

**Figure 2: Clean Z-interference channel**

**Theorem (Part of the capacity region (with Xia-Yazdanpanah ’15))**

For $\lambda \leq 1$,

$$
\max_{(R_1,R_2) \in \mathcal{C}} \lambda R_1 + R_2 = \max_{p_1(x_1)p_2(x_2)} \lambda I(X_1;Y_1) + H(X_2).
$$
What about weighted sum-rate?

Our proof of optimality of Han-Kobayashi region did not go through.

It is easy to see that for \( \lambda \geq 1 \)

\[
\max_{(R_1,R_2) \in \mathcal{HK}} \lambda R_1 + R_2
\]

\[
= \max_{p_1(x_1)p_2(x_2)} I(X_1, X_2; Y_1) + \mathcal{C}_{X_2}[(\lambda - 1)I(X_1; Y_1) + I(X_2; Y_2) - I(X_2; Y_1|X_1)]
\]
What about weighted sum-rate?

Our proof of optimality of Han-Kobayashi region did not go through.

It is easy to see that for $\lambda \geq 1$

$$\max_{(R_1, R_2) \in HK} \lambda R_1 + R_2$$

$$= \max_{p_1(x_1)p_2(x_2)} I(X_1, X_2; Y_1) + C_{X_2}[(\lambda - 1)I(X_1; Y_1) + I(X_2; Y_2) - I(X_2; Y_1 | X_1)]$$

**Question**: Can we numerically test the behavior of 2-letter region vs one-letter region?
Sub-optimality of the Han-Kobayashi region

Lots of counterexamples to the optimality of Han-Kobayashi region.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>channel</th>
<th>$\max_{R_{kb}}(\lambda R_1 + R_2)$</th>
<th>$\max_{R_{kno}}(\lambda R_1 + R_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\begin{bmatrix} 1 &amp; 0.5 \ 1 &amp; 0 \end{bmatrix}$</td>
<td>1.107516</td>
<td>1.108141</td>
</tr>
<tr>
<td>2.5</td>
<td>$\begin{bmatrix} 0.204581 &amp; 0.364813 \ 0.030209 &amp; 0.992978 \end{bmatrix}$</td>
<td>1.159383</td>
<td>1.169312</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 0.591419 &amp; 0.865901 \ 0.004021 &amp; 0.898113 \end{bmatrix}$</td>
<td>1.241521</td>
<td>1.255814</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 0.356166 &amp; 0.073253 \ 0.985504 &amp; 0.031707 \end{bmatrix}$</td>
<td>1.292172</td>
<td>1.311027</td>
</tr>
<tr>
<td>3</td>
<td>$\begin{bmatrix} 0.287272 &amp; 0.459966 \ 0.113711 &amp; 0.995405 \end{bmatrix}$</td>
<td>1.117253</td>
<td>1.123151</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 0.42804 &amp; 0.147712 \ 0.948192 &amp; 0.002848 \end{bmatrix}$</td>
<td>1.181392</td>
<td>1.196189</td>
</tr>
<tr>
<td>4</td>
<td>$\begin{bmatrix} 0.068730 &amp; 0.443630 \ 0.011377 &amp; 0.954887 \end{bmatrix}$</td>
<td>1.223409</td>
<td>1.243958</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0.969199 &amp; 0.564440 \ 0.950479 &amp; 0.061409 \end{bmatrix}$</td>
<td>1.351229</td>
<td>1.372191</td>
</tr>
<tr>
<td>5</td>
<td>$\begin{bmatrix} 0.943226 &amp; 0.447252 \ 0.950791 &amp; 0.024302 \end{bmatrix}$</td>
<td>1.231254</td>
<td>1.250564</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} 0.943292 &amp; 0.045996 \ 0.589551 &amp; 0.202487 \end{bmatrix}$</td>
<td>1.069405</td>
<td>1.076932</td>
</tr>
<tr>
<td>6</td>
<td>$\begin{bmatrix} 0.714431 &amp; 0.019375 \ 0.955918 &amp; 0.448539 \end{bmatrix}$</td>
<td>1.528508</td>
<td>1.541781</td>
</tr>
<tr>
<td>7</td>
<td>$\begin{bmatrix} 0.058449 &amp; 0.558649 \ 0.194915 &amp; 0.959172 \end{bmatrix}$</td>
<td>1.424974</td>
<td>1.452769</td>
</tr>
<tr>
<td>7</td>
<td>$\begin{bmatrix} 0.033312 &amp; 0.876087 \ 0.286125 &amp; 0.992825 \end{bmatrix}$</td>
<td>1.179438</td>
<td>1.187867</td>
</tr>
<tr>
<td>10</td>
<td>$\begin{bmatrix} 0.307723 &amp; 0.874843 \ 0.032090 &amp; 0.710535 \end{bmatrix}$</td>
<td>1.370830</td>
<td>1.388674</td>
</tr>
</tbody>
</table>
Sub-optimality of the Han-Kobayashi region

Lots of counterexamples to the optimality of Han-Kobayashi region.

**Generic behavior:**

- For $\lambda \leq 1$, treating interference as noise, i.e. $U_2 = 0$, was optimal.
- As you increase $\lambda$ from 1 until some $\lambda^*$, we observe that $U_2 = 0$, continues to be optimal for H-K.
- Starting from some $\lambda_1 \in (1, \lambda^*)$, we observe that 2-letter treating interference as noise region outperforms the 1-letter region.

**Note:** For the first example on left, we can calculate the concave envelope analytically.
A particular example

\[
\max_{R_{HK}} (2R_1 + R_2) = 1.1075163.. < 1.108035632 \leq \max_{2-R_{HK}} (2R_1 + R_2)
\]
A particular example

\[
\begin{align*}
X_1 & \quad Y_1 \\
0 & \quad 0 \\
1 & \quad 1
\end{align*}
\]

\(X_2 = 0\)

\[
\begin{align*}
X_1 & \quad Y_1 \\
0 & \quad 0 \\
\frac{1}{2} & \quad \frac{1}{2}
\end{align*}
\]

\(X_2 = 1\)

\[
\max(2R_1 + R_2) = 1.1075163.. < 1.108035632 \leq \max_{2-R_{HK}} (2R_1 + R_2)
\]

Remarks

- \(\lambda = 2\) helps a certain equation become quadratic; helpful for closed form computation.
- Perturbations along Markov chains improves the weighted rates \(\lambda = 2\). (G. Han)
Figures help.

Remarks
1. 1-letter = 2-letter for $\lambda \in [1, 1.83] \cup [3.35, \infty)$.
2. Corner point $(R_1, R_2) = (0.0488, 1)$
3. Capacity region matches 1-letter until $(R_1, R_2) = (0.0535, 0.9976)$ (sum-rate capacity).
Remarks

- \( \max_{\mathcal{R}_{HK}} (\lambda R_1 + R_2) = \lambda C_1, \text{ for } \lambda \geq \lambda^\dagger \approx 7.21. \)
  
  Here \( C_1 \) is the maximum possible rate to the receiver possible (corner point).

- On the other hand, recently we managed to show that such that
  \[ \max_{(R_1, R_2) \in C} (12R_1 + R_2) = 12C_1. \]
Figures help..

Remarks

- **Rather recently**: New outer bound developed by Polyanskiy-Wu using
  - Talagrand’s HWI inequality (for Gaussian noise)
  - Marton’s transportation inequality (for discrete variables)

The latter does not even provide a finite $\lambda$ s/t
$$\max_{(R_1,R_2) \in C} (\lambda R_1 + R_2) = \lambda C_1.$$
Remarks

- **"Simple" open problem:** What is the true corner-point slope for the interference channel?
- The similar question for broadcast channel was solved recently.
Recap

Showed sub-optimality of the Han-Kobayashi achievable region for the interference channel

- By constructing a channel where 2-letter region beats 1-letter region
- Difficulty was in evaluating the HK region (even for channels with binary inputs)
Recap

Showed sub-optimality of the Han-Kobayashi achievable region for the interference channel

- By constructing a channel where 2-letter region beats 1-letter region
- Difficulty was in evaluating the HK region (even for channels with binary inputs)

Optimality of Marton’s achievable region for the broadcast channel is **open**

- We now have results that help us evaluate Marton’s for any binary input broadcast channel
- Yet, unable to find examples where 2-letter beats 1-letter
- That means: **either** we have not burnt enough computational power **or** Marton’s region is optimal (solve our millennium problem)
Recap

There are still a lot of "simple" yet fundamental open problems in network information theory

- Unfortunately, not many people seem to know/care to work on these problems
- Some of these may be solved by just putting enough computing power (at least a part of the heavy lifting towards evaluation has been done)
- In some scenarios it seems that we have the right rate region
  - Establishing these would entail new mathematical ideas/arguments
Recap

There are still a lot of "simple" yet fundamental open problems in network information theory

- Unfortunately, not many people seem to know/care to work on these problems
- Some of these may be solved by just putting enough computing power (at least a part of the heavy lifting towards evaluation has been done)
- In some scenarios it seems that we have the right rate region
  - Establishing these would entail new mathematical ideas/arguments

Questions?

Thank You